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Chaotic difference equations in two variables and their multidimensional perturbations

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Abstract
We consider difference equations $\Phi_\lambda(y_n, y_{n+1}, \ldots, y_{n+m}) = 0$, $n \in \mathbb{Z}$, of order m with parameter λ close to that exceptional value λ_0 for which the function Φ depends on two variables: $\Phi_{\lambda_0}(x_0, \ldots, x_m) = \xi(x_N, x_{N+L})$ with $0 \leqslant N, N+L \leqslant m$. It is also assumed that for the equation $\xi(x, y) = 0$, there is a branch $y = \varphi(x)$ with positive topological entropy $h_{\mathrm{top}}(\varphi)$. Under these assumptions we prove that in the set of bi-infinite solutions of the difference equation with λ in some neighbourhood of λ_0, there is a closed (in the product topology) invariant set to which the restriction of the shift map has topological entropy arbitrarily close to $h_{\mathrm{top}}(\varphi)/|L|$, and moreover, orbits of this invariant set depend continuously on λ not only in the product topology but also in the uniform topology. We then apply this result to establish chaotic behaviour for Arneodo–Coullet–Tresser maps near degenerate ones, for quadratic volume preserving automorphisms of $\mathbb{R}^3$ and for several lattice models including the generalized cellular neural networks (CNNs), the time discrete version of the CNNs and coupled Chua's circuit.

Mathematics Subject Classification: 39A05, 37B45, 37B10, 54F15, 54H20

1. Introduction

In this paper, we continue the study from [13] on the chaotic behaviour of solutions for perturbed singular difference equations. Consider a family of difference equations of the form

$$\Phi_\lambda(y_n, y_{n+1}, \ldots, y_{n+m}) = 0, \qquad n \in \mathbb{Z}, \tag{1}$$

where λ is a parameter from some metric space. In [13], we assumed that at an exceptional value of the parameter, say λ_0, the difference equation depends on only one variable, i.e.

$$\Phi_{\lambda_0}(x_0, \ldots, x_m) = \varphi(x_N),$$

where $0 \leqslant N \leqslant m$, and we showed that among solutions for perturbed difference equation (1) with λ close to λ_0, there are topological k-horseshoes (full Bernoulli shifts with k symbols), provided that φ has $k \geqslant 2$ simple zeros; moreover, we proved that orbits in these horseshoes change continuously in the uniform topology as λ varies (see theorem 11 in the next section). In this paper, we consider similar problems in the situation when the difference equation at the exceptional value of the parameter depends on two variables, i.e.

$$\Phi_{\lambda_0}(x_0, \ldots, x_m) = \xi(x_N, x_{N+L}),$$

where N and $N + L$ are two distinct integers between 0 and m, and $\xi(x, y)$ is a function such that for the equation $\xi(x, y) = 0$ there is a branch $y = \varphi(x)$ with positive topological entropy, i.e. $\xi(x, \varphi(x)) = 0$ and $h_{\text{top}}(\varphi) > 0$. Notice that in the case when $L = 1$, the solutions of difference equation (1) with $\lambda = \lambda_0$ contain orbits of the one-dimensional map $x \mapsto \varphi(x)$. On the other hand, if $L > 1$, the solutions of (1) with $\lambda = \lambda_0$ contain orbits of a generalized one-dimensional transformation which can be regarded as the 'Lth root' of φ (see subsections 2.1 and 2.4 for details). For many cases, solutions of difference equations can be considered as orbits of a high dimensional map.

In view of more applications, we allow the functions Φ_λ and the local map φ to be not defined in some regions; more precisely, we suppose that Φ_λ and φ are defined on domains Q and Q^{m+1}, respectively, where $Q = [s_1, s_2] \setminus V$ for some fixed real numbers s_1, s_2 and open set V, the latter being the union of finitely many open intervals in $[s_1, s_2]$. Here s_1 and s_2 can be regarded as some fixed bounds (from below and from above, respectively) for coordinate projections of orbits we are interested in, while V stays for an escaping region which is never visited by those interesting orbits (for example, if $\varphi(x) = ax(1 - x)$ with $a > 4$ on the interval [0; 1], then V could contain the escaping interval $(\frac{1}{2} - \frac{\sqrt{a^2-4a}}{2a}, \frac{1}{2} + \frac{\sqrt{a^2-4a}}{2a})$). Also, one may include in V those intervals where the functions under consideration are nonsmooth or discontinuous, whenever one is interested only in the orbits (solutions for (1)) which never visit V. In this situation, the topological entropy for (1) (as a quantity to estimate chaotic behaviour of solutions) is defined as $h_{\text{top}}(\sigma)$ for the shift map σ restricted to the set of bi-infinite solutions $(x_n)_{n=-\infty}^{\infty}$ for (1) (with respect to the product topology) satisfying $x_n \in Q$ for all $n \in \mathbb{Z}$. Also, $h_{\text{top}}(\varphi)$ is meant as the topological entropy of φ restricted to $\bigcap_{n=0}^{\infty} \varphi^{-n}(Q)$ (see the next section for more precise definitions and comments). Our main result shows that if $h_{\text{top}}(\varphi) > 0$, then for λ sufficiently close to λ_0, one can find a closed σ-invariant subset Γ_λ of the set of solutions for (1) such that $h_{\text{top}}(\sigma|_{\Gamma_\lambda})$ is arbitrarily close to $h_{\text{top}}(\varphi)/|L|$. Roughly speaking, the perturbed multidimensional difference equations are chaotic provided that the one-dimensional map at the unperturbed value of the parameter has enough chaotic orbits which avoid prescribed regions. More precisely, we will prove the following.

Theorem 1. *Consider a family of difference equations of the form*

$$\Phi_\lambda(y_n, y_{n+1}, \ldots, y_{n+m}) = 0, \qquad n \in \mathbb{Z}, \tag{2}$$

with the function $\Phi_\lambda : Q^{m+1} \to \mathbb{R}$ *which is* C^1 *for each* λ *and is continuous in* λ *along with the partial derivatives* $\partial_i \Phi_\lambda$, $i = 1, \ldots, m + 1$, $Q = [s_1, s_2] \setminus V$ *for some (fixed) real numbers* $s_1 < s_2$ *and* V *is the union of finitely many open intervals in* $[s_1, s_2]$, *while parameter* λ *is from some neighbourhood of the unperturbed value* λ_0 *in some metric space. Assume that at* λ_0, *the function* Φ *depends on exactly two variables:*

$$\Phi_{\lambda_0}(x_0, x_1, \ldots, x_m) = \xi(x_N, x_{N+L}),$$

$0 \leqslant N, N + L \leqslant m$. *Assume, in addition, that for the equation* $\xi(x, y) = 0$ *there is a branch* $y = \varphi(x)$ *with positive topological entropy, where* $\varphi : Q \to [s_1, s_2]$ *is supposed to be a* *piecewise analytic function*[3]*. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that for each λ in the δ-neighbourhood of λ_0, there is a closed (in the product topology) σ-invariant subset Γ_λ of the set of solutions for (1) with $h_{\mathrm{top}}(\sigma|_{\Gamma_\lambda}) > h_{\mathrm{top}}(\varphi)/|L| - \epsilon$. Moreover, solutions from Γ_λ depend continuously on λ both in the product and in the uniform topologies.*

In comparison with the mentioned result from [13], which can be regarded as a multidimensional perturbation of zero-dimensional systems, the presented result is in a sense a multidimensional perturbation (in the difference equations settings) of generalized one-dimensional maps (also see [18] for a topological approach to perturbations of one-dimensional maps to multidimensional ones). To establish the persistence of chaotic behaviour from low dimensional systems to perturbed high dimensional ones, we find a suitable hyperbolic repelling invariant set[4] carrying almost all topological entropy of the low dimensional system, and we show how to 'continue' these hyperbolic orbits to orbits for the perturbed systems.

So the first problem that appears here is in dimension one. It is well known that in contrast to higher dimensions, for smooth one-dimensional maps one has commonly axiom A and hyperbolicity. In [15], Mañé showed that for a C^2 interval map f whose periodic points are hyperbolic, any compact f-invariant set away from critical points is hyperbolic repelling (see proposition 3 in the next section for the case of piecewise C^2 interval maps). Nevertheless, given a C^2 interval map, it is not easy to check the above assumption whether all the periodic orbits are hyperbolic. Also, given a compact invariant set, in order to ensure its hyperbolicity, one needs to check that this set is disjoint from some neighbourhood of the critical set. On the other hand, if the map f has positive topological entropy, it is reasonable to ask whether there is a compact f-invariant hyperbolic set whose topological entropy approximates $h_{\mathrm{top}}(f)$ with required accuracy (see similar problems in [26] for piecewise monotone piecewise C^1 intervals maps without critical points and in [10] for $C^{1+\varepsilon}$ surface diffeomorphisms; let us mention in this connection that for merely continuous surface homeomorphisms such an approximation need not take place, because Rees in [23] has constructed a minimal positive entropy homeomorphism of the 2-torus). In this context, we prove that given a piecewise monotone piecewise continuous map f on the interval with $h_{\mathrm{top}}(f) > 0$, for any $\epsilon > 0$, there exists a compact f-invariant set $M = M_\epsilon$ away from some neighbourhood of critical points and discontinuity points, such that $h_{\mathrm{top}}(f|_M) > h_{\mathrm{top}}(f) - \epsilon$ (see theorem 4). Then, with the help of Mañé's theorem, we can get rid of the redundant assumptions (both on the map and on the orbits of points from the invariant set) to ensure hyperbolicity for the restriction of a piecewise analytic map f to an appropriate set carrying entropy bigger than $h_{\mathrm{top}}(f) - \epsilon$. Finally, by using our technique from [13], we are able to 'continue' this hyperbolic set for perturbed difference equations.

In many applications, solutions of the difference equations for nonexceptional values of λ are actually associated with orbits of well-defined maps on a high dimensional space; refer to definition 3.1 of [13] and the appendix. For instance, polynomial maps on $\mathbb{R}^m$, under generic algebraic conditions, can be written as difference equations for x_is in $\mathbb{R}$. In [11], the description of the elimination process for polynomial maps which leads to such difference equations is given, and the problems on the uniqueness of the obtained difference equations are discussed by using the algebraic hypersurface approach by Milnor [17]. In section 3, we apply theorem 1 to establish chaotic behaviour for Arneodo–Coullet–Tresser maps near degenerate ones and

[3] The condition on piecewise analyticity of φ can be weakened to be piecewise C^2 with piecewise monotone derivative; but, for such an extension, some cumbersome techniques from [12] are needed because there might be an interval consisting of nonhyperbolic periodic points for φ.

[4] An invariant set J of an interval map f is said to be *hyperbolic repelling* if there exist constants $C > 0$ and $\mu > 1$ such that $|(f^n)'(x)| > C\mu^n$ for all $x \in J$ and $n \in \mathbb{N}$.

for quadratic volume preserving automorphisms of $\mathbb{R}^3$ at anti-integrable limits[5]. We also apply theorem 1 together with some results from [13] to study chaotic structures in stationary and travelling waves of several models including the generalized cellular neural network (CNN), the time discrete version of the CNNs, the coupled Chua's circuit and lattice models of an evolution equation.

The paper is organized as follows. In section 2, we prove our main result (theorem 1), and along with the proof, in section 2.1, we recall some techniques from [13]. In sections 2.2 and 2.3, we prove proposition 3 and theorem 4, the important one-dimensional ingredients for the main result. In section 2.4, we give a comparison of the continuation schemes for perturbations of singular difference equations depending on one variable (as in [13]) with those of two variables (as in this paper). Section 3 collects several applications. In the appendix, we give some basic terminologies.

2. Perturbations of difference equations in one and two variables

2.1. Proof of theorem 1

First we introduce the necessary notation. Let ℓ_∞ denote the Banach space of bounded real bi-sequences endowed with the norm $\|\underline{y}\| = \sup_{n\in\mathbb{Z}} |y_n|$, where $\underline{y} = (y_n)_{n=-\infty}^{\infty} \in \ell_\infty$. In what follows, we will consider both ℓ_∞ and its subsets not only in the above (uniform) topology but also in the product topology on $\mathbb{R}^{\mathbb{Z}}$, i.e. in the topology of pointwise convergence. In the latter case, to avoid misunderstanding, we will sometimes supply the notation of appropriate sets with the subscript prod, for example: $\ell_{\infty,\text{prod}}$ and B_{prod}. Let σ denote the shift map on ℓ_∞, i.e. $\sigma(\underline{y}) = \underline{y}'$ with $y'_n = y_{n+1}$ and $n \in \mathbb{Z}$, for any $\underline{y} \in \ell_\infty$. We will denote by $U(c, r)$ the open ball of radius r centred at c in an appropriate metric space.

For each λ from the parameter space, let Y_λ be the set of solutions of the difference equation (2), i.e. the set of bi-sequences $\underline{y} = (y_n)_{n=-\infty}^{\infty}$ such that for any $n \in \mathbb{Z}$, one has $y_n \in Q$ and, moreover, $(m+1)$ consecutive components $y_n, y_{n+1}, \ldots, y_{n+m}$ of $\underline{y}$ satisfy (2). In [13], we have shown that Y_λ is a σ-invariant closed subset of the space ℓ_∞ in the topology of uniform convergence, $Y_{\lambda,\text{prod}}$ is compact as a closed subset of $Q^{\mathbb{Z}}_{\text{prod}}$ and the restriction $\sigma|_{Y_{\lambda,\text{prod}}}$ is a homeomorphism. Thus, one can define the topological entropy for solutions of the difference equation (2) as $h_{\text{top}}(\sigma|_{Y_{\lambda,\text{prod}}})$.

The following special version of the implicit function theorem is proved in [13]; it serves as the main theoretical background for our construction of symbolic continuation both in [13] and in this paper.

Theorem 2 ([13, theorems 2.1 and 2.5]). *Let E be a metric space with metric ρ and let a be a point in E. Let B be a σ-invariant compact subset of $[s_1, s_2]^{\mathbb{Z}}_{\text{prod}}$ for some real numbers $s_1 < s_2$. Denote $V_0 = U(a, \delta_0)$ and $W_0 = \bigcup_{b\in B} U(b, \eta_0)$ (the latter balls being with respect to the ℓ_∞ metric) for some $\delta_0 > 0$ and $\eta_0 > 0$ and assume that $F : V_0 \times W_0 \to \ell_\infty$ is a function such that the following conditions hold:*

- *(i) $F(a, b) = 0$ for all $b \in B$;*
- *(ii) F is continuous and, moreover, the family of functions $F(\cdot, y)$ with the domain V_0 and parameter $y \in W_0$ is equicontinuous, i.e. for any $\epsilon > 0$ there is $\delta > 0$ such that for any $y \in W_0$ one has $\|F(x_1, y) - F(x_2, y)\| < \epsilon$ whenever $\rho(x_1, x_2) < \delta$;*

[5] The latter problem was motivated by a question posed to us in the referee report to our paper [13]. The referee asked whether the results of [13] could apply for quadratic volume preserving automorphisms on $\mathbb{R}^3$ in the generic form at anti-integrable limits. Actually, the difference equations for such limits depend on two variables (see section 3.2), and so, in order to establish chaotic behaviour in this situation, we have to use the presented new approach.

(*iii*) *the partial derivative operator with respect to the second variable,* $D_2F(x, y)$*, at any point* $(x, y) \in V_0 \times W_0$ *exists and is continuous at* $\{a\} \times B$ *uniformly in* $b \in B$ *in the following sense: for any* $\epsilon > 0$ *there exist* $\delta > 0$ *and* $\eta > 0$ *such that for any* $b \in B$, $\|D_2F(x, y) - D_2F(a, b)\| < \epsilon$ *whenever* $x \in U(a, \delta)$ *and* $y \in U(b, \eta)$*;*

(*iv*) *the operator* D_2F *at any point* $(a, b) \in \{a\} \times B$ *is invertible, and the inverse,* $(D_2F)^{-1}$*, is uniformly bounded, i.e. there is a constant* $M > 0$ *such that for any* $b \in B$, $\|(D_2F(a, b))^{-1}\| \leqslant M$ *and*

(*v*) *for any* $x \in U(a, \delta_0)$*, the function* $F(x, \cdot)$ *commutes with* σ *and is continuous with respect to the product topologies on the domain* W_0 *and codomain* ℓ_∞*.*

Then there exist $0 < \hat{\delta} < \delta_0$ *and* $0 < \hat{\eta} < \eta_0$ *such that for any* $x \in U(a, \hat{\delta})$*, there is a map from* B *to* $U(b, \hat{\eta})$*, given by* $b \mapsto \bar{\psi}_x(b) := \psi_b(x)$*, which conjugates* $\sigma|_{B_{\text{prod}}}$ *to* $\sigma|_{\bar{\psi}_x(B)_{\text{prod}}}$*. Moreover, the conjugacy map depends continuously in* x *not only in the product topology but also in the* ℓ_∞ *topology; more precisely, the family of maps* $x \mapsto \psi_b(x)$ *from* $U(a, \hat{\delta})$ *to* ℓ_∞ *forms an equicontinuous family in* $b \in B$*.*

In order to apply theorem 2, we consider E to be the parameter metric space and a to be the unperturbed parameter value λ_0 in theorem 1. Define $F : E \times Q^{\mathbb{Z}} \to \ell_\infty$ by

$$F(\lambda, \underline{y}) = (\Phi_\lambda(y_n, y_{n+1}, \ldots, y_{n+m}))_{n=-\infty}^{\infty}.$$

Then, Y_λ is precisely the zero-set of $F(\lambda, \cdot)$, i.e. Y_λ is the set of bi-sequences $\underline{y} \in Q^{\mathbb{Z}}$ satisfying $F(\lambda, \underline{y}) = \underline{0}$. Consider B to be a σ-invariant and compact subset of $\overline{Y_\lambda}$ (in the product topology), which will be specified later. Then assumption (i) of theorem 2 is satisfied. Since $E \times Q^{m+1}$ is compact, both $F(\lambda, \underline{y})$ and $D_2F(\lambda, \underline{y})$ are uniformly continuous on $E \times Q^{\mathbb{Z}}$. Therefore assumptions (ii), (iii) and (v) of theorem 2 are also satisfied. As for the most delicate assumption (iv), we need to specify a suitable subset B of Y_λ in order to guarantee this assumption.

Given $\underline{y} = (y_n) \in Q^{\mathbb{Z}}$, $\lambda \in E$, and integers $n \in \mathbb{Z}$, $1 \leqslant i \leqslant m + 1$, we denote for brevity $\partial_i\Phi_\lambda(\tilde{y}_n) = \partial_i\Phi_\lambda(y_n, y_{n+1}, \ldots, y_{n+m})$. Then by our assumptions on Φ_λ, we have that the partial derivative operator $D_2F(\lambda, \underline{y})$ exists at any point $(\lambda, \underline{y}) \in E \times Q^{\mathbb{Z}}$ and is represented by the following bi-infinite band matrix.

$$D_2F(\lambda, \underline{y})$$

$$= \begin{bmatrix} \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & \partial_1\Phi_\lambda(\tilde{y}_n) & \partial_2\Phi_\lambda(\tilde{y}_n) & \cdot & \cdots & \partial_{m+1}\Phi_\lambda(\tilde{y}_n) & 0 & \cdots \\ \cdots & 0 & \partial_1\Phi_\lambda(\tilde{y}_{n+1}) & \partial_2\Phi_\lambda(\tilde{y}_{n+1}) & \cdots & \cdot & \partial_{m+1}\Phi_\lambda(\tilde{y}_{n+1}) & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix} \begin{matrix} \text{the} \\ \leftarrow n\text{th.} \\ \text{row} \\ \end{matrix}$$

$\uparrow$ the nth column

Without loss of generality, we assume that $L > 0$, the case when $L < 0$ needs no additional treatment, because σ is a homeomorphism on the compact space $Y_{\lambda,\text{prod}}$ and $h_{\text{top}}(\sigma|_S) = h_{\text{top}}(\sigma^{-1}|_S)$ for any compact invariant set $S \subset Y_{\lambda,\text{prod}}$. Note that under our assumption on the branch $y = \varphi(x)$ of the equation $\xi(x, y) = 0$, at the exceptional value of the parameter, the difference equation (2) reads $y_{n+L} = \varphi(y_n)$, $n \in \mathbb{Z}$. Hence the difference equation at $\lambda = \lambda_0$ corresponds to the map $f_{\lambda_0} : Q^L \to [s_1, s_2]^L$ of the form

$$f_{\lambda_0}(x_1, x_2, \ldots, x_{L-1}, x_L) = (x_2, x_3, \ldots, x_L, \varphi(x_1));$$

see the appendix for what we mean by 'a difference equation corresponds to a map'. So f_{λ_0} can be regarded as the 'Lth root' (or '$1/L$th iterate') of the one-dimensional map φ. Also, it is easy to see that for each $k \in \mathbb{N}$, the iterate $f_{\lambda_0}^{kL}$ of f_{λ_0} is of the form

$$f_{\lambda_0}^{kL}(x_1, x_2, \ldots, x_{L-1}, x_L) = (\varphi^k(x_1), \varphi^k(x_2), \ldots, \varphi^k(x_L))$$

and corresponds to the difference equation

$$y_{kL+n} - \varphi^k(y_n) = 0, \qquad n \in \mathbb{Z}. \tag{3}$$

To guarantee assumption (iv) of theorem 2, we need a uniform estimate of the inverse of $D_2F(\lambda_0, \underline{y})$ on a suitable σ-invariant and compact subset B of Y_λ. To this end, we shall find a compact φ-invariant hyperbolic set (carrying enough topological entropy) which should be 'continued' by orbits of difference equation (2) for λ close to λ_0.

To estimate the topological entropy carried by such orbits, we will need some general results on the entropy of piecewise monotone (possibly discontinuous) maps. Let us give certain definitions and agreements about piecewise monotone maps on the interval. Without loss of generality, we put $I = [0, 1]$ for the interval. Let $f : I \to I$ be a piecewise monotone piecewise continuous map and let $\mathcal{Z}$ be its partition, i.e. $I = \bigcup_{Z \in \mathcal{Z}} \bar{Z}$ and $\mathcal{Z}$ consists of finitely many, say k, disjoint open intervals, denoted by $(0, d_1), (d_1, d_2), \ldots, (d_{k-1}, 1)$, on each of which the restriction of f is monotone and continuous. It is also assumed that a piecewise monotone map can have only finitely many constancy intervals, i.e. maximal subintervals at which f takes a constant value. Let us remark that the map f need not be strictly monotone on each interval in $\mathcal{Z}$ and moreover d_is need not be critical or discontinuity points. We say that a piecewise monotone map $g : I \to I$ is *piecewise* C^2 (respectively, *piecewise analytic*) if its partition can be chosen so that on each interval of the partition, g is C^2 (respectively, analytic). We will refer to the intervals of such a partition simply as *monotonicity intervals*. We will need the following lemma whose proof is given in section 2.2.

Proposition 3. *Let $g : I \to I$ be piecewise C^2 and let U be a neighbourhood of the set which consists of all critical points of g and all endpoints of monotonicity intervals. Then*

1. *all periodic orbits of g contained in $I \backslash U$ of sufficiently large periods are hyperbolic repelling,*
2. *if $M \subset I$ is a compact forward invariant set which contains neither attracting nor nonhyperbolic periodic points of g and is disjoint from U, then M is a hyperbolic repelling set.*

To define the topological entropy for piecewise continuous piecewise monotone maps, we use here the approach by doubling points construction, as in [19]; see section 2.3 for details. There are other definitions of topological entropy for these maps (via separated or spanned sets and also by counting the growth number for preturning points) and they are equivalent, as shown in [20].

The following result is an important ingredient for the proof of theorem 1. In its statement, by *strict f-invariance* of a set M we mean the equality $f(M) = M$.

Theorem 4. *Let f be a piecewise monotone piecewise continuous map on I with the partition $\mathcal{Z} = \{(0, d_1), (d_1, d_2), \ldots, (d_{k-1}, 1)\}$. If $h_{\mathrm{top}}(f) > 0$ then for any $\epsilon > 0$ there is a compact strictly f-invariant set $M \subset I$ and an open set $J \supset \{0, d_1, \ldots, d_{k-1}, 1\}$ such that $h_{\mathrm{top}}(f|_M) > h_{\mathrm{top}}(f) - \epsilon$ and $M \cap J = \emptyset$.*

The proof of theorem 4 is postponed to section 2.3. We continue the proof of theorem 1. Without loss of generality, we assume that the interval $[s_1, s_2]$ in theorem 1 is $I = [0, 1]$. Since we allowed the local map $\varphi : I \setminus V \to I$ to be not defined on V, which consists of finitely many open intervals, let us extend φ to become a self-map $\varphi : I \to I$ by taking at those 'exceptional' intervals the constant value equal to 1, the right endpoint of I (it is easily seen that this extension does not influence the value of the topological entropy of φ, see also [20]). We may apply theorem 4 to our piecewise analytic self-map $\varphi : I \to I$ because it is surely piecewise monotone and has only finitely many constancy intervals. Since φ has only finitely many critical points away from constancy intervals, we may include these points into the set D of endpoints of monotonicity and constancy intervals. Finally, let us agree that the (finitely many) values which φ takes at constancy intervals are also included in D. Then D induces a partition consisting of finitely many disjoint open intervals, on each of which the restriction of φ is monotone and continuous.

Given any small $\varepsilon > 0$, by theorem 4, one gets a compact strictly φ-invariant set, which is denoted now by $\bar{M}_\epsilon$, and an open set $J \supset D$ such that the φ-orbits of $\bar{M}_\epsilon$ are disjoint from J, and $h_{\text{top}}(\varphi|_{\bar{M}_\epsilon}) > h_{\text{top}}(\varphi) - \varepsilon/2$. Note that the φ-orbits of $\bar{M}_\epsilon$ are disjoint from each constancy interval for φ (if it exists). For convenience, redenote the open set J by adding all constancy open intervals. Then $h_{\text{top}}(\varphi|_{I\setminus J}) \geqslant h_{\text{top}}(\varphi|_{\bar{M}_\epsilon}) > h_{\text{top}}(\varphi) - \varepsilon/2$. By applying proposition 3 to φ with the neighbourhood J of the set of all critical points and endpoints of monotonicity intervals, one gets that all attracting and nonhyperbolic periodic points in $I \setminus J$ have bounded periods, say k_0. Thus, φ has finitely many attracting and nonhyperbolic periodic points in $I \setminus J$. Redenote the partition D by adding those attracting and nonhyperbolic periodic points in $I \setminus J$. Modify φ to get a new map $\tilde{\varphi}$ on I by taking the value 1 on the set J. Then $\tilde{\varphi}|_{I\setminus J} = \varphi|_{I\setminus J}$ and $h_{\text{top}}(\tilde{\varphi}) = h_{\text{top}}(\varphi|_{I\setminus J})$. By applying theorem 4 to $\tilde{\varphi}$ with the new partition D, we get a compact $\tilde{\varphi}$-invariant set M and an open set $\tilde{J} \supset D$ such that the $\tilde{\varphi}$-orbits of M are disjoint from $\tilde{J}$, and $h_{\text{top}}(\tilde{\varphi}|_M) > h_{\text{top}}(\tilde{\varphi}) - \varepsilon/2$. Then M is φ-invariant, the φ-orbits of M are disjoint from $\tilde{J}$ and

$$h_{\text{top}}(\varphi|_M) = h_{\text{top}}(\tilde{\varphi}|_M) > h_{\text{top}}(\tilde{\varphi}) - \varepsilon/2 = h_{\text{top}}(\varphi|_{I\setminus J}) - \varepsilon/2 > h_{\text{top}}(\varphi) - \varepsilon.$$

By the second item of proposition 3, M is a hyperbolic repelling set. Moreover, it can be shown (see lemma 2.1 of chapter III in [16]) that there is an integer $k_1 > k_0$ and a real number $\eta > 0$ such that $\|D\varphi^{k_1}(x)\| > 1 + \eta$ for all $x \in M$.

Now that we have found the set $M_\epsilon := M$, we are in a position to check assumption (iv) of theorem 2. Let $B = \overleftarrow{\lim}(M, \varphi) = \{(x_n)_{n=-\infty}^{\infty} : x_{n+1} = \varphi(x_n) \text{ and } x_n \in M \text{ for all } n \in \mathbb{Z}\}$. By replacing φ by φ^{k_1} and the difference equation (2) at λ_0 by (3) with φ^{k_1}, we get that the partial derivative operator $D_2F(\lambda_0, \underline{y})$ for all $\underline{y} \in B$ has the matrix of the form $\sigma^{k_1 L} \circ (I + \Lambda)$, where σ is the matrix of the shift operator, I is the identity matrix and Λ is a (shifted) one-diagonal matrix with entries bigger than $1+\eta$ in absolute value. By using the following lemma on the norm of two-diagonal infinite matrices, assumption (iv) of theorem 2 will be satisfied.

Lemma 5. *Let $A : \ell_\infty \to \ell_\infty$ be a linear operator given by $A = \sigma^k \circ (I + \Lambda)$, where $k \in \mathbb{Z}$, σ is the shift operator and Λ is associated with matrix of the form*

$$\Lambda_{ij} = \begin{cases} q_i, & \text{if } j = i + L, \\ 0, & \text{otherwise,} \end{cases}$$

for some sequence $(q_i)_{i=-\infty}^{\infty}$ satisfying $q := \inf_{i\in\mathbb{Z}} |q_i| > 1$. Then A is invertible and $\|A^{-1}\| < 1/(q-1)$.

Proof. Note that the operator Λ is invertible, and its inverse is represented by the matrix of the form

$$\Lambda^{-1}_{ij} = \begin{cases} 1/q_{i-1}, & \text{if } j = i - L, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\|\Lambda^{-1}\| \leqslant \frac{1}{q} < 1$. Then

$$A^{-1} = (I + \Lambda)^{-1} \circ \sigma^{-k} = \Lambda^{-1} \circ (I + \Lambda^{-1})^{-1} \circ \sigma^{-k} = \Lambda^{-1} \circ \left(\sum_{i=0}^{\infty} (-1)^i \Lambda^{-i} \right) \circ \sigma^{-k}.$$

Since the shift operator σ is invariant with respect to the ℓ_∞-norm, i.e. $\|\sigma \circ T\| = \|T \circ \sigma^{-1}\| = \|T\|$ for any T, and since $\|\Lambda^{-1}\| < 1$, it follows that the last series converges in the operator norm. So we have

$$\|A^{-1}\| = \|\Lambda^{-1} \circ \left(\sum_{i=0}^{\infty} (-1)^i \Lambda^{-i} \right) \circ \sigma^{-k}\| \leqslant \|\Lambda^{-1}\| \cdot \left(\sum_{i=0}^{\infty} \|\Lambda^{-1}\|^i \right)$$
$$\leqslant \frac{1}{q} \cdot \frac{1}{1 - q^{-1}} = \frac{1}{q - 1}.$$

The proof of the lemma is completed. □

Hence, theorem 2 implies that we may use the conjugacy $\bar{\psi}_\lambda|_B$ to get a closed σ-invariant subset $\Gamma_\lambda := \bar{\psi}_\lambda(B)$ of $Y_{\lambda,\text{prod}}$ such that $\sigma|_B$ is topologically conjugate to $\sigma^L|_{\Gamma_\lambda}$ (both in the product topology). For details about the conjugacy map $\bar{\psi}_\lambda$ and its properties see section 2.4, especially diagram (15). Therefore, $h_{\text{top}}(\sigma|_{\Gamma_\lambda}) = h_{\text{top}}(\sigma|_B)/|L| = h_{\text{top}}(\varphi|_M)/|L|$, which is arbitrarily close to $h_{\text{top}}(\varphi)/|L|$. This completes the proof of theorem 1.

2.2. *Proof of proposition 3*

Proposition 3 itself and its proof are adapted from the following results by Mañé.

Theorem 6 ([15], see also [16]). *Let I be a compact interval in $\mathbb{R}$ and $g : I \to I$ be a C^2 map. Let U be a neighbourhood of the set of critical points of g. Then*

1. *all periodic orbits of g contained in $I \backslash U$ of sufficiently large periods are hyperbolic repelling,*
2. *if all the periodic orbits of g contained in $I \backslash U$ are hyperbolic repelling, then there exist $C > 0$ and $\mu > 1$ such that $\|Dg^n(x)\| \geqslant C\mu^n$, whenever $g^i(x) \in I\backslash(U \cup B_0)$ for all $0 \leqslant i \leqslant n - 1$, where B_0 is the union of the immediate basins of the attracting periodic orbits of g contained in $I \backslash U$.*

The above theorem implies the following important corollary.

Corollary 7 ([16, corollary III.5.1]). *Let I be a compact interval in $\mathbb{R}$, $g : I \to I$ be a C^2 map and $M \subset I$ be a compact forward invariant set. If M does not contain critical points, attracting periodic points and nonhyperbolic periodic points of g, then it is a hyperbolic repelling set.*

Let us prove proposition 3 by applying theorem 6 and corollary 7 as follows. In order to obtain a C^2 map, we can modify the map g inside small neighbourhoods of endpoints of monotonicity intervals so that such neighbourhoods are contained in U. Denote the obtained map by G. By applying theorem 6 to G, we get that any periodic orbit of G contained in $I \setminus U$ of sufficiently large period is hyperbolic repelling. On the other hand, by the construction of G, any g-periodic orbit contained in $I \setminus U$ coincides with the G-periodic orbit and is away from U. This proves the first item. The second item follows from corollary 7 being applied for G, because here we use again that G and g coincide on $I \backslash U$.

2.3. Proof of theorem 4

The proof of theorem 4 contains several lemmas. We will need the so-called doubling points construction (see [19] for instance). Let $I = [0, 1]$ and $D = \{d_1, d_2, \dots, d_{k-1}\}$. Let us emphasize that we do not care about values of f at the points from D because only one-sided limits of f at these points are of use (see also [20], where the authors proved that the values of the map at the endpoints of intervals of continuity are irrelevant for calculation of topological entropy). Define the set

$$W := \left(\bigcup_{i=0}^{\infty} f^{-i}(D)\right) \setminus \{0, 1\}$$

(which could be thought of as 'the set of preturning points'). Now consider the following set $\hat{I}$ which contains 'doubling preturning points' rather than single ones:

$$\hat{I} := (I \setminus W) \cup \{w^-, w^+ : w \in W\}.$$

This means that we have doubled (i.e. separated by moving apart) all points of D along with all their inverse images; the order and the topology in this new set are as follows. The set $\hat{I} = \hat{I}(f)$ is endowed with the natural (full) order so that if $y < w < z$ in I and $w \in W$, then $y < w^- < w^+ < z$. Then it is supposed that $\hat{I}$ is endowed with the order topology (note that $\hat{I}$ is a totally disconnected space provided f has no homtervals). It is also convenient sometimes to include the points $\{0, 1\}$ in D, in which case the ('half-open') intervals $[0, 0^+)$ and $(1^-, 1]$ are included in $\hat{I}$. We will call $\hat{I}$ the *doubling construction space* for f.

Let $\pi : \hat{I} \to I$ denote the map by

$$\pi(y) = w \quad \text{for } y \in \{w^-, w^+\} \text{ with } w \in W \text{ and } \pi(y) = y \quad \text{for } y \in I \setminus W. \tag{4}$$

For a subset $A \subset I$, let $\text{clos}_{\hat{I}} A$ denote the closure of $\pi^{-1}(A \setminus W)$ in $\hat{I}$. Let $\hat{\mathcal{Z}} = \{\text{clos}_{\hat{I}} A : A \in \mathcal{Z}\}$. The restriction $f|_{I\setminus W}$ can be uniquely extended to a continuous piecewise monotone map $\hat{f} : \hat{I} \to \hat{I}$. We will call $\hat{f}$ the *doubling extension* of f.

For $x, y \in \hat{I}$, let $\ell(x, y)$ be the minimal nonnegative integer ℓ such that $\hat{f}^\ell(x)$ and $\hat{f}^\ell(y)$ belong to different elements of $\hat{\mathcal{Z}}$, and set $\ell(x, y) = +\infty$ if for any n, $\hat{f}^n(x)$ and $\hat{f}^n(y)$ belong to the same element (depending on n) of $\hat{\mathcal{Z}}$. Then the order topology on $\hat{I}$ is induced by the metric $\hat{\rho}$ on $\hat{I}$ defined by the formula

$$\hat{\rho}(x, y) := \frac{1}{\ell(x, y) + 1} + |\pi(x) - \pi(y)|. \tag{5}$$

We remark that $\hat{\rho}(d^-, d^+) = 1$ for any $d \in D$ (even in the case when f is continuous at d). The map π is continuous on $\hat{I}$ and it is a semiconjugacy from $\hat{f}$ to f in the following sense: $f \circ \pi(x) = \pi \circ \hat{f}(x)$ for all $x \in \hat{I} \setminus \{d^-, d^+ : d \in D\}$, and if $x = d^+$ (respectively, $x = d^-$) for some $d \in D$, then $\pi \circ \hat{f}(x) = \lim_{y \searrow d} f \circ \pi(y)$ (respectively, $\pi \circ \hat{f}(x) = \lim_{y \nearrow d} f \circ \pi(y)$). So, if P is an $\hat{f}$-invariant set disjoint from $\{w^-, w^+ : w \in W\}$ then π conjugates $\hat{f}|_P$ to $f|_{\pi(P)}$.

According to the full order in $\hat{I}$, we can consider intervals in $\hat{I}$ of the form (a, b), $(a, b]$, $[a, b)$, or $[a, b]$ (the latter possibly with $a = b$), $a, b \in \hat{I}$. Let $c \in \hat{I}$, $\epsilon > 0$ and let $U_\epsilon(c)$ denote the open ball of radius ϵ centred at c : $U_\epsilon(c) = \{x \in \hat{I} : \hat{\rho}(x, c) < \epsilon\}$. Note that $U_\epsilon(c)$ is an interval in $\hat{I}$ which might have any of the four above forms, and $\pi(U_\epsilon(c))$ is an interval in I which contains $\pi(c)$, but $\pi(c)$ need not be the middle point of this interval. Nevertheless, it is easily seen that $\pi(U_\epsilon(c))$ tends to $\{c\}$ as $\epsilon \to 0$.

Following [19], we define the topological entropy, $h_{\text{top}}(f)$, of the initial map f to be $h_{\text{top}}(\hat{f})$, the usual topological entropy of the continuous map $\hat{f}$ on the compact space $\hat{I}$. Given an $\epsilon > 0$ and $a \in \hat{I}$, let $M_{\epsilon,a}$ denote the set

$$M_{\epsilon,a} := \hat{I} \setminus \bigcup_{n=0}^{\infty} \hat{f}^{-n}(U_\epsilon(a)). \tag{6}$$

So $M_{\epsilon,a}$ is a compact $\hat{f}$-invariant subset of $\hat{I}$ (in the order topology), and $\hat{f}$-orbits of points from $M_{\epsilon,a}$ never visit $U_\epsilon(a)$. Let us fix (for a while) an ϵ with

$$0 < \epsilon < \min\{1, \min\{|d - d'|/2 : d, d' \in D, d \neq d'\}\} \tag{7}$$

and take a point $a \in \{d^-, d^+\}$ for some $d \in D$. We now consider, without loss of generality, the case when $a = d^-$. Then, because of (7), the subinterval $U_\epsilon(a)$ of $\hat{I}$ is of the form either $(y, d^-]$ with $y \in I \setminus W$ or $[y, d^-]$ with $y = w^+$ for some $w \in W$. We denote such a left endpoint y by d_ϵ^-. Note that (7) also implies that these $2(m-1)$ subintervals $\{U_\epsilon(d^-), U_\epsilon(d^+) : d \in D\}$ are disjoint (because the distance $\hat{\rho}$ between points on $\hat{I}$ is bigger than or equal to the distance on I between the π-image of these points).

Define the following map $\tilde{f}_{\epsilon,d^-}$ on $\hat{I}$ by

$$\tilde{f}_{\epsilon,d^-}(x) = \begin{cases} \hat{f}(x), & \text{if } x \notin U_\epsilon(d^-), \\ \hat{f}(d_\epsilon^-), & \text{otherwise}, \end{cases} \tag{8}$$

and call it the *left ϵ-truncation of $\hat{f}$ at d.*

Lemma 8. *The map $\tilde{f}_{\epsilon,d^-} : \hat{I} \to \hat{I}$ is continuous and $h_{\mathrm{top}}(\tilde{f}_{\epsilon,d^-}) = h_{\mathrm{top}}(\tilde{f}_{\epsilon,d^-}|_{M_{\epsilon,d^-}}) = h_{\mathrm{top}}(\hat{f}|_{M_{\epsilon,d^-}})$.*

Proof. The map $\tilde{f}_{\epsilon,d^-}$ differs from $\hat{f}$ only in the interval $U_\epsilon(d^-)$, which equals either $(d_\epsilon^-, d^-]$ or $[d_\epsilon^-, d^-]$. Since $\hat{f}$ is continuous, it follows that in both cases $\tilde{f}_{\epsilon,d^-}$ is continuous at d_ϵ^- (because of the definition by (8)). Moreover, $\tilde{f}_{\epsilon,d^-}$ is continuous at d^- because d^- is isolated in $\hat{I}$ from the right.

Let $\Omega(\tilde{f}_{\epsilon,d^-})$ be the nonwandering set of $\tilde{f}_{\epsilon,d^-}$. If $\hat{f}^n(d_\epsilon^-) \cap U_\epsilon(d^-) = \varnothing$ for every $n \geqslant 1$, then $\Omega(\tilde{f}_{\epsilon,d^-}) \subset M_{\epsilon,d^-}$ because $U_\epsilon(d^-)$ consists of a wandering point for $\tilde{f}_{\epsilon,d^-}$, and if we suppose, in contrast, that there is a point in $\Omega(\tilde{f}_{\epsilon,d^-}) \setminus M_{\epsilon,d^-}$, then we would have a contradiction to the fact that the nonwandering set is invariant. Therefore,

$$h_{\mathrm{top}}(\tilde{f}_{\epsilon,d^-}) = h_{\mathrm{top}}(\tilde{f}_{\epsilon,d^-}|_{\Omega(\tilde{f}_{\epsilon,d^-})}) = h_{\mathrm{top}}(\tilde{f}_{\epsilon,d^-}|_{M_{\epsilon,d^-}}) = h_{\mathrm{top}}(\hat{f}|_{M_{\epsilon,d^-}}), \tag{9}$$

where the last equality holds because the restriction of $\hat{f}$ to M_{ϵ,d^-} coincides with $\tilde{f}_{\epsilon,d^-}$. In the case when $\hat{f}^{n_0}(d_\epsilon^-) \in U_\epsilon(d^-)$ for some $n_0 \geqslant 1$, it is easily seen that the set $\Omega(\tilde{f}_{\epsilon,d^-}) \setminus M_{\epsilon,d^-}$ consists precisely of one periodic orbit of period n_0. Thus we have as before, $h_{\mathrm{top}}(\tilde{f}_{\epsilon,d^-}|_{\Omega(\tilde{f}_{\epsilon,d^-})}) = h_{\mathrm{top}}(\tilde{f}_{\epsilon,d^-}|_{M_{\epsilon,d^-}})$ because the topological entropy on a finite set is zero. So in this case the equalities in (9) are true. □

Similar to the left ϵ-truncation, we can consider the right ϵ-truncation, $\tilde{f}_{\epsilon,d^+}$, and get $h_{\mathrm{top}}(\hat{f}|_{M_{\epsilon,d^+}}) = h_{\mathrm{top}}(\tilde{f}_{\epsilon,d^+}|_{M_{\epsilon,d^+}})$. Furthermore, we consider the ϵ-truncation for all $d \in D$ simultaneously. To do this, we define the map $\tilde{f}_\epsilon$ on $\hat{I}$ by

$$\tilde{f}_\epsilon(x) = \begin{cases} \hat{f}(x), & \text{if } x \notin U_\epsilon(d^-) \cup U_\epsilon(d^+) \text{ for all } d \in D, \\ \hat{f}(d_\epsilon^-), & \text{if } x \in U_\epsilon(d^-) \text{ with } d \in D, \\ \hat{f}(d_\epsilon^+), & \text{if } x \in U_\epsilon(d^+) \text{ with } d \in D. \end{cases} \tag{10}$$

Also let $\tilde{M}_\epsilon := \bigcap_{d\in D}(M_{\epsilon,d^-} \cap M_{\epsilon,d^+})$. Then $\tilde{M}_\epsilon$ is a compact $\hat{f}$-invariant subset of $\hat{I}$ whose $\hat{f}$-orbits never visit the ϵ-neighbourhood of the set $\bigcup_{d\in D}\{d^-, d^+\}$. Since D is a finite set, a result similar to lemma 8 readily follows.

Lemma 9. *The map $\tilde{f}_\epsilon : \hat{I} \to \hat{I}$ is continuous and $h_{\rm top}(\tilde{f}_\epsilon) = h_{\rm top}(\tilde{f}_\epsilon|_{\tilde{M}_\epsilon}) = h_{\rm top}(\hat{f}|_{\tilde{M}_\epsilon})$.*

In order to relate the above properties of continuous maps on $\hat{I}$ to the properties of piecewise continuous maps on I (in other words, 'to project' the constructed truncations to maps on I), we need to introduce an intermediate space. To do this, we identify those pairs of points $\{w^-, w^+\}$, $w \in W$, which under some iterate of $\hat{f}$ belong to the same ϵ-neighbourhood of either d^- or d^+ for some $d \in D$. More precisely, consider the following equivalence relation $\sim$ on $\hat{I}$:

$$x \sim y \iff \begin{array}{l} x = y \text{ or } \{x, y\} = \{w^-, w^+\},\ w \in W, \text{ and there exist} \\ n \geqslant 0 \text{ and } \tilde{d} \in \bigcup_{d\in D}\{d^-, d^+\} \text{ such that } \{\hat{f}^n(x), \hat{f}^n(y)\} \subset U_\epsilon(\tilde{d}). \end{array}$$

Note that by the above definition, the relation $\sim$ is preserved by $\tilde{f}_\epsilon$, i.e. if $x \sim y$ then $\tilde{f}_\epsilon(x) \sim \tilde{f}_\epsilon(y)$. Let $\hat{I}_\epsilon$ be the quotient space with respect to this relation $\sim$ and let $\hat{\pi}_\epsilon : \hat{I} \to \hat{I}_\epsilon$ be the corresponding quotient map. Clearly, $\hat{\pi}_\epsilon$ is at most two-to-one, order preserving, and is continuous with respect to the order topologies on $\hat{I}$ and $\hat{I}_\epsilon$. If two points w^-, w^+ are collapsed by $\hat{\pi}_\epsilon$ (i.e. $w^- \sim w^+$), we will denote their common image simply by w. Let W_ϵ denote the subset of W which consists of 'noncollapsed' points by $\hat{\pi}_\epsilon$, i.e.

$$W_\epsilon = \{w \in W : \sharp\{\hat{\pi}_\epsilon^{-1}(w)\} = 1\}.$$

Then $\hat{I}_\epsilon$ can be represented as $\hat{I}_\epsilon = (I \setminus W_\epsilon) \cup \{w^-, w^+ : w \in W_\epsilon\}$. By the definition of $\hat{\pi}_\epsilon$, one easily gets $\pi_\epsilon \circ \hat{\pi}_\epsilon = \pi$, where $\pi_\epsilon : \hat{I}_\epsilon \to I$ is defined just as π by (4) (for π_ϵ we use the subscript ϵ in order to mention that it acts on the space different from $\hat{I}$).

Let $g : \hat{I} \to \hat{I}$ be a continuous map which preserves the relation $\sim$, i.e. g satisfies the assumption that $g(w^-) = g(w^+)$ for every $w \in W$ with $\hat{\pi}_\epsilon(w^-) = \hat{\pi}_\epsilon(w^+)$. Then g projects to a continuous map on $\hat{I}_\epsilon$. Indeed, consider the map $g^\dagger : \hat{I}_\epsilon \to \hat{I}_\epsilon$ defined by

$$g^\dagger(x) = \hat{\pi}_\epsilon \circ g(\hat{\pi}_\epsilon^{-1}(x)),$$

where by $\hat{\pi}_\epsilon^{-1}(x)$ we mean the full preimage of x; note that although $\hat{\pi}_\epsilon^{-1}(x)$ may consist of two points, $g(\hat{\pi}_\epsilon^{-1}(x))$ is a single point because of the above assumption on g. By its definition, $g^\dagger$ is continuous and satisfies $g^\dagger \circ \hat{\pi}_\epsilon = \hat{\pi}_\epsilon \circ g$. We may apply the above construction to the map $g = \tilde{f}_\epsilon$ because it is continuous on $\hat{I}$ and preserves the relation $\sim$; hence we have the continuous map $\tilde{f}_\epsilon^\dagger$ on $\hat{I}_\epsilon$.

Now we return to the 'initial phase space' I. Let $M_\epsilon^{'} := \pi(\tilde{M}_\epsilon)$.

Lemma 10. *The following statements hold:*

1. *the set $M_\epsilon^{'}$ is a compact f-invariant subset of I;*
2. *there is a neighbourhood of D such that for any point $x_0 \in M_\epsilon^{'}$, its f-orbit is disjoint from this neighbourhood.*

Proof. Let us recall that $\pi : \hat{I} \to I$ is continuous and the restriction of π to any subset of $\hat{I}$ disjoint from $\{w^-, w^+ : w \in W\}$ is a one-to-one map which satisfies $f \circ \pi(x) = \pi \circ \hat{f}(x)$. This implies the first statement of the lemma. Let

$$\delta_0 = \min_{d\in D} \min\{d - \pi(d_\epsilon^-), \pi(d_\epsilon^+) - d\}.$$

Then the f-orbit of any initial point in $M_\epsilon^{'}$ is away from D by the distance of at least δ_0. □

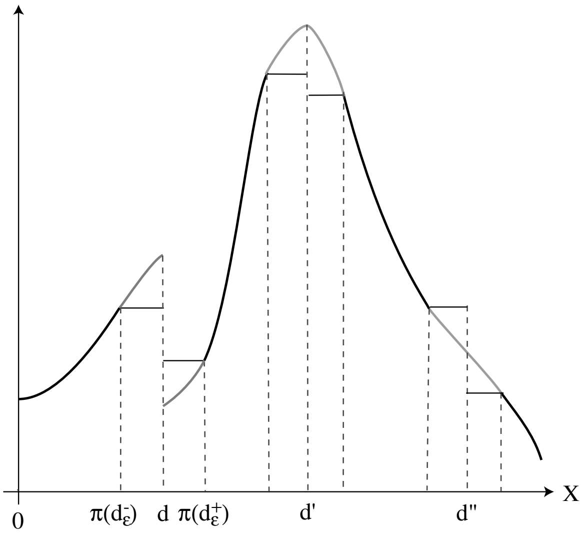

Figure 1. The graph of the ϵ-truncation map f_ϵ.

So, in particular, $M_\epsilon^{'}$ is disjoint from W and therefore, by notation in (4), $M_\epsilon^{'}$ is the same as $\tilde{M}'_\epsilon$ (they are homeomorphic metric spaces with respect to the usual length on M_ϵ and the metric $\hat{\rho}$ on $\tilde{M}_\epsilon$). Next, we define the ϵ-truncation map $f_\epsilon : I \to I$ by

$$f_\epsilon(x) = \begin{cases} f(x), & \text{if } x \notin (\pi(d_\epsilon^-), \pi(d_\epsilon^+)) \text{ for all } d \in D, \\ f(\pi(d_\epsilon^-)), & \text{if } x \in (\pi(d_\epsilon^-), d) \text{ with } d \in D, \\ f(\pi(d_\epsilon^+)), & \text{if } x \in (d, \pi(d_\epsilon^+))d \text{ with } D \end{cases} \tag{11}$$

(see figure 1).

It is easily seen that f_ϵ is continuous at the points $\pi(d_\epsilon^-), \pi(d_\epsilon^+)$ for any $d \in D$. Hence, f_ϵ is a piecewise continuous and piecewise monotone map with the partition $\mathcal{Z}$. Then it is easily checked that the doubling construction space for f_ϵ is precisely $\hat{I}_\epsilon$ (i.e. if we consider f_ϵ as the initial map then $\hat{I}_{f_\epsilon}$, its doubling construction space, coincides with $\hat{I}_\epsilon$), while the doubling extension of f_ϵ is precisely $\tilde{f}_\epsilon^\dagger$, i.e. $\hat{f}_\epsilon = \tilde{f}_\epsilon^\dagger$. So we have the following commutative diagrams:

$$\begin{array}{ccccc} \hat{I} & \xrightarrow{\hat{\pi}_\epsilon} & \hat{I}_\epsilon & \xrightarrow{\pi_\epsilon} & I \\ \tilde{f}_\epsilon \big\downarrow & & \tilde{f}_\epsilon^\dagger \big\downarrow \hat{f}_\epsilon & & \big\downarrow f_\epsilon . \\ \hat{I} & \xrightarrow[\hat{\pi}_\epsilon]{} & \hat{I}_\epsilon & \xrightarrow[\pi_\epsilon]{} & I \end{array} \tag{12}$$

Now we are in position to prove theorem 4.

Proof. Since $\tilde{f}_\epsilon$ is semiconjugate to $\hat{f}_\epsilon$ by the map $\hat{\pi}_\epsilon$, which is at most two-to-one, we have $h_{\text{top}}(\tilde{f}_\epsilon) = h_{\text{top}}(\hat{f}_\epsilon)$. Note that if we consider the restrictions $\tilde{f}_\epsilon|_{M'_\epsilon}$, $\hat{f}_\epsilon|_{\hat{\pi}_\epsilon(M'_\epsilon)}$ and $f_\epsilon|_{M'_\epsilon}$ in diagram (12), then the semiconjugacies $\hat{\pi}_\epsilon$ and π_ϵ become in fact conjugacies. Hence $h_{\text{top}}(\tilde{f}_\epsilon|_{M'_\epsilon}) = h_{\text{top}}(f_\epsilon|_{M'_\epsilon})$. So by using lemma 9, we have

$$h_{\text{top}}(\hat{f}_\epsilon) = h_{\text{top}}(\tilde{f}_\epsilon) = h_{\text{top}}(\tilde{f}_\epsilon|_{M'_\epsilon}) = h_{\text{top}}(f_\epsilon|_{M'_\epsilon}).$$

Since $\hat{f}_\epsilon$ is the doubling extension of f_ϵ, it follows from the definition of topological entropy for piecewise monotone maps that $h_{\rm top}(f_\epsilon) = h_{\rm top}(\hat{f}_\epsilon)$. Now, using the fact that the restrictions to M_ϵ of the maps f and f_ϵ coincide, we have

$$h_{\rm top}(f_\epsilon) = h_{\rm top}(f_\epsilon|_{M'_\epsilon}) = h_{\rm top}(f|_{M'_\epsilon}). \tag{13}$$

Since each interval $(\pi(d_\epsilon^-), \pi(d_\epsilon^+))$ tends to $\{d\}$ as $\epsilon \to 0$, we have that the Hausdorff distance between graphs of f_ϵ and f tends to zero. Thus by the lower semi-continuity property of the topological entropy function on the set of piecewise monotone piecewise continuous maps with the given number of monotone intervals (see [20]), we have $\liminf_{\epsilon\to 0} h_{\rm top}(f_\epsilon) \geqslant h_{\rm top}(f)$. On the other hand, it is easily seen that for any $\epsilon > 0$, $h_{\rm top}(f_\epsilon) \leqslant h_{\rm top}(f)$. So we get that $\lim_{\epsilon\to 0} h_{\rm top}(f|_{M'_\epsilon}) = h_{\rm top}(f)$ and thus, by (13), $\lim_{\epsilon\to 0} h_{\rm top}(f|_{M'_\epsilon}) = h_{\rm top}(f)$. So, given a $\delta > 0$ we can find $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, $h_{\rm top}(f|_{M'_\epsilon}) > h_{\rm top}(f) - \delta$. Finally, we let $M := \bigcap_{n=0}^\infty f^n(M'_\epsilon)$ and $J := \bigcup_{d\in D}(\pi(d_\epsilon^-), \pi(d_\epsilon^+))$ according to notations in the statement of theorem 4. Then the strict f-invariance of M follows from compactness of M'_ϵ and continuity of the restriction $f|_{M'_\epsilon}$. Thus, by corollary 8.6.1 of [25], we have $h_{\rm top}(f|_M) = h_{\rm top}(f|\bigcap_{n=0}^\infty f^n(M'_\epsilon)) = h_{\rm top}(f|_{M'_\epsilon})$. □

2.4. *Properties of the conjugacy $\bar{\psi}_\lambda$ and comparison with the result in [13]*

In [13], we considered the case when the unperturbed difference equation involved only one variable: $\Phi_{\lambda_0}(x_0, x_1, \ldots, x_m) = \varphi(x_N)$, and the local map φ had multiple simple zeros, say $\{d_1, \ldots, d_k\} \in$ int Q. In this case, we let $B = \{d_1, \ldots, d_k\}^{\mathbb{Z}}$, then the infinite matrix of $D_2F(\lambda, \underline{y})$ for $\lambda = \lambda_0$ and $\underline{y} \in B$ contains only one nonzero diagonal with finitely many values of the form $\varphi'(d_i)$. This implies that assumption (iv) of theorem 2 is satisfied. As a result, we got the following (some terminologies used in the following statement are given in the appendix).

Theorem 11 ([13]). *Under the above assumptions, there exists $\bar{\delta} > 0$ such that for any $\lambda \in U(\lambda_0, \bar{\delta})$ there is a closed (in the product topology) σ-invariant subset $\Gamma_\lambda \subset Y_\lambda$ and the following holds.*

- (i) *$\sigma|_{\Gamma_\lambda}$ is topologically conjugate to $\sigma|_{\Sigma_k}$ by the conjugacy map $\bar{\psi}_\lambda : \Sigma_k \to \Gamma_\lambda$, where $\bar{\psi}_\lambda$ is from theorem 2 and $\Sigma_k = \{1, 2, \ldots, k\}^{\mathbb{Z}}$; moreover, one has the commutative relations in the first three columns in diagram (14).*
- (ii) *The conjugacy map $\bar{\psi}_\lambda$ is the identity map for $\lambda = \lambda_0$ and is continuous in λ; moreover, the map $\lambda \mapsto \bar{\psi}_\lambda(\underline{x})$ from $U(\lambda_0, \bar{\delta})$ to ℓ_∞ (in the uniform topology) forms an equicontinuous family in $\underline{x} \in \Sigma_k$.*
- (iii) *If, in addition, the difference equation (2) for given λ corresponds to a map $f_\lambda : P_\lambda \to \mathbb{R}^m$, then one has the following commutative diagram:*

$$\begin{array}{ccccccccc}
\Sigma_k & \xrightarrow{\bar{\psi}_\lambda} & \Gamma_\lambda & \overset{i}{\hookrightarrow} & Y_\lambda & \xrightarrow{T_\lambda} & \tilde{P}_\lambda & \xrightarrow{\pi_0} & K_\lambda \\
\sigma\downarrow & & \sigma\downarrow & & \sigma\downarrow & & \sigma_m\downarrow & & f_\lambda\downarrow, \\
\Sigma_k & \xrightarrow[\bar{\psi}_\lambda]{} & \Gamma_\lambda & \underset{i}{\hookrightarrow} & Y_\lambda & \xrightarrow[T_\lambda]{} & \tilde{P}_\lambda & \xrightarrow[\pi_0]{} & K_\lambda
\end{array} \tag{14}$$

where all the maps involved are continuous (in the product topology on the symbolic spaces), $\bar{\psi}_\lambda$ is injective, T_λ is bijective, the notation $\overset{i}{\hookrightarrow}$ denotes the embedding and π_0 is a (surjective) projection which is entropy preserving and is bijective when restricted to the set of periodic points; here $\tilde{P}_\lambda = \{\underline{p} = (p_n)_{n=-\infty}^\infty \in P_\lambda^{\mathbb{Z}} : p_{n+1} = f_\lambda(p_n)\}$ and

$K_\lambda = \bigcap_{i=0}^{\infty} f_\lambda^i(\bigcap_{n=0}^{\infty} f_\lambda^{-n}(P_\lambda))$. *In particular, the map* $\theta_\lambda := \pi_0 \circ T_\lambda \circ \bar{\psi}_\lambda$ *semiconjugates* $\sigma|_{\Sigma_k}$ *with the restriction of* f_λ *to the closed* f_λ*-invariant set* $\Lambda_\lambda := \theta_\lambda(\Sigma_k)$*. Moreover, the inverse limit of* $f_\lambda|_{\Lambda_\lambda}$ *is conjugate to* $\sigma|_{\Sigma_k}$*. If* $f_\lambda : P_\lambda \to \mathbb{R}^m$ *is injective, then the above semiconjugacy* $\theta_\lambda|_{\Sigma_k}$ *is in fact a conjugacy.*

In the settings of this paper, we show similar results using the same approach by theorem 2. The only difference is that instead of the full horseshoe Σ_k, we continue with an appropriate hyperbolic, strictly φ-invariant set M_ϵ (i.e. $\varphi(M_\epsilon) = M_\epsilon$) with enough entropy and consider the set B in theorem 2 to be $\varprojlim(M_\epsilon, \varphi)$, the inverse limit of $\varphi|_{M_\epsilon}$ (rather than M_ϵ itself); because, by the definition of Y_λ, it consists of bi-infinite sequences, i.e. we need to recover all preimages inside M_ϵ. Moreover, the set M_ϵ is chosen so that it is away from some neighbourhood of the intervals of V and endpoints of Q. It follows that, in terms of notations in theorem 2, the function F is well defined in some uniform neighbourhood of $\lambda_0 \times B$, i.e. on the domain of the form $\bigcup_{b\in B} U(\lambda_0, \delta_0) \times U(b, \eta_0)$ for some fixed positive radii δ_0 and η_0. As a result we will have, instead of (14), the following commutative diagram:

$$\begin{array}{ccccccccc} \varprojlim(M_\epsilon, \varphi) & \xrightarrow{\bar{\psi}_\lambda} & \Gamma_\lambda & \overset{i}{\hookrightarrow} & Y_\lambda & \xrightarrow{T_\lambda} & \tilde{P}_\lambda & \xrightarrow{\pi_0} & K_\lambda \\ \sigma\big\downarrow & & \sigma\big\downarrow & & \sigma\big\downarrow & & \sigma_m\big\downarrow & & f_\lambda\big\downarrow, \\ \varprojlim(M_\epsilon, \varphi) & \xrightarrow[\bar{\psi}_\lambda]{} & \Gamma_\lambda & \underset{i}{\hookrightarrow} & Y_\lambda & \xrightarrow[T_\lambda]{} & \tilde{P}_\lambda & \xrightarrow[\pi_0]{} & K_\lambda \end{array} \tag{15}$$

where $\Gamma_\lambda \subset Y_\lambda$ is a closed (in the product topology) σ-invariant set, which is a continuation of $B = \varprojlim(M_\epsilon, \varphi)$ for perturbations.

3. Applications

In this section, we apply our perturbation results to several families of maps. We also obtain chaotic structures in stationary and travelling waves in lattice models and consider spatially homogeneous solutions of these systems by checking their local maps[6].

3.1. The Hénon map

First let us discuss a simple (and well known) example of the standard Hénon map $H(x, y) = (y, ay(1-y) - bx)$; it has a corresponding difference equation of the form

$$x_{n+2} - ax_{n+1}(1 - x_{n+1}) + bx_n = 0.$$

In order to apply theorems 1 and 11, one can fix $s_1 \leqslant 0$, $s_2 \geqslant 1$ (and if the inequalities are strict then the 'exceptional' set V must contain the intervals $[s_1, 0)$ and $(1, s_2]$). Here, we give two choices for the exceptional value of λ_0 depending on how the values of a and b play the role of a parameter.

First, in the case when b plays the role of the parameter λ with $\lambda_0 = 0$, it is well known that for small b the standard Hénon map behaves like the one-dimensional logistic map $\varphi(x) = ax(1-x)$, which is precisely the local map in our notation. In particular, the Hénon map is chaotic for a close to 4, and if $a > 4$, its nonwandering set is the Smale horseshoe. This agrees with our results. Indeed, for $a > 3.569\ldots$, we may use theorem 1 to assure the chaotic behaviour of orbits for the Hénon map (with b small enough), because $h_{\text{top}}(\varphi) > 0$. Also, the statement of theorem 1 on the continuous dependence of perturbed solutions in the uniform topology assures that the orbits starting in $[0, 1]^2$ under $H = H_{a,b}$ with

[6] See also [4, 5] for applications to economic models.

$3.569\ldots < a < 4$ and small b will stay infinite times in a narrow strip around the parabola $y = ax(1-x)$, $x \in [0, 1]$. On the other hand, for $a > 4$ sufficiently big, in order to establish a full 2-horseshoe structure we may use theorem 11 also (see [8] and [13] for more details on generalized Hénon maps).

Next, consider other regions of parameters for the Hénon map. Let a and b be sufficiently big. More precisely, let a and b tend to ∞ in such a way that $a/b \to \lambda_0$ for some constant $\lambda_0 > 0$. Then after dividing the difference equation by b, we will have the limit equation of the form

$$x_n - \lambda_0 x_{n+1}(1 - x_{n+1}) = 0.$$

Note that this equation corresponds to the value of delay $L = -1$ in our notation, and it is easy to see that by reversing the time, we have again the same situation, with the roles of x and y interchanged. So in the regions of parameters with a and b sufficiently big, we also have a chaotic structure of orbits when a/b is bigger than $3.569\ldots$, and for $a/b > 4$ we have again the full 2-horseshoe.

3.2. *The Arneodo–Coullet–Tresser maps*

Consider the family of the so-called ACT maps $f : \mathbb{R}^3 \to \mathbb{R}^3$ (due to Arneodo, Coullet and Tresser; refer to [6]); they are of the form

$$f(x, y, z) = (ax - b(y - z), bx + a(y - z), cx - dx^k + ez),$$

where $a, b, c, d, e \in \mathbb{R}$ are parameters and $k \geqslant 2$. If $(a^2+b^2)e \neq 0$, then f is a diffeomorphism with the inverse

$$f^{-1}(x, y, z) = \left(\hat{x}, \frac{-bx + ay}{a^2 + b^2} + \hat{z}, \hat{z}\right), \qquad \text{where } \hat{x} = \frac{ax + by}{a^2 + b^2} \text{ and } \hat{z} = \frac{z - c\hat{x} + d\hat{x}^k}{e}.$$

If $bd \neq 0$, then there are interesting dynamical properties and bifurcations in several regions of the parameter space; see [6]. For an initial point $p = (x_0, y_0, z_0)$, denote the nth iteration of p under f by (x_n, y_n, z_n). Then we have the following difference equation which in fact is an equivalent form for defining the ACT map (see [7, 13]):

$$dx_{n+1}^k - \frac{a^2e + b^2e}{b} x_n + \frac{a^2 + b^2 - bc + 2ae}{b} x_{n+1} - \frac{2a + e}{b} x_{n+2} + \frac{1}{b} x_{n+3} = 0.$$

Now we consider the pair of coefficients (a, e) as the parameter λ, and for the singular value $\lambda_0 = (0, 0)$ we have the difference equation in two variables with $L = 2$:

$$dx_{n+1}^k + (b - c)x_{n+1} + \frac{1}{b} x_{n+3} = 0.$$

At $\lambda = \lambda_0 = (0, 0)$ the ACT map f_λ is not invertible and its Jacobian becomes zero, while for small nonzero a, e, f_λ is a diffeomorphism. So we are able to apply theorem 1 in order to show chaotic behaviour of f_λ for a, e small; namely, it is sufficient to check that the one-dimensional map

$$\varphi(x) = (bc - b^2)x - bdx^k \qquad (16)$$

has positive topological entropy. In [6] and [7], sufficient conditions for the existence of full horseshoes (Bernoulli shifts on two or three symbols depending on the evenness of k) for ACT maps in the above situation, i.e. near degenerate ACT maps with $a = e = 0$, were obtained. Now our result provides other regions of parameters in which one has a chaotic structure with positive entropy, which need not be full horseshoe. For instance, it applies for small b whenever c/b is rather big, while in [6,7], the condition $|b| > 1$ was needed. So the result may be regarded

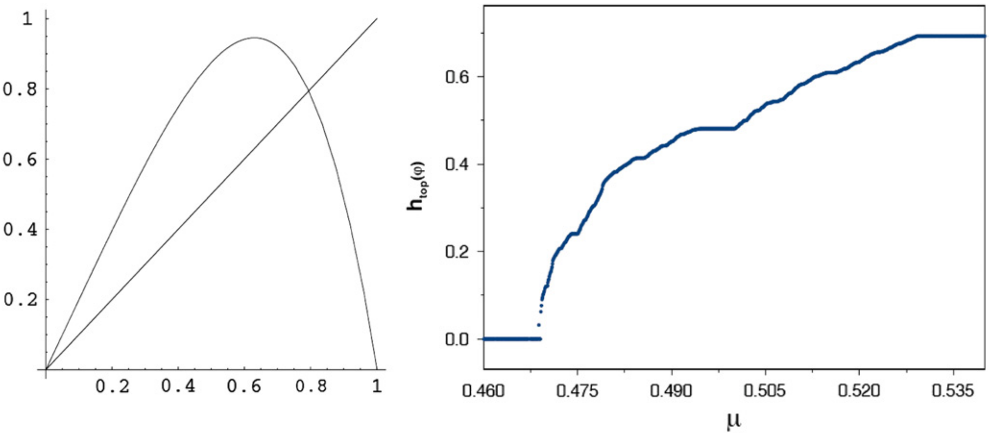

Figure 2. The graph of map (16) with $k = 4, b = 2, d = 1$ and $c = 1.5b$ and the topological entropy as a function of μ, when $c = (1 + \mu)b$ varies.

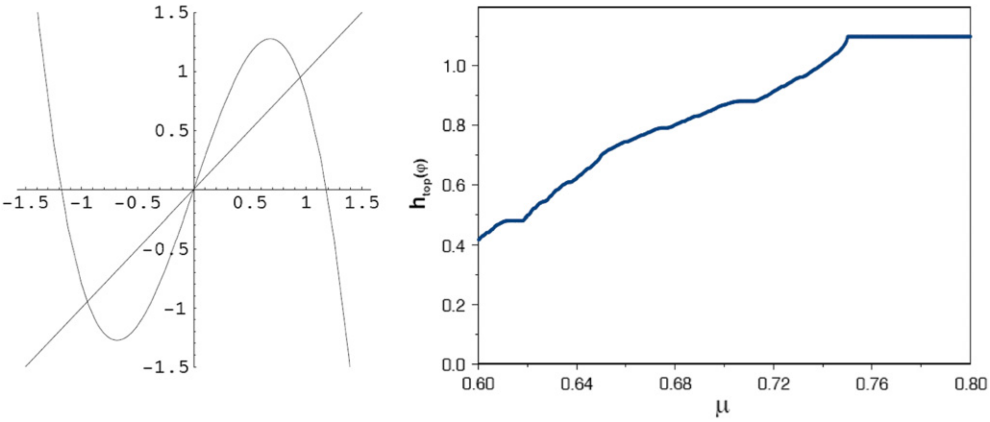

Figure 3. The graph of map (16) with $k = 3, b = 2, d = 1$ and $c = 1.7b$ and its topological entropy as a function of μ, when $c = (1 + \mu)b$ varies.

as a continuation of 'finer horseshoes' (actually, they are horseshoes under some iterates of the map, and it is well known that such horseshoes are contained necessarily in the nonwandering set of one-dimensional maps with positive topological entropy). See figures 2 and 3 which show some regions of parameters with positive $h_{\text{top}}(\varphi)$; note that since $L = 2$, the topological entropy of the perturbed ACT maps is bounded below approximately by $h_{\text{top}}(\varphi)/2$.

3.3. Quadratic volume preserving maps

In this subsection, we consider the family of maps $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(x, y, z) = (\eta + \alpha x + \beta y + z + Q(x, y), x, y), \tag{17}$$

where $\eta, \alpha, \beta, \gamma$ are real parameters and $Q(x, y) = ax^2 + bxy + cy^2$ is a quadratic form. As shown by Lomeli and Meiss in [14], generically every quadratic automorphism, i.e. volume preserving diffeomorphism of $\mathbb{R}^3$ which has a quadratic inverse, is topologically conjugate to a map (17) with $a + b + c = 1$ (note that in [14], it is also shown that the parameter β can be eliminated by an appropriate change of coordinates; however, we do not assume $\beta = 0$ because this parameter will be of use for our considerations). It is easily seen that the corresponding difference equation for (17) is of the form

$$\eta + \alpha x_n + \beta x_{n-1} + x_{n-2} + a x_n^2 + b x_n x_{n-1} + c x_{n-1}^2 - x_{n+1} = 0.$$

By using theorem 11, we have proved in [13] that if the parameters η, β, b, c in (17) are fixed while $\alpha \to \infty$ and $a \to \infty$ in such a way that $a/\alpha = \text{constant} \neq 0$, then for $|a|$ sufficiently big, the map $f = f_a$ has a closed invariant set Λ_a such that $f_a|_{\Lambda_a}$ is conjugate to the full shift on two symbols. (In fact, to have the same conclusion it is sufficient that all the parameters η, β, b, c in (17) are $o(\alpha)$ while $\alpha \to \infty$ and $a \to \infty$ in such a way that $a/\alpha \to \text{constant} \neq 0$).

Note that theorem 11 does not apply to quadratic automorphisms in the generic form with $a + b + c = 1$ because the latter equality implies that both coordinates x_n and x_{n-1} should be involved in the limit difference equation. But now by using theorem 1, we are able to get a chaotic structure of quadratic automorphisms in the generic form. Indeed, for the case when $\beta \neq 0$, let the parameter β tend to ∞, b tend to 0 and $a + c$ tend to 1 with $a > 0, c > 0$ and $a + b + c = 1$, while other parameters remain constant; or more generally, let us have (as $\beta \to \infty$) the following: $b = o(1)$, $\alpha = o(\beta)$, $\eta = o(\beta^2)$ and $a + c$ converges to 1 with $a > 0, c > 0$ and $a + b + c = 1$. Then after scaling by β we get for the new coordinates $\bar{x}_n = -x_n/\beta$, the following difference equation at the limit as $\beta \to \infty$ and $b \to 0$:

$$(1 - c)\bar{x}_n^2 + c\bar{x}_{n-1}^2 - \bar{x}_{n-1} = 0, \tag{18}$$

whose upper branch is the map

$$\varphi(x) = \sqrt{\frac{1}{4(1-c)c} - \frac{c}{1-c}\left(x - \frac{1}{2c}\right)^2}. \tag{19}$$

It is a unimodal map (with an upper semi-ellipse for the graph) and, thus, one has a positive topological entropy whenever c is close to 0.8 (and also for $c > 0.8$, in which case we have the full 2-horseshoe). In figure 4, it is shown that the topological entropy of the local map φ becomes positive beginning with $c \approx 0.791$. Therefore, by using theorem 1 with $L = 1$, the system in (17) has chaotic dynamics for sufficiently large β and sufficiently small b.

Furthermore, we can consider the case when $\beta = 0$ in the generic form of quadratic volume preserving automorphisms. Indeed, there is chaotic dynamics for sufficiently large α and sufficiently small b by using theorem 1 with $L = -1$, because in the new coordinates $\bar{x}_n = -x_n/\alpha$, we have at the limit as $\alpha \to \infty$ and $b \to 0$, the following difference equation

$$(1 - a)\bar{x}_{n-1}^2 + a\bar{x}_n^2 - \bar{x}_n = 0,$$

which is the same as (18) with only the roles of a and c and also the roles of $\bar{x}_{n-1}$ and $\bar{x}_n$ interchanged. So the chaotic dynamics here takes place for a near 0.8.

3.4. The generalized CNN models

We consider the one-dimensional generalized CNNs of the following form, which was introduced by Itoh, Julián and Chua [9],

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = -\boldsymbol{G}(\boldsymbol{x}) + \boldsymbol{A}\boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{u} + z, \tag{20}$$

where

$\boldsymbol{x} = (x_i)_{i\in\mathbb{Z}}, \boldsymbol{G}(\boldsymbol{x}) = (g(x_i))_{i\in\mathbb{Z}}, \boldsymbol{A} = (a_{-m}, \ldots, a_0, \ldots, a_m),$

$\boldsymbol{F}(\boldsymbol{x}) = ((f(x_{i-m}), \ldots, f(x_i), \ldots, f(x_{i+m}))^{\mathrm{T}})_{i\in\mathbb{Z}}, \boldsymbol{B} = (b_{-m}, \ldots, b_0, \ldots, b_m),$

$\boldsymbol{u} = ((u_{i-m}, \ldots, u_i, \ldots, u_{i+m})^{\mathrm{T}})_{i\in\mathbb{Z}}, \boldsymbol{z} = (z_i)_{i\in\mathbb{Z}}.$

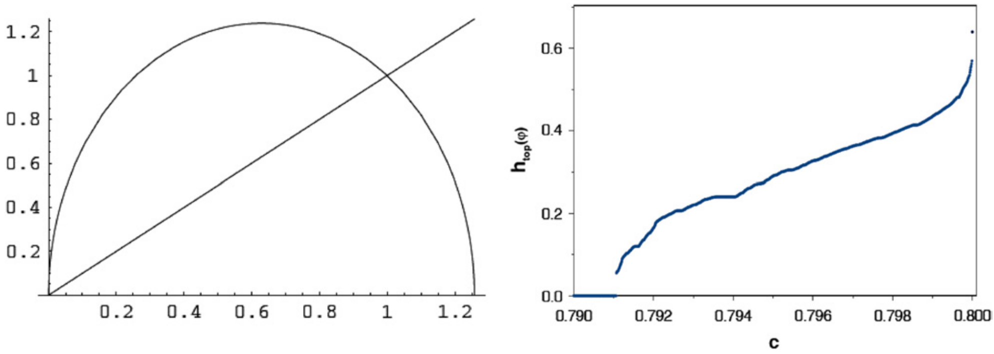

Figure 4. The graph of map (19) with $c = 0.795$ and its topological entropy as a function of c (which is presumed to be continuous at finer scales of c).

Here, $\boldsymbol{x}$ denotes the state, $\boldsymbol{G(x)}$ denotes the v–i characteristic, $\boldsymbol{F(x)}$ denotes the output, $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{z}$ denote the feedback, control and threshold template parameters, respectively, and m is the neighbourhood radius of each cell. As usual, the output function $\boldsymbol{F(x)}$ is related through the piecewise-linear saturation function

$$f(x) = \tfrac{1}{2}(|x+1| - |x-1|),$$

and the v–i characteristic function $\boldsymbol{G(x)}$ is related to the piecewise-linear function

$$g(x) = \alpha x + \gamma(|x - V_\mathrm{p}| - |x - V_\mathrm{v}|) - \gamma(|x + V_\mathrm{p}| - |x + V_\mathrm{v}|),$$

where α and γ are constants and V_p and V_v are the peak and valley voltages.

Here, we will allow the output and v–i characteristic functions to have more deviations. One can rewrite the generalized CNN model (20) as a system of state equations:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -g(x_i) + \sum_{j=-m}^{m} a_j f(x_{i+j}) + \sum_{-m}^{m} b_j u_j + z_i.$$

Let $h(x) = g(x) - a_0 f(x)$. Then we arrive at the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -h(x_i) + \sum_{j=-m, j\neq 0}^{m} a_j f(x_{i+j}) + \sum_{j=-m}^{m} b_j u_j + z_i. \tag{21}$$

Standard CNNs and resonant tunnelling diode (RTD) based CNNs are special cases of the generalized CNN model (21). For example, model (21) with $h(x) = x - a_0 f(x)$ is the original standard CNN model, with $h(x) = x - a_0 \tilde{f}(x)$ the modified standard CNN model, with $h(x) = g(x) - a_0 x$ and $f(x) = x$ the original RTD-based CNN model and with $h(x) = g(x) - ax$ and $f(x) = g(x) - \alpha x - \beta$ the modified RTD-based CNN model. Circuit implementations of these CNNs have been explicitly illustrated in [9].

Here, we assume that the template parameters have a common parameter λ; more precisely, the generalized CNN model (21) becomes of the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -h(x_i) + \lambda \left[\sum_{j=-m, j\neq 0}^{m} a_j f(x_{i+j}) + \sum_{j=-m}^{m} b_j u_j + z_i \right]. \tag{22}$$

In order to demonstrate the spatial chaos phenomenon, we consider the set of stationary solutions, i.e. solutions which do not depend on 'time': $x_i(t) = x_i$. Thus letting the right-hand side of (22) be zero (i.e. the exceptional value $\lambda_0 = 0$), we have the stationary solutions of (22) given by

$$h(x_i) - \lambda \left[\sum_{j=-m, j\neq 0}^{m} a_j f(x_{i+j}) + \sum_{j=-m}^{m} b_j u_j + z_i \right] = 0.$$

Thus, by using theorem 11, we have the following result on the chaotic structure of stationary solutions.

Corollary 12. *If h has k simple zeros and both f and h are C^1 in a neighbourhood U of these zeros, then for any sufficiently small λ, there exists a closed (in the product topology) σ-invariant sunset Λ_λ of the set of stationary solutions for (22) such that $\sigma|_{\Lambda_\lambda}$ is topologically conjugate to $\sigma|_{\Sigma_k}$, the full shift on k symbols and the conjugacy map can be chosen to be continuous in λ in the uniform topology.*

3.5. Time discrete version of the CNN system

One can also consider the time discrete version of the CNN system introduced by Sbitnev and Chua in [24] as follows

$$x_i^{n+1} = x_i^n - \delta h(x_i^n) + \delta\lambda \left[\sum_{j=-m, j\neq 0}^{m} a_j f(x_{i+j}^n) + \sum_{j=-m}^{m} b_j u_j^n + z_i^n \right], \qquad (23)$$

where δ and λ are nonzero constants. Stationary solutions do not depend on the 'time' coordinate n, i.e. $x_i^n = x_i$ for these solutions. In this case, equation (23) is reduced to the equalities

$$h(x_i) + \lambda \left[\sum_{j=-m, j\neq 0}^{m} a_j f(x_{i+j}) + \sum_{j=-m}^{m} b_j u_j + z_i = 0 \right].$$

Theorem 11 implies exactly the same result on the chaotic structure of stationary solutions for (23) as corollary 12.

Let $p, q \in \mathbb{Z}$ with $(p, q) = 1$. A travelling wave solution with velocity q/p of equation (23) is a solution of the form

$$x_i^n = x_{pi+qn}, \qquad \text{where } y : \mathbb{Z} \to \mathbb{R}^k \text{ and write } y_\ell = y(\ell).$$

Let $r = pi + qn + q$ be the travelling wave coordinate, i.e. set $x_i^{n+1} = y_r = y_{pi+qn+q}$. Then the travelling wave solutions are given by

$$y_r = y_{r-q} - \delta h(y_{r-q}) + \delta\lambda \left[\sum_{j=-m, j\neq 0}^{m} a_j f(y_{r-q-pj}^n) + \sum_{j=-m}^{m} b_j u_{r-q-pi+pj} + z_{r-q} \right].$$

Spatially homogeneous solutions do not depend on the 'space' coordinates i, i.e. $x_i^n = y_n$ for these solutions. In this case, equation (23) is reduced to the equalities:

$$y_{n+1} = y_n - \delta h(y_n) + \delta\lambda \left[\sum_{j=-m, j\neq 0}^{m} a_j f(y_n) + \sum_{j=-m}^{m} b_j u_n + z_n \right].$$

Thus, by theorem 1, we have the following result.

Corollary 13. *Suppose that the function $x \mapsto x - \delta h(x)$ from $Q = [s_1, s_2]\backslash V$ to $[s_1, s_2]$, for some $s_1 < s_2$ and $V \subset [s_1, s_2]$ open, is piecewise analytic and has positive topological entropy. If f is C^1 on Q, then for any sufficiently small λ, there exists a subset Λ_λ of travelling wave solutions (or spatially homogeneous solutions) for (23) such that Λ_λ is invariant under the spatial translation σ and $h_{\text{top}}(\sigma|_{\Lambda_\lambda}) > 0$.*

3.6. Steady state of Chua's circuit

The equations for the coupled Chua's circuit in [21] are

$$\begin{aligned}
\dot{x}_i &= \alpha y_i - \alpha(x_i + g(x_i)) + d(x_{i+1} - 2x_i + x_{i-1}),\\
\dot{y}_i &= x_i - y_i + z_i, \qquad (24)\\
\dot{z}_i &= -\beta y_i - \gamma z_i,
\end{aligned}$$

where α, β, γ, d are positive parameters and $g(x) = m_1 x + \frac{m_0 - m_1}{2}(|x+1| - |x-1|)$. Considering the stationary solutions, equation (24) yields

$$-\frac{\alpha\beta}{\gamma+\beta}x_i - \alpha g(x_i) + d(x_{i+1} - 2x_i + x_{i-1}) = 0.$$

From theorem 11, we have the following corollary.

Corollary 14. *If the function $x \mapsto \frac{\beta}{\gamma+\beta}x + g(x)$ has k simple zeros and g is C^1 in a neighbourhood U of these zeros, then for any sufficiently small d, there exists a closed (in the product topology) σ-invariant sunset Λ_d of the set of stationary solutions for (24) such that $\sigma|_{\Lambda_d}$ is topologically conjugate to $\sigma|_{\Sigma_k}$ and the conjugacy map is continuous in d in the uniform topology.*

3.7. Lattice models of an evolution equation

Finally, we consider solutions in lattice models of an evolution equation of which some motions can be described by discrete versions of reaction–diffusion equations (for lattice models and their chaotic and stability properties see [1,2,3,22]). Consider the lattice models of an evolution equation of the form

$$u_n^{t+1} = f(u_n^t) + \epsilon g(u_{n-s}^t, u_{n-s+1}^t, \ldots, u_{n+s}^t), \qquad (25)$$

where $t \in Z$ is the time variable, $n \in Z$ is the space one and $\epsilon \geqslant 0$ usually stands for the diffusion coefficient. The function f is called the local map and g is called the interaction of finite size s.

If we look for the steady state (or stationary) solutions u_n^t of (25), then u_n^t must be independent of the time coordinate t, i.e. $u_n^t := x_n$ for all $t \in \mathbb{Z}$. In this case, equation (25) can be reduced to the difference equation

$$x_n = f(x_n) + \epsilon g(x_{n-s}, x_{n-s+1}, \ldots, x_{n+s}), \qquad n \in \mathbb{Z}.$$

Thus, by using theorem 11, we have the following result on the chaotic structure of steady state solutions.

Corollary 15. *Let the function $x \mapsto x - f(x)$ have k simple zeros and be of class C^1 in a neighbourhood U of these zeros and let g be of class C^1 in U^{2s+1}. Then for sufficiently small ϵ, there exists a closed (in the product topology) σ-invariant subset Λ_ϵ of the set of steady state solutions for (25) such that $\sigma|_{\Lambda_\epsilon}$ is topologically conjugate to $\sigma|_{\Sigma_k}$ and the conjugacy map is continuous in ϵ in the uniform topology.*

Let $p, q \in \mathbb{Z}$ with $(p, q) = 1$. A travelling wave solution with velocity q/p of equation (25) is a solution of the form

$$u_m^n = x_{pm+qn}, \qquad \text{where } x : \mathbb{Z} \to \mathbb{R}^k \text{ and write } x_\ell = x(\ell).$$

Let $i = pm + qn + q$ be the travelling wave coordinate, i.e. set $u_m^{n+1} = x_i = x_{pm+qn+q}$. Then we obtain the following equation for x_i:

$$x_i = f(x_{i-q}) + \epsilon g(x_{i-q-ps}, x_{i-q-ps+p}, \ldots, x_{i-q+ps}).$$

Spatially homogeneous solutions do not depend on the 'space' coordinate m, i.e. $u_m^n = x_n$ for these solutions. In this case, equation (25) is reduced to the equalities

$$x_{n+1} = f(x_n) + \epsilon g(x_n, x_n, \ldots, x_n).$$

As a consequence of theorem 1, we have the following result.

Corollary 16. *Suppose that f is a C^2 map from $Q = [s_1, s_2]\backslash V$ to $[s_1, s_2]$, where $s_1 < s_2$ and $V \subset [s_1, s_2]$ is open, such that f has no critical points and has positive entropy. If g is C^1 on Q, then for any sufficiently small ϵ, there exists a subset Λ_ϵ of travelling wave solutions (or spatially homogeneous solutions) for (25) with parameter ϵ such that Λ_ϵ is invariant under the spatial translation σ and $h_{\text{top}}(\sigma|_{\Lambda_\epsilon}) > 0$.*

Let us remark that unlike in [2, 22] we do not require the functions f and g to be defined and smooth on the whole $\mathbb{R}$ and $\mathbb{R}^{2s+1}$, respectively, and to have bounded partial derivatives.

Appendix

In our applications, the difference equations for nonexceptional values of λ are actually associated with well-defined maps. We say that for a given λ, *the difference equation (2) corresponds to a map* $f_\lambda : P_\lambda \to \mathbb{R}^m$, where P_λ is a compact subset of $\mathbb{R}^m$ and f_λ is continuous, if $\sigma|_{Y_{\lambda,\text{prod}}}$ is conjugate to the restriction of the shift map $\sigma_m : (\mathbb{R}^m)^{\mathbb{Z}} \to (\mathbb{R}^m)^{\mathbb{Z}}$ to the space $\tilde{P}_\lambda = \{\underline{p} = (p_n)_{n=-\infty}^{\infty} \in P_\lambda^{\mathbb{Z}} : p_{n+1} = f_\lambda(p_n)$ for all $n \in \mathbb{Z}\}$ of full orbits under f_λ contained in P_λ with respect to the product topology.

In the case when the difference equation (1) corresponds to a map $f_\lambda : P_\lambda \to \mathbb{R}^m$, the set $P_{\lambda,\text{prod}}$ is a compact, σ_m-invariant set, and if we denote $K_\lambda^+ := \bigcap_{n=0}^{\infty} f_\lambda^{-n}(P_\lambda)$, then $K_{\lambda,\text{prod}}^+$ is compact and f_λ-invariant, and the restriction $\sigma_m|_{\tilde{P}_{\lambda,\text{prod}}}$ can be regarded as the inverse limit $\underleftarrow{\lim}(K_\lambda^+, f_\lambda)$ by identifying points $(\ldots, p_{-2}, p_{-1}, p_0) \in \underleftarrow{\lim}(K_\lambda^+, f_\lambda)$ with corresponding ones $(\ldots, p_{-2}, p_{-1}, p_0, f_\lambda(p_0), f_\lambda^2(p_0), \ldots) \in \tilde{P}_\lambda$. Put $K_\lambda := \pi_0(\tilde{P}_\lambda)$, where π_0 denotes the projection to the 0th coordinate. Then $K_\lambda = \bigcap_{n=0}^{\infty} f_\lambda^n(K_\lambda^+)$ because of compactness of K_λ^+ in the product topology. Obviously, π_0 is a semiconjugacy map from $\sigma|_{\tilde{P}_\lambda}$ to $f_\lambda|_{K_\lambda}$ (here we have omitted the subscript prod, and so we will do later if there are no doubts). It is easy to see that if, in addition, $f_\lambda : P_\lambda \to \mathbb{R}^m$ is one-to-one then π_0 is in fact a conjugacy. We have the following commutative diagram:

$$\begin{array}{ccccc} Y_\lambda & \xrightarrow{T_\lambda} & \tilde{P}_\lambda & \xrightarrow{\pi_0} & K_\lambda \\ \sigma\big\downarrow & & \sigma_m\big\downarrow & & f_\lambda\big\downarrow \\ Y_\lambda & \xrightarrow[T_\lambda]{} & \tilde{P}_\lambda & \xrightarrow[\pi_0]{} & K_\lambda \end{array}.$$

The topological entropy $h_{\text{top}}(\sigma|_{Y_{\lambda,\text{prod}}})$ is equal to $h_{\text{top}}(f_\lambda)$, the topological entropy of the map f_λ, which is meant as $h_{\text{top}}(f_\lambda|_{K_\lambda^+})$, the topological entropy of the restriction of f_λ to the maximal f_λ-(forward) invariant set K_λ^+.

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