

O.R. Applications

# A distributed computation algorithm for solving portfolio problems with integer variables

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## Abstract

A portfolio problem with integer variables can facilitate the use of complex models, including models containing discrete asset values, transaction costs, and logical constraints. This study proposes a distributed algorithm for solving a portfolio program to obtain a global optimum. For a portfolio problem with  $n$  integer variables, the objective function first is converted into an ellipse function containing  $n$  separated quadratic terms. Next, the problem is decomposed into  $m$  equal-size separable programming problems solvable by a distributed computation system composed of  $m$  personal computers linked via the Internet. The numerical examples illustrate that the proposed method can obtain the global optimum effectively for large scale portfolio problems involving integral variables.

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## 1. Introduction

The mean-variance portfolio (MVP) model stems from the work of Markowitz (1952). Most MVP models have been formulated as quadratic programming programs. As Young (1998) remarked, the prospects for developing practical applications of portfolio analysis would be greatly enhanced if the MVP program could involve integer variables. Allowing an MVP program containing integer variables can facilitate the use of complex models, including models with discrete asset or stock values and models with transaction costs and logical constraints such as “IF...Then” and “OR” conditions. Owing to the obvious merits of casting portfolio management in an integer programming framework, many authors have sought to make such a connection feasible. Notably, Sharpe (1971) and Stone (1973) employed a piecewise linear approximation to the quadratic

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term in the mean-variance quadratic program. [Bienstock \(1996\)](#) developed a branch-and-cut algorithm to solve an MVP program with integer variables. Moreover, [Yamakozi and Konno \(1991\)](#) derived an absolute deviation selection approach, in which the sum of absolute deviations about the mean (a L1 measure) is used to measure risk, to solve an MVP problem. Additionally, [Young \(1998\)](#) developed a minimax selection approach, in which minimum return (a  $L_\infty$  measure) rather than variance serves as a measure of risk, to minimize the maximum loss in an MVP problem. The MVP model discussed in this study is formulated as follows:

**Problem 1**

$$\begin{aligned}
 \text{Minimize } f(\mathbf{x}) &= \sum_{i \neq j} a_{ij} d_i d_j x_i x_j + \sum_{i=1}^n b_i (d_i x_i)^2, \\
 \text{subject to (C1)} \quad &\sum_{i=1}^n (c_i d_i x_i - p_i \theta_i) \geq R, \\
 \text{(C2)} \quad &\sum_i (d_i x_i - p_i \theta_i) \leq q, \\
 \text{(C3)} \quad &x_i \leq q \theta_i \quad \text{for } i = 1, 2, \dots, n, \\
 \text{(C4)} \quad &\mathbf{x} = (x_1, x_2, \dots, x_n), 0 \leq x_i \leq \bar{x}_i, \quad i = 1, 2, \dots, n, \quad x_i \text{ are integers,}
 \end{aligned} \tag{1.1}$$

where  $x_i$  the number of shares allocation to security  $i$ ,  $a_{ij}$  the covariance between securities  $i$  and  $j$ ,  $d_i$  the price per share in dollars of security  $i$ ,  $b_i$  the variance of security  $i$ ,  $b_i \geq 0$ ,  $c_i$  the expected return in percentages on security  $i$ ,  $c_i > 0$ ,  $\theta_i$  the (0 – 1) variables;  $\theta_i = 1$  if  $x_i > 0$ , and  $\theta_i = 0$  if  $x_i = 0$ ,  $p_i$  the fixed charge transaction costs paid at the beginning for purchasing a security  $i$ ,  $q$  the total allocation budget in dollars,  $\bar{x}_i$  the upper bound of investing in security  $i$ ,  $R$  the net return in dollars.

$f(\mathbf{x})$  represents the variance of the portfolio. The first constraint is the expected return constraint considering transaction costs. The second constraint is a budget constraint where all portfolio allocation to securities is integral and the transaction costs are paid at the beginning of purchasing the securities. The third constraint is a logical constraint for the fixed charge transaction costs and if  $x_i > 0$  then  $\theta_i = 1$ , and if  $x_i = 0$  then  $\theta_i = 0$ . **Problem 1** is a quadratic integer programming (QIP) problem for which the methods of [Young \(1998\)](#) and [Yamakozi and Konno \(1991\)](#) are unable to reach a global optimum. [Khachian \(1979\)](#) presented an ellipsoid method to solve a convex problem for locating a global solution. [Karmarkar \(1984\)](#) also proposed an interior point method whose worst case complexity bound was better than that of the simplex and the ellipsoid methods. However, the above mentioned methods can only treat a problem with convex real-valued functions. There are three approaches usable for solving **Problem 1**, as described below:

- (i) Dual Lagrangean relaxation approach ([Michelon and Maculan, 1991](#); [Guignard and Kim, 1987](#)): This approach decomposes **Problem 1** into two sub-problems which share the constraints of the original problem and yields a dual Lagrangean relaxation. The lower bound of **Problem 1** is found by solving a dual Lagrangean relaxation program, while the upper bound is reached using heuristic or enumerative methods. By decreasing iteratively the primal-dual gap using cutting plane techniques, this approach can eventually find a global optimum. Although this approach can converge to a final result, the rate of convergence is quite slow. The experiments conducted here show that for a portfolio problem with 30 assets, a personal computer takes several hours to identify a final solution with 0.5% tolerance.
- (ii) Linear terms transformation approach ([Li and Chang, 1998](#)): This approach first expresses  $x_i$  by a set of 0–1 variables  $u_{i1}, u_{i2}, \dots, u_{imi}$ . A product term  $x_i x_j$  is then replaced by a set of cross terms  $u_{ik} u_{jk}$  which can be linearized conveniently. The problem with this approach is that the transformed linear 0–1 model requires many binary variables, which may cause a heavy computational burden.
- (iii) Separable terms transformation approach ([Sharpe, 1971](#); [Stone, 1973](#)): By rewriting a cross term  $x_i x_j$  as follows

$$x_i x_j = \frac{1}{2}(x_i + x_j)^2 - \frac{1}{2}x_i^2 - \frac{1}{2}x_j^2, \tag{1.2}$$

$f(\mathbf{x})$  in (1.1) then becomes

$$f(\mathbf{x}) = \sum_{i=1}^n b_i x_i^2 + \sum_{i \neq j} a_{ij} z_{ij}^2, \quad (1.3)$$

where  $z_{ij} = x_i + x_j$  and  $b_i$  and  $a_{ij}$  are constants with unrestricted signs.

Since (1.3) contains separated terms only, various separable programming techniques (Bazaraa et al., 1993; Horst et al., 2000) can be applied to solve it. A major difficulty of this approach (Sharpe, 1971; Stone, 1973) is that the transformed separable program contains too many quadratic terms. For the objective function in (1.1), which contains  $n$  variables, the corresponding number of separated terms in (1.3) is  $\frac{1}{2}(n^2 + n)$ . For instance, if  $n = 3$ , then  $f(\mathbf{x}) = b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 + a_{12} z_{12}^2 + a_{13} z_{13}^2 + a_{23} z_{23}^2$  with six quadratic terms have to be approximated.

Solving a large scale MVP problem with integer variables is very time-consuming. During the past decade, many distributed/parallel algorithms have been developed for different applications to increase computational efficiency. For example, Miller and Pekny (1989) presented results obtained from solving the asymmetric traveling salesman problem using a parallel branch and bound algorithm. Cannon and Hoffman (1990) proposed a solution methodology for solving large scale 0–1 linear programming problems over a network with distributed computation. Moreover, Bienstock (1996) presented parallel computation experience with a branch-and-cut algorithm to solve mixed-integer quadratic programming problems. Verkama et al. (1996) developed distributed algorithms for  $n$ -player game problems involving the computation of Pareto optimal solutions. Keyser and Davis (1998) also designed different distributed computer schemes for manufacturing scheduling problems.

This study proposes a novel technique for solving Problem 1. Based on the eigenvector of a convex quadratic function,  $f(\mathbf{x})$  is reformulated as an ellipse function with  $n$  axes  $y_1, y_2, \dots, y_n$ , where  $-\bar{y}_i \leq y_i < \bar{y}_i$ .  $f(\mathbf{x})$  can then be rewritten as the sum of  $n$  separated terms. Branching the whole search region by  $0 \leq y_i \leq \bar{y}_i$  and  $-\bar{y}_i \leq y_i < 0$  for some  $i$ , Problem 1 is decomposed into some equal size separable programming programs. These separable programs are solvable conveniently by a distributed computation system composed of many personal computers linked via the Internet.

To illustrate the computational efficiency of the proposed method, this study tests several portfolio problems. The experiments show that by solving a problem involving up to 50 stocks by the proposed distributed system composed of eight personal computers, the globally optimal solution can be determined in 12 seconds with 0.5% tolerance.

## 2. Theoretical developments

Problem 1 with a strictly convex objective function is a constrained QIP program which can be rewritten below:

$$\begin{aligned} \text{Minimize } & f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \mathbf{Q} \mathbf{x}, \\ \text{subject to } & (C1), (C2), (C3), (C4), \end{aligned} \quad (2.1)$$

where  $\mathbf{Q}$  denotes a positive definite matrix, and all other variables are defined as in Problem 1.

Since  $\mathbf{Q}$  is positive definite, an  $n \times n$  orthonormal matrix  $\mathbf{B}$  exists, such that

$$\mathbf{Q} = \mathbf{B}^T \text{diag}(\lambda_1, \dots, \lambda_n) \mathbf{B}, \quad (2.2)$$

where  $\lambda_i$  are the eigenvalues of  $\mathbf{Q}$  for  $i = 1, 2, \dots, n$  and  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ .

Let  $\mathbf{y} = (y_1, y_2, \dots, y_n) = \mathbf{B}(\mathbf{x} - \mathbf{x}^0)$ , where  $\mathbf{x}^0$  denotes the unconstrained continuous optimum of  $f(\mathbf{x})$  and

$$y_i = \sum_{j=1}^n b_{ij}(x_j - x_j^0). \tag{2.3}$$

Consider the following propositions:

**Proposition 1** (Referring to Sun and Li, 2001). *Given  $\mathbf{x}^0$ , the unconstrained continuous optimum of  $f(\mathbf{x})$ , for any  $\mathbf{x} \in R^n$ , the difference between  $f(\mathbf{x})$  and  $f(\mathbf{x}^0)$  is computed as*

$$f(\mathbf{x}) - f(\mathbf{x}^0) = \frac{1}{2} \sum_{i=1}^n [\lambda_i y_i^2], \tag{2.4}$$

where  $y_i$  and  $\lambda_i$  are specified in (2.2) and (2.3).

**Proof.** Since  $f(\mathbf{x}) - f(\mathbf{x}^0) = \frac{1}{2}(\mathbf{x} - \mathbf{x}^0)^T \mathbf{Q}(\mathbf{x} - \mathbf{x}^0) = \frac{1}{2} \mathbf{y}^T \text{diag}(\lambda_1, \dots, \lambda_n) \mathbf{y}$ , and  $\|\mathbf{y}\| = \|\mathbf{x} - \mathbf{x}^0\|_2$ , clearly  $f(\mathbf{x}) - f(\mathbf{x}^0) = \frac{1}{2} \sum_{i=1}^n [\lambda_i y_i^2]$ .  $\square$

**Proposition 2.** *Expression (2.4) can be rewritten as the following ellipse equation*

$$\sum_{i=1}^n \frac{y_i^2}{d/\lambda_i} = 1, \tag{2.5}$$

where  $d = 2[f(\mathbf{x}) - f(\mathbf{x}^0)]$  and the ellipse is centered at  $\mathbf{x}^0$ .

**Proof.** By referring to (2.4),  $\sum_{i=1}^n \lambda_i y_i^2 = d$ , where  $y_i = \sum_{j=1}^n b_{ij}(x_j - x_j^0)$ , which is converted into a standard ellipse equation,  $\sum_{i=1}^n \frac{y_i^2}{d/\lambda_i} = 1$ , centered at  $\mathbf{y} = (0, 0, \dots, 0)^T$ . Since  $\mathbf{y} = \mathbf{B}(\mathbf{x} - \mathbf{x}^0)$ , point  $\mathbf{y} = (0, 0, \dots, 0)^T$  on the  $Y$  axis is the same as the point  $\mathbf{x} = (x_1^0, x_2^0, \dots, x_n^0)$  on the  $X$  axis. Thus the proposition is proven.  $\square$

**Proposition 3.** *Given a reference integer point  $\mathbf{x}^A \in R^n$ , let  $d^A = 2[f(\mathbf{x}^A) - f(\mathbf{x}^0)]$ , the area of the search region for optimal solution based on  $\mathbf{x}^A$ , denoted as  $r^A$ , becomes*

$$r^A = 2^n \sqrt{\frac{(d^A)^n}{\lambda_1 \lambda_2 \dots \lambda_n}}. \tag{2.6}$$

**Proof.** Let  $d^A = 2[f(\mathbf{x}^A) - f(\mathbf{x}^0)]$ , the axis lengths for the ellipse in (2.5) are  $(\sqrt{d^A/\lambda_1}, \sqrt{d^A/\lambda_2}, \dots, \sqrt{d^A/\lambda_n})$ . Moreover, the bounds for  $y_i$  are specified as

$$-\sqrt{d^A/\lambda_i} \leq y_i \leq \sqrt{d^A/\lambda_i}, \quad \text{for } i = 1, 2, \dots, n. \tag{2.7}$$

The area of the search region for the optimal solution of Problem 1 then becomes  $(2\sqrt{\frac{d^A}{\lambda_1}})(2\sqrt{\frac{d^A}{\lambda_2}}) \dots (2\sqrt{\frac{d^A}{\lambda_n}}) = 2^n \sqrt{\frac{(d^A)^n}{\lambda_1 \lambda_2 \dots \lambda_n}}$ .  $\square$

**Remark 1.** If  $\lambda_1 = \lambda_2 = \dots = \lambda_n = \lambda$ , then the area of the search region becomes  $2^n (d^A/\lambda)^{\frac{n}{2}}$ .

Suppose there exists a set of connected computers comprising one host computer and  $m = 2^s$  ( $s$  is an integer) slave computers for distributed computation. The region  $r^A$  in (2.6) can be divided into  $2^s$  equal-size sub-regions to perform a distributed search to locate an optimum. One convenient method of achieving this division is to split the whole region into equal-size sub-regions, as described below.

**Proposition 4.** *The area of the search region  $r^A$  in (2.6) can be divided into equal-size sub-regions  $SR_1, SR_2, \dots, SR_{2^s}$ , where*

- (i) each sub-region has the same area  $2^{n-s} \sqrt{(d^A)^n / \lambda_1 \lambda_2 \dots \lambda_n}$ ,

(ii) the upper and lower bounds for  $SR_k$  are specified as  $(u_{ik} - 1)\sqrt{d^A/\lambda_i} \leq y_i \leq u_{ik}\sqrt{d^A/\lambda_i}$  for  $k = 1, 2, \dots, 2^s$  and  $i \leq s, -\sqrt{d^A/\lambda_i} \leq y_i \leq \sqrt{d^A/\lambda_i}$  for  $i > s$ , where  $u_{1k}, u_{2k}, \dots, u_{nk} \in \{0, 1\}$  and  $(u_{1k}, u_{2k}, \dots, u_{nk}) \neq (u_{1l}, u_{2l}, \dots, u_{nl})$  for  $k \neq l$  and  $k, l \in \{1, 2, \dots, 2^s\}$ .

**Proof.** Since  $0 \leq y_i \leq \sqrt{d^A/\lambda_i}$  or  $-\sqrt{d^A/\lambda_i} \leq y_i \leq 0$ , for  $i = 1, 2, \dots, n$  and  $s \leq n$ , the area for each  $SR_k$  is  $2^{n-s} \sqrt{\frac{d^A}{\lambda_1}} \sqrt{\frac{d^A}{\lambda_2}} \dots \sqrt{\frac{d^A}{\lambda_n}} = 2^{n-s} \sqrt{\frac{(d^A)^n}{\lambda_1 \lambda_2 \dots \lambda_n}}$ .

For instance, if there are 32 personal computers, then the search region for the first PC can be  $0 \leq y_i \leq \sqrt{d^A/\lambda_i}$  for  $i = 1, 2, \dots, 5$  and  $-\sqrt{d^A/\lambda_i} \leq y_i \leq \sqrt{d^A/\lambda_i}$  for  $i = 6, 7, \dots, n$ . The search region for the second PC can be  $-\sqrt{d^A/\lambda_1} \leq y_1 \leq 0, 0 \leq y_i \leq \sqrt{d^A/\lambda_i}$  for  $i = 1, 2, \dots, 5$  and  $-\sqrt{d^A/\lambda_i} \leq y_i \leq \sqrt{d^A/\lambda_i}$  for  $i = 6, 7, \dots, n$ . Fig. 1 illustrates the search region of a QIP problem with two integer variables by the proposed method. □

According to the above propositions, finding an optimum of a mean-variance portfolio problem is equivalent to solving the following separable program:

**Problem 2**

$$\begin{aligned} \text{Minimize } f(\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^n \frac{y_i^2}{d^A/\lambda_i}, \\ \text{subject to } &(C1), (C2), (C3), (C4), \quad \mathbf{y} = \mathbf{B}(\mathbf{x} - \mathbf{x}^0), \end{aligned} \tag{2.8}$$

where all the variables are defined as described previously.

Problem 2 is a separable integer program solvable to find an optimal solution by linearizing the quadratic terms  $y_i^2$  using piecewise linearization techniques described below.

**Proposition 5** [Piecewise Linearization, Li et al., 2002]. Denote  $L(f(x))$  as a piecewise linear function of  $f(x)$ , where  $a_j, j = 1, 2, \dots, m$  represents the break points of  $L(f(x))$ , and  $s_j$  is the slopes of line segments between  $a_j$  and  $a_{j+1}$  for  $j = 1, 2, \dots, n$ .  $L(f(x))$  is expressed as follows:

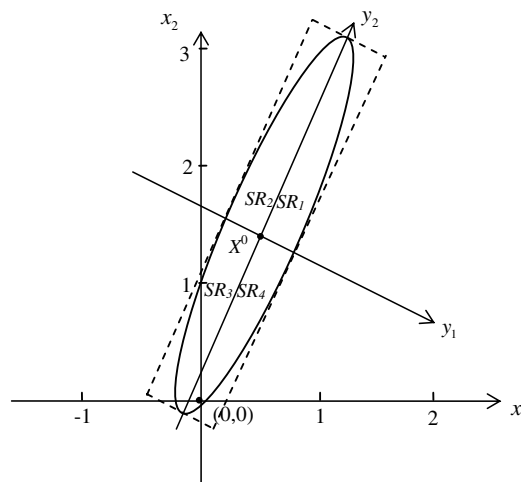


Fig. 1. Graphic illustration of the search region after the transformation.

$$f(x) \cong L(f(x)) = f(a_1) + s_1(x - a_1) + \sum_{j=2}^m \frac{s_j - s_{j-1}}{2} (|x - a_j| + x - a_j), \tag{2.9}$$

where  $|x|$  denotes the absolute value of  $x$  and  $\cong$  represents approximation.

Since the quadratic terms  $\frac{s_j^2}{d^{\lambda_j}}$  in (2.8) are convex, the following proposition can be used to effectively transform the problem into a linear program.

**Proposition 6** (Referring to Li, 1996). Consider the program below:

$$\begin{aligned} \text{Minimize } z &= \sum_{j=1}^m (|x - b_j| + x - b_j), \\ \text{subject to } x &\in F, \end{aligned} \tag{2.10}$$

where  $F$  is a feasible set,  $x$  is unrestricted in sign, and  $b_j$  are constants,  $0 < b_1 < b_2 < \dots < b_m$ .

This program can be linearized as follows:

$$\begin{aligned} \text{Minimize } z &= 2 \sum_{j=1}^m \left( x - b_j + \sum_{k=1}^j d_k \right), \\ \text{subject to } x + \sum_{k=1}^m d_k &\geq b_m, \\ 0 &\leq d_1 \leq b_1, \\ 0 &\leq d_k \leq b_k - b_{k-1} \quad \text{for } k = 2, 3, \dots, m, \\ x &\in F \quad (\text{where } F \text{ is a feasible set}). \end{aligned} \tag{2.11}$$

### 3. Solution process with distributed computation

Suppose there exists a set of connected computers comprising one host computer and  $2^s$  slave computers. A distributed algorithm using  $m + 1$  computers to solve Problem 1 is described below:

- Step 1: Solve an integer relaxation program of Problem 1, where  $\mathbf{x}$  is considered a vector of continuous variables. Denote the solution as  $\mathbf{x}^0$ .
- Step 2: Solve Problem 1 by the genetic algorithm. Let  $\mathbf{x}^d$  denote the best integer solution found by the algorithm.
- Step 3: Find out the upper and lower bounds of variables  $x_i$  by the following programs.

$$\text{Max/Min} \left\{ x_i \mid \frac{1}{2} \mathbf{x}^T \mathbf{Q} \mathbf{x} \leq f(\mathbf{x}^d), (C1), (C2), (C3), (C4) \right\},$$

where  $\mathbf{x}$  is treated as a vector of continuous variables.

Denote the obtained upper and lower bounds of  $x_i$  as  $\bar{x}_i$  and  $\underline{x}_i$ . The integral upper and lower bounds of  $x_i$  then are  $\lceil \bar{x}_i \rceil$  and  $\lfloor \underline{x}_i \rfloor$ .

- Step 4: Formulate an ellipse equation centered at  $\mathbf{x}^0$ .
- Step 5: Partition the area of the search region  $r^d = 2^n \sqrt{(d^d)^n / \lambda_1 \lambda_2 \dots \lambda_n}$  into  $2^s$  sub-regions according to Proposition 4.
- Step 6: Transform the separable QIP program with search region  $SR_k$  into a linear integer program according to Propositions 5 and 6. The host computer then assigns the transformed linear integer program with search region  $SR_k$  to a slave computer to find the optimal solution.

*Step 7:* If no integer solution better than  $\mathbf{x}^d$  is found after searching all the sub-regions, then  $\mathbf{x}^d$  is the global optimum of **Problem 1**. If an integer solution  $\mathbf{x}^*$  exists, where one of the slave computers finds  $f(\mathbf{x}^*) < f(\mathbf{x}^d)$ , then terminate the other processes and replace  $\mathbf{x}^d$  by  $\mathbf{x}^*$ .

Reiterate Steps 3–7.

After searching the  $2^s$  sub-regions,  $SR_1, SR_2, \dots, SR_{2^s}$ , the obtained solution can be guaranteed to be the global optimum of **Problem 1**.

#### 4. Numerical examples

**Example 1.** This study uses the example from Markowitz (1952) to demonstrate the process of solving a portfolio problem with integer variables. Table 1 displays the 12 year data from 1943 to 1954 for three stocks, ATT, GMC, and USX. Meanwhile, Table 2 lists the expected returns of the three stocks.

The associated variance-covariance matrix of Table 1 is computed as shown in Table 3.

Given a total allocation budget of 100, annual returns of 15%, and no transaction costs, the portfolio optimization model is formulated below:

$$\text{Min } 0.0108x_1^2 + 0.0584x_2^2 + 0.0942x_3^2 + 0.0248x_1x_2 + 0.0262x_1x_3 + 0.1108x_2x_3 \quad (4.1)$$

$$\text{subject to } 1.089x_1 + 1.214x_2 + 1.235x_3 \geq 115, \quad (4.2)$$

$$x_1 + x_2 + x_3 = 100, \quad (4.3)$$

$$\mathbf{x} = (x_1, x_2, x_3), \quad x_i \text{ are integral.}$$

This example can be solved by the proposed algorithm with four computers connected via the Internet, as follows:

*Iteration 1:*

*Step 1:* The optimal solution of the continuous relaxation of (4.1) is  $(x_1^0, x_2^0, x_3^0) = (53, 35.64, 11.34)$  with  $f(\mathbf{x}^0) = 224.105$ . The eigenvalues of  $\mathbf{Q}$  are  $\lambda_1 = 0.274111$ ,  $\lambda_2 = 0.0371779$ , and  $\lambda_3 = 0.0155112$ .

*Step 2:* Let an initially feasible solution obtained by the genetic algorithm be  $\mathbf{x}^d = (52, 37, 11)$  with  $f(\mathbf{x}^d) = 228.348$ .

*Step 3:* The found upper and lower bounds of  $x_i$  are:

$$51.35 \leq x_1 \leq 54.65, \quad 24.22 \leq x_2 \leq 47.08, \quad \text{and} \quad 1.55 \leq x_3 \leq 21.13.$$

Moreover, the integral upper and lower bounds of  $x_i$  are:

$$52 \leq x_1 \leq 54, \quad 25 \leq x_2 \leq 47, \quad \text{and} \quad 2 \leq x_3 \leq 21.$$

Table 1  
Returns of three selected stocks from 1943 to 1954

Year	ATT	GMC	USX
43	1.300	1.225	1.149
44	1.103	1.290	1.260
45	1.216	1.216	1.419
46	0.954	0.728	0.922
47	0.929	1.144	1.169
48	1.056	1.107	0.965
49	1.038	1.321	1.133
50	1.089	1.305	1.732
51	1.090	1.195	1.021
52	1.083	1.390	1.131
53	1.035	0.928	1.006
54	1.176	1.715	1.908

Table 2  
Expected returns of the selected stocks

	ATT	GMC	USX
Expected return	1.089	1.214	1.235

Table 3  
Variance-covariance matrix

	ATT	GMC	USX
ATT	0.0108	0.0124	0.0131
GMC	0.0124	0.0584	0.0554
USX	0.0131	0.0554	0.0942

Step 4: The obtained **B** is

$$\begin{pmatrix} -0.14 & -0.585 & -0.799 \\ -0.214 & -0.77 & 0.601 \\ 0.967 & -0.256 & 0.017 \end{pmatrix}.$$

$y_1, y_2,$  and  $y_3$  are expressed as

$$\begin{aligned} y_1 &= -0.14(x_1 - 53) - 0.585(x_2 - 35.64) - 0.799(x_3 - 11.34), \\ y_2 &= -0.214(x_1 - 53) - 0.77(x_2 - 35.64) + 0.601(x_3 - 11.34), \\ y_3 &= 0.967(x_1 - 53) - 0.256(x_2 - 35.64) + 0.017(x_3 - 11.34). \end{aligned}$$

The related ellipse equation, based on  $\mathbf{x}^d = (52, 37, 11)$ , is

$$\frac{y_1^2}{30.96} + \frac{y_2^2}{228.25} + \frac{y_3^2}{547.09} = 1.$$

$$\sqrt{d^A/\lambda_1} = 5.56, \quad \sqrt{d^A/\lambda_2} = 15.1 \quad \text{and} \quad \sqrt{d^A/\lambda_3} = 23.39.$$

Step 5: The search region is divided into four sub-regions  $SR_k$  for  $k = 1, 2, 3, 4$ . The search region for each computer is listed below.

For 1st PC:  $0 \leq y_1 \leq 5.56, 0 \leq y_2 \leq 15.1, -23.39 \leq y_3 \leq 23.39$ .

2nd PC:  $-5.56 \leq y_1 \leq 0, 0 \leq y_2 \leq 15.1, -23.39 \leq y_3 \leq 23.39$ .

3rd PC:  $0 \leq y_1 \leq 5.56, -15.1 \leq y_2 \leq 0, -23.39 \leq y_3 \leq 23.39$ .

4th PC:  $-5.56 \leq y_1 \leq 0, -15.1 \leq y_2 \leq 0, -23.39 \leq y_3 \leq 23.39$ .

Step 6: Taking  $SR_1$  for instance, the program is formulated as:

Minimize  $\frac{L(y_1^2)}{30.96} + \frac{L(y_2^2)}{228.25} + \frac{L(y_3^2)}{547.09}$

subject to (4.2), (4.3),

$$y_1 = -0.14(x_1 - 53) - 0.585(x_2 - 35.64) - 0.799(x_3 - 11.34), \tag{4.4}$$

$$y_2 = -0.214(x_1 - 53) - 0.77(x_2 - 35.64) + 0.601(x_3 - 11.34), \tag{4.5}$$

$$y_3 = 0.967(x_1 - 53) - 0.256(x_2 - 35.64) + 0.017(x_3 - 11.34), \tag{4.6}$$

$$0 \leq y_1 \leq 5.56, \quad 0 \leq y_2 \leq 15.1, \quad -23.39 \leq y_3 \leq 23.39,$$

$$52 \leq x_1 \leq 54, \quad 25 \leq x_2 \leq 47, \quad 2 \leq x_3 \leq 21,$$



where  $L(y_1^2)$ ,  $L(y_2^2)$ , and  $L(y_3^2)$  are piecewise linear functions of  $y_1^2$ ,  $y_2^2$ , and  $y_3^2$ , respectively, and  $x_1$ ,  $x_2$ , and  $x_3$  are integral.

*Step 7:* The execution time for the three sub-problems submitted in the first iteration is almost identical and the solutions obtained for these sub-problems are listed below:

$SR_1$ : No feasible solution.

$SR_2$ :  $(x_1, x_2, x_3) = (52, 39, 9)$ ,  $f(\mathbf{x}) = 227.1066$ .

$SR_3$ :  $(x_1, x_2, x_3) = (53, 35, 12)$ ,  $f(\mathbf{x}) = 224.645$ .

Since the solution of  $SR_3$  is better than  $\mathbf{x}^A = (52, 37, 11)$ ,  $\mathbf{x}^A$  is replaced by  $(53, 35, 12)$ . Reiterate Steps 3–7.

*Iteration 2:* Only one sub-problem is left to be solved in this iteration. The search region of this sub-problem is smaller than  $SR_4$  because a better integer solution  $\mathbf{x}^A = (53, 35, 12)$  is found in the first iteration.

*Step 3':* The integral upper and lower bounds of  $x_i$  become:

$$x_1 = 53, \quad 32 \leq x_2 \leq 39, \quad \text{and} \quad 8 \leq x_3 \leq 14.$$

*Step 4':* The related ellipse equation, based on  $\mathbf{x}^A = (53, 35, 12)$ , is

$$\frac{y_1^2}{3.94} + \frac{y_2^2}{20.05} + \frac{y_3^2}{69.63} = 1.$$

*Step 6':* The host computer assigns the transformed sub-problem to a slave computer and the sub-problem is formulated as:

$$\text{Minimize} \quad \frac{L(y_1^2)}{3.94} + \frac{L(y_2^2)}{20.05} + \frac{L(y_3^2)}{69.63}$$

subject to (4.2), (4.3), (4.4), (4.5), (4.6),

$$-1.98 \leq y_1 \leq 0, \quad -4.48 \leq y_2 \leq 0, \quad -8.34 \leq y_3 \leq 8.34,$$

$$x_1 = 53, \quad 32 \leq x_2 \leq 39, \quad 8 \leq x_3 \leq 14,$$

where all variables are defined as before.

*Step 7':* Solving the sub-problem reveals no feasible solution better than  $\mathbf{x}^A = (53, 35, 12)$ . The whole solution process is completed after this step.

From the above solution process, the globally optimal solution found for this example is  $\mathbf{x}^* = (53, 35, 12)$  with  $f(\mathbf{x}^*) = 224.645$  and  $\mathbf{y}^* = (-0.153, 0.889, 0.175)$ . This is exactly the global optimum of this example.

Most stock markets require invested units to be integral. However, traditional mean variance portfolio models can only find solutions with continuous values. Such models will then round off the obtained continuous values to integral values. However, the rounded integral values may not be the optimal solution for the original program. In this example, a global solution with 0.5% tolerance identified by the Lagrangian relaxation techniques (Geoffrion, 1974; Fisher, 1981) is  $(53, 35.64, 11.34)$  and rounding off the continuous solution to an integral solution may obtain  $(53, 36, 11)$ , which is neither a global solution nor a feasible solution.

Table 4  
Empirical results from different portfolio sizes (Tolerance:  $\varepsilon = 0.5\%$ )

Number of stocks	CPU time	
	Single PC (second)	Distributed computation (8 PCs, (second))
10	2	<1
30	9	3
50	58	12

**Example 2.** This example is used to test the execution time of the proposed method with distributed computation for various sizes of **Problem 1**, where  $a_{ij}$  and  $b_i$  are generated randomly.

The same data sets are run by the proposed model with a total allocation budget of 100, annual returns of 15% and transaction costs of 1%. **Table 4** lists the computational results, which demonstrates that the proposed method can find optimal integer solutions within 12 seconds for  $n \leq 50$  with a network of eight personal computers. Using the proposed method with a distributed computation mechanism significantly reduces computational time.

## 5. Conclusions

This study proposes a distributed algorithm for solving a portfolio problem with integer variables. By converting the objective function of the portfolio problem into a related ellipse function, the original problem is transformed into a linear mixed integer program. The search region is partitioned equally into several sub-regions for distributed computation. By searching these sub-regions, a global optimum is guaranteed to be reached. The numerical examples illustrate that the proposed distributed algorithm can find the global optimum of a portfolio problem effectively.

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