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Strong Menger connectivity with conditional faults on the class of hypercube-like networks ☆

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Abstract

In this paper, we study the Menger property on a class of hypercube-like networks. We show that in all n-dimensional hypercube-like networks with $n-2$ vertices removed, every pair of unremoved vertices u and v are connected by $\min\{\deg(u), \deg(v)\}$ vertex-disjoint paths, where $\deg(u)$ and $\deg(v)$ are the remaining degree of vertices u and v, respectively. Furthermore, under the restricted condition that each vertex has at least two fault-free adjacent vertices, all hypercube-like networks still have the strong Menger property, even if there are up to $2n-5$ vertex faults.

Keywords: Strong Menger connectivity; Conditional faults; Hypercube-like network; Interconnection networks

1. Introduction

Interconnection networks have been widely studied recently. The architecture of an interconnection network is usually denoted as an undirected graph G. Among all fundamental properties for interconnection networks, the (vertex) connectivity is a major parameter widely discussed for the connection status of networks. A basic definition of the connectivity $\kappa(G)$ of a graph G is defined as the minimum number of vertices whose removal from G produces a disconnected graph. In contrast to this concept, Menger [5] provided a local point of view, and define the connectivity of any two vertices as the minimum number of internally vertex-disjoint paths between them.

In this paper, we study the Menger property on a class of hypercube-like networks [9], which is a variation of the classical hypercube network by twisting some pairs of links in it. We show that in all n-dimensional hypercube-like networks with some vertices removed, every pair of unremoved vertices u and v are connected by $\min\{\deg(u), \deg(v)\}$ vertex-disjoint paths, where $\deg(u)$ and $\deg(v)$ are the remaining degree of vertices u and v, respectively. This concept is firstly applied on hypercubes and stars by Oh and Chen [6–8]. In this paper, we give a simpler proof of this result. Furthermore, if we restrict a condition such that each vertex has at least two fault-free adjacent vertices, all hypercube-like networks still have this strong Menger

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property, even if there are up to $2n-5$ vertex faults. The bound of $2n-5$ is sharp.

2. Preliminary

The topology of a multiprocessor system can be modeled as an undirected graph $G=(V,E)$, where $V(G)$ represents the set of all processors and $E(G)$ represents the set of all connecting links between the processors. For a subset of vertices $F\subset V(G)$, the induced graph obtained by deleting the vertices of F from G is denoted by $G-F$. Let u be a vertex, we use $N(u)$ to denote the set of vertices adjacent to u, and use $\deg(u)$ to denote the cardinality of $N(u)$. For a set of vertices V', the neighborhood of V' is defined as the set $N(V')=\{\bigcup_{v\in V'}N(v)\}-V'$. Let G be a graph with a set F of faulty vertices, the number of fault-free neighbors of u in $G-F$ is denoted by $\deg_{G-F}(u)$.

Let $G_0=(V_0,E_0)$ and $G_1=(V_1,E_1)$ be two disjoint graphs with the same number of vertices. A one-to-one connection between $V(G_0)$ and $V(G_1)$ is defined as an edge set $M=\{(v,\phi(v))\mid v\in V_0, \phi(v)\in V_1$ and $\phi: V_0\to V_1$ is a bijection$\}$. We use $G_0\oplus_M G_1$ to denote the graph $G=(V_0\cup V_1, E_0\cup E_1\cup M)$. Different bijection functions ϕ lead to different operations $\oplus_M$ and generate different graphs.

The *hypercube* network is one of the popular topologies in interconnection networks. Several variants of hypercubes are proposed by twisting some pairs of links in hypercubes, including twisted cubes [1,4], Möbius cubes [2], and crossed cubes [3], to name a few. To make a unified study on these variants, Vaidya et al. [9] proposed a class of graphs, called *a class of hypercube-like networks.* We now give a recursive definition of the n-dimensional hypercube-like networks HL_n as follows: (1) $HL_0=K_1$, where K_1 is a trivial graph in the sense that it has only one vertex; and (2) $G\in HL_n$ if and only if $G=G_0\oplus_M G_1$ for some $G_0,G_1\in HL_{n-1}$. By the definitions above if G is a graph in HL_n, then G is a composition of $G_0\oplus_M G_1$ with both G_0 and G_1 in HL_{n-1}, $n\geqslant 1$. Each vertex in G_0 has exactly one neighbor in G_1.

A graph G is r-regular if the degree of every vertex in G is r. We say that a graph G is *connected* if there is a path between every pair of two distinct vertices. A subset S of $V(G)$ is a *cut set* if $G-S$ is disconnected. The *connectivity* of G, written as $\kappa(G)$, is defined as the minimum size of a vertex cut if G is not a complete graph, and $\kappa(G)=|V(G)|-1$ if otherwise. We say that a graph G is *k-connected* if $k\leqslant\kappa(G)$. In addition, a graph has *connectivity k* if it is k-connected but not $(k+1)$-connected.

A classical theorem about connectivity was provided by Menger as follows.

Theorem 1. *(See [5].) Let x and y be two distinct vertices of a graph G and $(x,y)\notin E(G)$. The minimum size of an x,y-cut equals the maximum number of pairwise internally disjoint x,y-paths.*

Following this theorem, Oh and Chen [7] gave a definition to extend the Menger's theorem.

Definition 1. (See [7].) A k-regular graph G is *strongly Menger-connected* if for any subgraph $G-F$ of G with at most $k-2$ vertices removed, each pair of vertices u and v in $G-F$ are connected by $\min\{\deg_{G-F}(u), \deg_{G-F}(v)\}$ vertex-disjoint fault-free paths in $G-F$, where $\deg_{G-F}(u)$ and $\deg_{G-F}(v)$ are the degree of u and v in $G-F$, respectively.

By Definition 1, Oh and Chen [6–8] showed that an n-dimensional star graph S_n (respectively, an n-dimensional hypercube Q_n) with at most $n-3$ (respectively, $n-2$) vertices removed is strongly Menger-connected. In order to be consistent with Definition 1, we say that a graph G possess the strongly Menger-connected property with respect to a vertex set F if, after deleting F from G, there are $\min\{\deg_{G-F}(u),\deg_{G-F}(v)\}$ vertex-disjoint fault-free paths connecting u and v, for each pair of vertices u and v in $G-F$. Throughout this paper, we shall call a graph "strongly Menger-connected", and omit the description of the remaining structure $G-F$ of the graph, if there is no ambiguous.

It is known that the connectivity of an n-dimensional hypercube-like network HL_n is n [9]. To extend the connectivity result of HL_n further, we study the strongly Menger-connected property of HL_n with at most $n-2$ vertices deleted. Moreover, if we restrict a condition such that each vertex has at least two fault-free adjacent vertices, HL_n still have the strong Menger property, even if there are up to $2n-5$ vertex faults.

3. Strong Menger connectivity

In this section, we will prove that all graphs in the class of n-dimensional hypercube-like networks are strongly Menger-connected if there are at most $n-2$ vertex faults. Before proving this main result, we need the following lemma, essentially it says that every n-dimensional hypercube-like network with no more than $2n-3$ vertex faults, still contains a large connected component.

Lemma 1. *Let $G \in HL_n$ be an n-dimensional hypercube-like network, and S be a set of vertices with $|S| \leqslant 2n-3$, for $n \geqslant 2$. There exists a connected component C in $G-S$ such that $|V(C)| \geqslant 2^n - |S| - 1$.*

Proof. We prove this statement by induction on n. For $n = 2$, HL_2 is a cycle of length four, the result is trivially true. Assume this lemma holds for $n-1$, for some $n \geqslant 3$, we will prove that it is true for n.

Let G be an n-dimensional hypercube-like network, $G = G_0 \oplus_M G_1$, and $G_0, G_1 \in HL_{n-1}$. Let S be a set of vertices with $|S| \leqslant 2n-3$, for $n \geqslant 3$, and let S_0 and S_1 be subsets of set S in G_0 and G_1, respectively. Then $|S_0| + |S_1| = |S| \leqslant 2n-3$. Without loss of generality, we assume $|S_0| \leqslant |S_1|$. The proof is divided into two major cases:

Case 1: $0 \leqslant |S_0| \leqslant 1$.

Since G_0 is $(n-1)$-connected, $G_0 - S_0$ is connected, for $n \geqslant 3$. All the vertices in $G_0 - S_0$ are connected and form a connected component C_0 with $|V(C_0)| = 2^{n-1} - S_0$. By definition, all the vertices in $G_1 - S_1$ are adjacent to the vertices in $G_0 = C_0 \cup S_0$. Thus, $G - S$ contains a connected component C such that the number of vertices in C is greater than $|V(G_0) - S_0| + |V(G_1) - S_1| - |S_0| = |V(G)| - |S| - |S_0| \geqslant 2^n - |S| - 1$. (See Fig. 1.)

Case 2: $|S_0| \geqslant 2$ and consequently $|S_1| \leqslant 2n-5$.

Since $2 \leqslant |S_0| \leqslant |S_1| \leqslant 2n-5$, so $|S_0| \leqslant n-2$ and $n \geqslant 4$. By induction hypothesis, there exists a connected component C_1 in $G_1 - S_1$, and $|V(C_1)| \geqslant 2^{n-1} - |S_1| - 1$. Since the connectivity of G_0 is $n-1$ and $|S_0| \leqslant n-2$, $G_0 - S_0$ is connected. Then $G - S$ contains a connected component C such that the number of vertices in C is greater than $|V(G_0) - S_0| + (|V(G_1) - S_1| - 1) = |V(G)| - |S| - 1 = 2^n - |S| - 1$. □

By Lemma 1, we have the following corollary.

Corollary 1. *Let G be an n-dimensional hypercube-like network, $n \geqslant 2$, and let V' be a set of vertices in G with $|V'| = 2$. Then $|N(V')| \geqslant 2n-2$.*

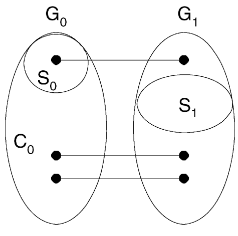

Fig. 1. The illustration of the proof of Case 1 in Lemma 1.

In the following, we show that with up to $n-2$ vertex faults, an n-dimensional hypercube-like network has strongly Menger-connected property. Referring to the relative study proposed by Oh [6], the strong Menger connectivity of regular hypercube networks has been proved. Here we provide a significantly simpler proof for the general hypercube-like networks.

Theorem 2. *Consider an n-dimensional hypercube-like network $G \in HL_n$, for $n \geqslant 2$. Let F be a set of faulty vertices with $|F| \leqslant n-2$. Then each pair of vertices u and v in $G-F$ are connected by* $\min\{\deg_{G-F}(u), \deg_{G-F}(v)\}$ *vertex-disjoint fault-free paths, where* $\deg_{G-F}(u)$ *and* $\deg_{G-F}(v)$ *are the remaining degree of u and v in $G-F$, respectively.*

Proof. Let G be an n-dimensional hypercube-like network, and u and v be two fault-free vertices in $G-F$. We first assume, without loss of generality, that $\deg_{G-F}(u) \leqslant \deg_{G-F}(v)$, so $\min\{\deg_{G-F}(u), \deg_{G-F}(v)\} = \deg_{G-F}(u)$. We now show that u is connected to v if the number of vertices deleted is smaller than $\deg_{G-F}(u) - 1$ in $G-F$. By Theorem 1, this implies that each pair of vertices u and v in $G-F$ are connected by $\deg_{G-F}(u)$ vertex-disjoint fault-free paths, where $|F| \leqslant n-2$.

For the sake of contradiction, suppose that u and v are separated by deleting a set of vertices V_f, where $|V_f| \leqslant \deg_{G-F}(u) - 1$. As a consequence, $|V_f| \leqslant n-1$ because of $\deg_{G-F}(u) \leqslant \deg(u) \leqslant n$. Then, the summation of the cardinality of these two sets F and V_f is $|F| + |V_f| \leqslant 2n-3$. Let $S = F \cup V_f$. By Lemma 1, there exists a connected component C in $G-S$ such that $|V(C)| \geqslant 2^n - |S| - 1$. It means that (i) either $G-S$ is connected, or (ii) $G-S$ has two components, one of which contains only one vertex. If $G-S$ is connected, it contradicts to the assumption that u and v are disconnected. Otherwise, if $G-S$ has two component and one of which contains only one vertex x. Since we assume that u and v are separated, one of u and v is the vertex x, say $u = x$. Thus, the set V_f must be the neighborhood of u and $|V_f| = \deg_{G-F}(u)$, which is also a contradiction. Then, u is connected to v when the number of vertices deleted is smaller than $\deg_{G-F}(u) - 1$ in $G-F$.

The proof is complete. □

4. Strong Menger connectivity with conditional faults

As proved in the last section, an n-dimensional hypercube-like network with at most $n-2$ faulty vertices is strongly Menger-connected. But the result can-not be guaranteed, if there are $n-1$ faulty vertices and all these faulty vertices are adjacent to the same vertex. In most circumstances, the possibility of all the neighbors of a vertex being faulty simultaneously is very small. Motivated by the deficiency of traditional fault tolerance, we consider a measure of conditional faults by restricting that every vertex has at least two fault-free neighboring vertices.

Under this condition, we claim that for every n-dimensional hypercube-like network with at most $2n-5$ faulty vertices and $n \geqslant 5$, the resulting network is still strongly Menger-connected. We have an example to show that this result does not hold for $n=4$. Consider a 4-dimensional HL_4, this network may not be strongly Menger-connected, if the number of conditional faults is 3. (See Fig. 2. The remaining degrees of nodes u and v are both four, with three vertices deleted as indicated in the graph. But the number of vertex-disjoint paths between u and v is three.) So we can only expect the result holds for $n \geqslant 5$.

To prove this result, we need some preliminary lemma. In the following, we show that an n-dimensional hypercube-like network with at most $3n-6$ vertex faults S has a connected component having at least $2^n - |S| - 2$ vertices.

The proof is by induction, and the case for $n=5$ is proved in the following two lemmas.

Lemma 2. *Let V' be a set of vertices in a 4-dimensional hypercube-like network with $|V'| = 3$. Then, $|N(V')| \geqslant 7$.*

Proof. Let G be a 4-dimensional hypercube-like network. G is a composition of two 3-dimensional hypercube-like networks G_0 and G_1, $G = G_0 \oplus_M G_1$, for a matching operation $\oplus_M$. Without loss of generality, let V' be a subset of $V(G)$ containing three vertices $\{x, y, z\}$. If x, y, z are all in G_0, by Lemma 1, $\{x, y, z\}$ has at least 4 neighboring vertices in G_0. Besides, $\{x, y, z\}$ has 3 neighboring vertices in G_1. Then, $|N(\{x, y, z\})| \geqslant 4+3=7$. If x, y are in G_0, and z is in G_1, by Lemma 1, $\{x, y\}$ has at least 4 neighboring vertices in G_0. In addition, $\{z\}$ has 3 neighboring vertices in G_1. Then, $|N(\{x, y, z\})| \geqslant 4+3=7$. □

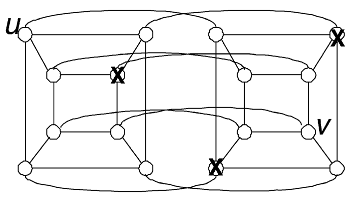

Fig. 2. An example showing that an HL_4 is not strongly Menger-connected.

Lemma 3. *Let G be a 5-dimensional hypercube-like network and S be a set of vertices with $|S| \leqslant 9$. ($3n-6=9$, for $n=5$.) There exists a connected component C in $G-S$ such that $|V(C)| \geqslant 2^5 - |S| - 2$.*

Proof. Let G be a 5-dimensional hypercube-like network, $G_0, G_1 \in HL_4$, and $G = G_0 \oplus_M G_1$, for a matching operation $\oplus_M$. Let S be a set of vertices with $|S| \leqslant 3n-6=9$, for $n=5$, and let S_0 and S_1 be subsets of S in G_0 and G_1, respectively. Without loss of generality, we assume $|S_0| \leqslant |S_1|$. (Note that $|S| \leqslant 9$, so $|S_0| \leqslant 4$.) We then consider three cases:

Case 1: $0 \leqslant |S_0| \leqslant 2$.

Since G_0 is $(n-1)$-connected, $G_0 - S_0$ is connected, for $n=4$. So $G_0 - S_0$ has only one connected component C_0 with $|V(C_0)| = 2^4 - S_0$. By definitions, all vertices in $G_1 - S_1$ are adjacent to the vertices of $G_0 = C \cup S_0$. Let C be the connected component of $G-S$ containing C_0. Then the number of vertices in C is greater than $|V(G_0) - S_0| + |V(G_1) - S_1| - |S_0| = |V(G)| - |S| - |S_0| \geqslant 2^5 - |S| - 2$.

Case 2: $|S_0| = 3$ and therefore $|S_1| \leqslant 6$.

$G_0 - S_0$ is connected by the fact that G_0 is $(n-1)$-connected, for $n \geqslant 4$. Thus, $G_0 - S_0$ has only one connected component C_0 with $|V(C_0)| = 2^4 - S_0$. Then, all vertices in G_1 are connected to component C_0, except for the three vertices in G_1 adjacent to the vertices in S_0. Since $|S_1| \leqslant 6$ and by Lemma 2, at least one of these three vertices is connected to component $G_1 - S_1$. So at least $2^4 - |S_1| - 2$ vertices are connected to component C_0. Let C be the connected component of $G-S$ containing C_0. Then, the number of vertices in C is $|V(C)| \geqslant |V(G_0) - S_0| + |V(G_1) - S_1 - 2| = |V(G)| - |S| - 2 = 2^5 - |S| - 2$.

Case 3: $|S_0| = 4$ and consequently $4 \leqslant |S_1| \leqslant 5$.

Since $5 \leqslant 2n-3$, for $n \geqslant 4$. By Lemma 1, there exists a connected components C_0 (respectively, C_1) in $G_0 - S_0$ (respectively, $G_1 - S_1$) such that $|V(C_0)| \geqslant 2^4 - |S_0| - 1$ (respectively, $|V(C_1)| \geqslant 2^4 - |S_1| - 1$). Thus, there exists a connected component C in $G-S$ such that $|V(C)| \geqslant |V(G_0) - S_0 - 1| + |V(G_1) - S_1 - 1| = |V(G)| - |S| - 2 = 2^5 - |S| - 2$. □

Based on Lemma 3, the general case for $n \geqslant 5$ is stated as follows.

Lemma 4. *Let G be an n-dimensional hypercube-like network, and S be a set of vertices with $|S| \leqslant 3n-6$, for*

$n \geqslant 5$. There exists a connected component C in $G - S$ such that $|V(C)| \geqslant 2^n - |S| - 2$.

Proof. We prove this statement by induction on n. By Lemma 3, the result holds for $n = 5$. Assume the lemma holds for $n - 1$, for some $n \geqslant 6$. We now show that it is true for n.

Let G be an n-dimensional hypercube-like network, $G_0, G_1 \in HL_{n-1}$, and $G = G_0 \oplus_M G_1$, for some matching operation $\oplus_M$. Let S be a set of vertices with $|S| \leqslant 3n - 6$, for $n \geqslant 6$, and let S_0 and S_1 be subsets of S in G_0 and G_1, respectively. Therefore, $|S_0| + |S_1| = |S| \leqslant 3n - 6$. Without loss of generality, we assume $|S_0| \leqslant |S_1|$. The proof is divided into two major cases:

Case 1: $0 \leqslant |S_0| \leqslant 2$.

Since G_0 is $(n - 1)$-connected, $G_0 - S_0$ is connected, for $n \geqslant 6$. Let $C_0 = G_0 - S_0$, C_0 is a connected component with $|V(C_0)| \geqslant 2^{n-1} - S_0$. By definitions, all vertices in $G_1 - S_1$ are adjacent to the vertices in $G_0 = C_0 \cup S_0$. Let C be the connected component of $G - S$ containing C_0. The number of vertices in C is greater than $|V(G_0) - S_0| + |V(G_1) - S_1| - |S_0| = |V(G)| - |S| - |S_0| \geqslant 2^n - |S| - 2$.

Case 2: $|S_0| \geqslant 3$ and consequently $|S_1| \leqslant 3n - 9$.

By induction hypothesis, there are two connected components C_0 and C_1 in $G_0 - S_0$ and $G_1 - S_1$, and $|V(C_0)| \geqslant 2^{n-1} - |S_0| - 2$ and $|V(C_1)| \geqslant 2^{n-1} - |S_1| - 2$, respectively. Without loss of generality, we assume that $|V(C_0)| \geqslant |V(C_1)|$. Now we focus on the number of vertices in the component C_1, and discuss two situations. First, suppose $|V(C_1)| = 2^{n-1} - |S_1| - 2$. By Corollary 1, $|S_1| \geqslant 2(n - 1) - 2 = 2n - 4$. So $|S_0| = |S| - |S_1| \leqslant n - 2$. Since G_0 is $(n - 1)$-connected, $G_0 - S_0$ is connected. $G_0 - S_0$ has only one connected component C_0 and $|V(C_0)| = 2^{n-1} - |S_0|$. Let C be the connected component containing C_0. Then $|V(C)| = |V(C_0)| + |V(C_1)| \geqslant 2^{n-1} - |S_0| + 2^{n-1} - |S_1| - 2 \geqslant 2^n - |S| - 2$. Second, suppose that $|V(C_1)| \geqslant 2^{n-1} - |S_1| - 1$. Since $|V(C_0)| \geqslant |V(C_1)| \geqslant 2^{n-1} - |S_1| - 1$, there exists a connected component C containing C_0 such that $|V(C)| = |V(C_0)| + |V(C_1)| \geqslant 2^{n-1} - |S_0| - 1 + 2^{n-1} - |S_1| - 1 \geqslant 2^n - |S| - 2$. □

Corollary 2. *Let G be an n-dimensional hypercube-like network, $n \geqslant 5$, and let V' be a set of vertices in G with $|V'| = 3$. Then $|N(V')| \geqslant 3n - 5$.*

As stated in the last section, we showed that every n-dimensional hypercube-like network with at most $n - 2$ vertex faults is strongly Menger-connected. In the following, we will show another main result that, by restricting every vertex having at least two fault-free neighboring vertices, every n-dimensional hypercube-like network with up to $2n - 5$ vertex faults is still strongly Menger-connected.

For the next theorem, we define a set of vertices F_c in graph G to be a *conditional* faulty vertex set if, in the induced subgraph $G - F_c$, every vertex has at least two fault-free neighboring vertices. We also call the subgraph $G - F_c$ a *conditional* faulty graph.

Theorem 3. *Consider an n-dimensional hypercube-like network $G \in HL_n$, for $n \geqslant 5$. Let F_c be a set of conditional faulty vertices with $|F_c| \leqslant 2n - 5$. Then each pair of vertices u and v in $G - F_c$ are connected by $\min\{\deg_{G-F_c}(u), \deg_{G-F_c}(v)\}$ vertex-disjoint fault-free paths, where $\deg_{G-F_c}(u)$ and $\deg_{G-F_c}(v)$ are the degree of u and v in $G - F_c$, respectively.*

Proof. Without loss of generality, we assume $\deg_{G-F_c}(u) \leqslant \deg_{G-F_c}(v)$, and therefore

$$\min\{\deg_{G-F_c}(u), \deg_{G-F_c}(v)\} = \deg_{G-F_c}(u).$$

We want to prove that each pair of vertices u and v in $G - F_c$ are connected by $\deg_{G-F_c}(u)$ vertex-disjoint fault-free paths, for $|F_c| \leqslant 2n - 5$. We are going to show that u is connected to v if the number of vertices deleted is smaller than $\deg_{G-F_c}(u) - 1$ in $G - F_c$, where $|F_c| \leqslant 2n - 5$.

Suppose on the contrary that u and v are separated by deleting a set of vertices V_{f_c}, where $|V_{f_c}| \leqslant \deg_{G-F_c}(u) - 1$. By $\deg_{G-F_c}(u) \leqslant \deg(u) \leqslant n$, we have $|V_{f_c}| \leqslant n - 1$. We sum up the cardinality of these two sets F_c and V_{f_c}. Since $|F_c| \leqslant 2n - 5$ and $|V_{f_c}| \leqslant n - 1$, then $|F_c| + |V_{f_c}| \leqslant 3n - 6$. Let $S = F_c \cup V_{f_c}$. By Lemma 4, there exits a connected component C in $G - S$ such that $|V(C)| \geqslant 2^n - |S| - 2$ and $|S| \leqslant 3n - 6$. It means that there are at most two vertices in $G - S$ not belonging to C. We then consider three cases:

Case 1: $|V(C)| = 2^n - |S|$. It means that all vertices in $G - S$ are connected, which contradicts to the assumption that u and v are disconnected.

Case 2: $|V(C)| = 2^n - |S| - 1$. Only one vertex is disconnected to $G - S$. Since $|V_{f_c}| \leqslant \deg_{G-F_c}(u) - 1 \leqslant \deg_{G-F_c}(v) - 1$, neither u nor v can be the only one disconnected vertex, a contradiction.

Case 3: $|V(C)| = 2^n - |S| - 2$. Let a and b be the two vertices in $G - S$ not belonging to C. We consider two situations. (i) Suppose first that $u \in C$. If $v \in C$, then u and v are connected, a contradiction. If $v \in \{a, b\}$, since $|V_{f_c}| \leqslant \deg_{G-F_c}(v) - 1$, v is connected to at least one vertex in component C, a contradiction. (ii) Suppose $u \in \{a, b\}$. We without loss of generality let $u = a$, and consider the adjacency between a and b.

Subcase 1: Suppose that a is not adjacent to b. By the assumption that u and v are separated by deleting a set of vertices V_{f_c} with $|V_{f_c}| = \deg_{G-F_c}(u) - 1$. Let V_{f_c} be a subset of the neighborhood of u, that is, $V_{f_c} \subset N(u)$. Since $|V_{f_c}| < |N(u)|$, vertex u and component C are connected, which is a contradiction.

Subcase 2: Suppose that a is adjacent to b. Let $V_{f_c} = N(u) - \{b\}$. Since $G - F_c$ is a conditional faulty graph, one of the neighbors of b is in C. Then, b is connected to C, which is a contradiction.

Therefore, vertex u and v are still connected with up to $\deg_{G-F_c}(u) - 1$ vertex faults. By Theorem 1, this implies that each pair of vertices u and v in $G - F_c$ are connected by $\min\{\deg_{G-F_c}(u), \deg_{G-F_c}(v)\}$ vertex-disjoint fault-free paths, where $|F_c| \leqslant 2n - 5$. The proof is complete. □

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