行政院國家科學委員會專題研究計畫成果報告

球面上預設均曲率之超曲面
Hypersurfaces with Prescribed Mean Curvature in Spheres

計畫編號：NSC 88-2115-M-009-011
執行期限：87 年 8 月 1 日至 88 年 7 月 31 日
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摘要

假設 H 為定義於 n+1 維球面上之平滑實值函數。本計畫探討相關於 n+1 維球面上以 H 為均曲率之超曲面存在問題所衍生的流動方程。利用 Schauder 估計，本報告證明：當 H 滿足適當的條件時，該流動方程之解存在。

關鍵詞：均曲率、超曲面、球面

Abstract

Let H be a given smooth real valued function defined on the (n+1)-sphere. The problem of prescribed mean curvature is to find certain conditions on H so that there exists a hypersurface M of spherical type whose mean curvature is the given function H.The purpose of this report is to show the existence of solutions of a related evolution equation. Using the Schauder fixed point theorem, we consider a linearized equation, and prove that under the assumption that the given function H satisfies certain conditions, a apriori estimates of maximum norm and gradient of the solutions for this linearized equation hold. It follows that the solution of this evolution equation exists.

Keywords: Mean curvature, hypersurface, spheres

1. Introduction

Let M and N be complete Riemannian manifolds dim(M)= n, dim(N) = n+1. Let H be a preassigned smooth function defined on N. We consider the problem of prescribed mean curvature, that is, find conditions on H so that there exists an embedding Y from M into N whose mean curvature is the given function H (see [Y]). For N being the Euclidean space and M being a sphere, significant works in this direction have been studied in [BK] and [TW]. Based on the theory of elliptic partial differential equation, Treibergs and Wei showed the existence and uniqueness of the problem of prescribed mean curvature if H decays faster than the mean curvature of two concentric spheres [TW]. For both of N and M being spheres, using the theory of elliptic partial differential equation [GT], we also showed the existence of the problem of prescribed mean curvature if H satisfies certain growth conditions.

In this report, we want to show the existence of the solution for a evolution flow related to the problem of prescribed mean curvature. Let Y be a map from the n-sphere into the (n+1)-sphere which is given by

$$Y(X) = \frac{1}{\sqrt{1+u^2}} X + \frac{u}{\sqrt{1+u^2}} E_{n+2},$$
$$where \ X \in S^n, \ E_{n+2} = (0,0,\cdots,0,1).$$

We consider the following related evolution equation

$$Y_t = (H - M)N,$$
$$where \ \text{N is the unit normal and}$$
$$M \ \text{is the mean curvature of Y.}$$

Then the corresponding quasilinear parabolic equation is

$$u_t = \frac{1+u^2}{1+u^2+|\nabla u|^2} \frac{1}{n} a_{ij} u_{ij} + b \ \text{ in } \Re_+ \times S^n,$$
$$u|_{t=0} = u_0 \ \text{ on } S^n, \qquad (*)$$
$$where \ a_{ij} = (1+u^2+|\nabla u|^2)\delta_{ij} - u_i u_j,$$
$$b = u(1+u^2) - \frac{1}{n} \frac{1+u^2}{1+u^2+|\nabla u|^2} u|\nabla u|^2$$
$$-\sqrt{1+u^2}\sqrt{1+u^2+|\nabla u|^2} H.$$

To show the existence of solutions for such a equation, we need make a apriori estimates for the solutions of the following linearized equation

$$u_t = \frac{1+u^2}{1+u^2+|\nabla u|^2} \frac{1}{n} \bar{a}_{ij} u_{ij} + \bar{b} \ \text{ in } \Re_+ \times S^n,$$
$$u|_{t=0} = u_0 \ \text{ on } S^n,$$
$$where \ \bar{a}_{ij} = (1+u^2+|\nabla u|^2)\delta_{ij} - u_i u_j,$$
$$\bar{b} = -\frac{1}{n} \frac{1+u^2}{1+u^2+|\nabla u|^2} u +$$
$$\sigma(-\sqrt{1+u^2}\sqrt{1+u^2+|\nabla u|^2} H +$$
$$(1+u^2)u - \frac{1}{n} \frac{1+u^2}{1+u^2+|\nabla u|^2} u|\nabla u|^2$$
$$+ \frac{1}{n} \frac{1+u^2}{1+u^2+|\nabla u|^2} u),$$
$$\sigma \in [0,1].$$

We show that if H satisfies the following three conditions :

$$(H1) \ H\left(\frac{1}{\sqrt{1+\alpha^2}} X + \frac{\alpha}{\sqrt{1+\alpha^2}} E_{n+2}\right) < \alpha,$$
$$for \ \alpha < -c_1$$

$$(H2) \ H\left(\frac{1}{\sqrt{1+\alpha^2}} X + \frac{\alpha}{\sqrt{1+\alpha^2}} E_{n+2}\right) > \alpha,$$
$$for \ \alpha > c_2$$
$$(H3) \ H_\alpha\left(\frac{1}{\sqrt{1+\alpha^2}} X + \frac{\alpha}{\sqrt{1+\alpha^2}} E_{n+2}\right)$$
$$> -\frac{\alpha}{1+\alpha^2} H\left(\frac{1}{\sqrt{1+\alpha^2}} X + \frac{\alpha}{\sqrt{1+\alpha^2}} E_{n+2}\right),$$

for some positive constants c_1 and c_2, then a priori estimates of the solutions for this linearized equation hold. Theose conditions are essentially growth condition. We then can state the main result of this report as follows

Theorem. Let H be a function defined on the (n+1)-sphere satisfying (H1), (H2) and (H3). Then there exists a solution u in $C^{2,1}$ of the evolution equation (*).

2. Proof of Theorem

We separate the proof into two parts. In the first part we show that the maximum estimate follows from (H1) and (H2). In the second part we show that the gradient estimate follows from (H3).

We only show that the lower bound estimate follows from (H1), similar arugument show that upper bound estimate follows from (H2). Let x be the infimum of u. We may assume that x< 0. Claim that: x is not less than $-c_1$. Suppose that x is less than $-c_1$, by using (H1), we have

$$0 \geq u_t = \frac{1+u^2}{1+u^2+|\nabla u|^2}\frac{1}{n}\bar{a}_{ij}u_{ij} + \bar{b}$$
$$\geq . -\frac{1}{n}u + \sigma(-(1+u^2)H + (1+u^2)u + \frac{1}{n}u)$$
$$> 0$$

we get a contraction.

To make the gradient estimate, after rescale the time variable t and omit the lower order term, which does not play any rule in the following argument, we may assume that

$$u_t = \bar{a}_{ij}u_{ij} + \bar{b} \text{ in } \Re_+ \times S^n,$$
$$u|_{t=0} = u_0 \text{ on } S^n,$$
$$where\ \bar{a}_{ij} = (1+u^2)\delta_{ij} - u_i u_j,$$
$$\bar{b} = \sigma(-\sqrt{1+u^2}\sqrt{1+u^2+|\nabla u|^2}H),$$
$$\sigma \in [0, n].$$

Let

$$\varphi = \frac{|\nabla u|^2}{(1+u^2)^n}.$$

Then the following inequality follows from the evolution equation of this function

$$\varphi_t \leq a_{ij}\varphi_{ij} - O(|\nabla u|^3)(H_u + \frac{u}{1+u^2}H)$$
$$+ O(\nabla\varphi, |\nabla u|^2),$$

where the second term in the right hand side is negative. The gradient estimate then follows from the maximum principle .

3. Final Comments

In this report we show that the evolution equation related to the problem of prescribed mean curvature has a solution. A natural question which should be discussed in the next work is that whether this solution converge to a solution of prescribed mean curvature. That is, the behavior of the stated state must be studied. Since we have the elliptic type result, it is possible that the stated state is just our desired hypersphere. What would be particularly interesting here is an answer to the following question: Can one find any singularities at finite time or infinite time when one of these conditions (H1), (H2) and (H3) false ? What is the behavior of the singular set ? Is there any results analogue to well known results, such as Ricci flow, mean curvature flow etc. ? these problem are also interesting to us.

4. References

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