Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 198 (2008) 414–432
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Panpositionable hamiltonicity and panconnectivity of the arrangement graphs ☆

Yuan-Hsiang Teng [a], Jimmy J.M. Tan [a,*], Lih-Hsing Hsu [b]

[a] *Department of Computer Science, National Chiao Tung University, Hsinchu City 300, Taiwan, ROC*
[b] *Department of Computer Science and Information Engineering, Providence University, Taichung County 433, Taiwan, ROC*

Abstract

The arrangement graph $A_{n,k}$ is a generalization of the star graph. It is more flexible in its size than the star graph. There are some results concerning hamiltonicity and pancyclicity of the arrangement graphs. In this paper, we propose a new concept called panpositionable hamiltonicity. A hamiltonian graph G is panpositionable if for any two different vertices x and y of G and for any integer l satisfying $d(x,y) \leqslant l \leqslant |V(G)| - d(x,y)$, there exists a hamiltonian cycle C of G such that the relative distance between x and y on C is l. A graph G is panconnected if there exists a path of length l joining any two different vertices x and y with $d(x,y) \leqslant l \leqslant |V(G)| - 1$. We show that $A_{n,k}$ is panpositionable hamiltonian and panconnected if $k \geqslant 1$ and $n - k \geqslant 2$.

Keywords: Arrangement graph; Panpositionable hamiltonian; Panconnected; Interconnection network

1. Introduction

Network topology is a crucial factor for an interconnection network since it determines the performance of the network and the distributed systems. Many interconnection network topologies have been proposed in literature for the purpose of connecting a large number of processing elements and the design of a parallel computing systems [1–6]. The hypercube [5] and the star graph [1,7] are two examples. The hypercube possesses many good properties and is implemented as many multiprocessor systems [8]. Akers et al. [1] proposed the star graph, which has smaller degree, diameter, and average distance than the hypercube while reserving symmetry properties and desirable fault-tolerant characteristics. As a result, the star graph has been recognized as an alternative to the hypercube. However, the hypercube and the star are less flexible in adjusting their sizes.

The arrangement graph [2] was proposed by Day and Tripathi as a generalization of the star graph. It is more flexible in its size than the star graph. Given two positive integers n and k with $n > k$, the

☆ This work was supported in part by the National Science Council of the Republic of China under Contract NSC 94-2213-E-009-138.
* Corresponding author.
E-mail address: jmtan@cs.nctu.edu.tw (J.J.M. Tan).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2007.08.073

(n,k)-arrangement graph $A_{n,k}$ is the graph (V,E), where $V = \{p|p$ is an arrangement of k elements out of the symbols : $1,2,\ldots,n\}$ and $E = \{(p,q)|p,q \in V$ and p,q differ in exactly one position$\}$. A more precise definition and an example will be given in the following section. $A_{n,k}$ is a regular graph of degree $k(n-k)$ with $\frac{n!}{(n-k)!}$ vertices. The diameter of $A_{n,k}$ is $\lfloor\frac{3k}{2}\rfloor$. $A_{n,1}$ is isomorphic to the complete graph K_n and $A_{n,n-1}$ is isomorphic to the n-dimensional star graph. Moreover, $A_{n,k}$ is vertex symmetric and edge symmetric [2]. Many related works about the arrangement graph have appeared in the literature [2,9–13]. Day and Tripathi showed that there exist vertex-disjoint paths between any two vertices in the arrangement graph [9]. The existence of hamiltonian cycles and the cycles of all lengths ranging between three to the size of the graph was proved in [10]. It was further proved that the multidimensional grids, hypercubes and spanning trees all with constant dilations can be embedded in the arrangement graph [11]. Hsieh et al. [12] and Hsu et al. [13] studied the fault tolerant hamiltonian property of the arrangement graph to enhance the reliability of the specific interconnection network.

Computer network topologies are usually represented by graphs where vertices represent processors and edges represent links between processors. In this paper, a network is represented as an undirected graph. For the graph definitions and notation, we follow [14]. Let $G=(V,E)$ be a *graph* if V is a finite set and E is a subset of $\{(u,v)|(u,v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set* of G. Two vertices u and v are *adjacent* if $(u,v)\in E$. A *path* is a sequence of vertices such that two consecutive vertices are adjacent. A path is represented by $\langle v_0,v_1,v_2,\ldots,v_n\rangle$. The *length* of a path P is the number of edges in P, denoted by $L(P)$. We sometimes write the path $\langle v_0,v_1,v_2,\ldots,v_k\rangle$ as $\langle v_0,P_1,v_i,v_{i+1},\ldots,v_j,P_2,v_t,\ldots,v_k\rangle$, where P_1 is the path $\langle v_0,v_1,\ldots,v_i\rangle$ and P_2 is the path $\langle v_j,v_{j+1},\ldots,v_t\rangle$. It is possible to write a path $\langle v_0,v_1,P,v_1,v_2,\ldots,v_k\rangle$ if $L(P)=0$. We use $d_G(u,v)$, or simply $d(u,v)$ if there is no ambiguity, to denote the distance between u and v in a graph G, i.e., the length of shortest path joining u and v in G. We use $d_C(u,v)$ and $D_C(u,v)$ to denote the shorter and the longer distance between u and v on a cycle C of G, respectively. It is possible that $D_C(u,v)=d_C(u,v)$ if the lengths of the two disjoint paths joining u and v in C are equal. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last one. A *hamiltonian path* is a path such that its vertices are distinct and span V. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two vertices of G. A *hamiltonian cycle* is a cycle such that its vertices are distinct except for the first vertex and the last vertex and span V. A *hamiltonian graph* is a graph with a hamiltonian cycle.

For designing a good interconnection network, there are several desired properties we have to consider. The hamiltonian property is one of the major requirements in designing an interconnection network because the property is related to the reliability and the performance of a distributed system. A high-reliability network can be designed by embedding a hamiltonian cycle in it. Many related works have appeared in the literature [10,12,13,15]. Further attempts at hamiltonian problems led researches into the study of super-hamiltonian graphs, such as pancyclic graphs and panconnected graphs. A graph G is *pancyclic* if it contains a cycle of length l for each l satisfying $3\leqslant l\leqslant |V(G)|$. The concept of pancyclic graphs is proposed by Bondy [16]. A graph G is *panconnected* if there exists a path of length l joining any two different vertices x and y with $d(x,y)\leqslant l\leqslant |V(G)|-1$. The concept of panconnected graphs is proposed by Alavi and Williamson [17]. There are some studies concerning panconnectivity and pancyclicity of some interconnection network [18–20].

We propose a new concept called *panpositionable hamiltonicity*. A hamiltonian graph G is *panpositionable* if for any two different vertices x and y of G and for any integer l satisfying $d(x,y)\leqslant l\leqslant |V(G)|-d(x,y)$, there exists a hamiltonian cycle C of G such that the relative distance between x and y on C is l; more precisely, $d_C(x,y)=l$ if $l\leqslant\lfloor\frac{|V(G)|}{2}\rfloor$ or $D_C(x,y)=l$ if $l>\frac{|V(G)|}{2}$. Given a hamiltonian cycle C, if $d_C(x,y)=l$, we have $D_C(x,y)=|V(G)|-d_C(x,y)$. Therefore, a graph is panpositionable hamiltonian if for any integer l with $d(x,y)\leqslant l\leqslant\frac{|V(G)|}{2}$, there exists a hamiltonian cycle C of G with $d_C(x,y)=l$. One example, the alternating group graph is proved to be panpositionable hamiltonian [21]. Similar to the importance of hamiltonicity for the communication between processors in an interconnection network, panpositionable hamiltonicity allows more flexible communication in a hamiltonian network. The panpositionable hamiltonian property inherits the hamiltonian property and advances it further. The concept is interesting and useful in the study of interconnection networks. In [21], an example was given to show that a panconnected graph is not necessarily panpositionable hamiltonian. Therefore, the panpositionable hamiltonian property is a stronger property for an interconnection network.

In this paper, we study the panpositionable hamiltonicity of the arrangement graph $A_{n,k}$. For $n-k=1$, $A_{n,n-1}$ is isomorphic to the n-dimensional star graph, which is bipartite and clearly is not panpositionable hamiltonian. Thus, throughout this paper, we only consider the case that $n-k \geqslant 2$. We show that the arrangement graph is panpositionable hamiltonian for all $k \geqslant 1$ and $n-k \geqslant 2$, and we find that it is closely related to its panconnected and pancyclic properties. Applying our result, we can show that the arrangement graph is panconnected and pancyclic. In the following section, we discuss some basic properties of the arrangement graphs. In Section 3, we prove that $A_{n,1}$ and $A_{n,2}$ are panpositionable hamiltonian if $n-k \geqslant 2$. In Section 4, we prove that $A_{n,k}$ is panpositionable hamiltonian and panconnected for all $k \geqslant 1$ and $n-k \geqslant 2$. In the final section, we present our conclusion and derive some relationship between the panpositionable hamiltonicity and the other useful properties for a interconnection network.

2. Some properties of the arrangement graphs

Let n and k be two positive integers with $n > k$. And, let $\langle n \rangle$ and $\langle k \rangle$ denote the sets $\{1, 2, \ldots, n\}$ and $\{1, 2, \ldots, k\}$, respectively. Then, the vertex set of the arrangement graph $A_{n,k}$, $V(A_{n,k}) = \{p | p = p_1 p_2 \ldots p_k$ with $p_i \in \langle n \rangle$ for $1 \leqslant i \leqslant k$ and $p_i \neq p_j$ if $i \neq j\}$ and the edge set of $A_{n,k}$, $E(A_{n,k}) = \{(p, q) | p, q \in V(A_{n,k}), p$ and q differ in exactly one position$\}$. Fig. 1 illustrates $A_{4,2}$.

Let i and j be two positive integers with $1 \leqslant i, j \leqslant n$. And, let $V(A_{n,k}^{(j:i)}) = \{p | p = p_1 p_2 \ldots p_k$ and $p_j = i\}$. It is the set of all vertices with the jth position being i. For a fixed position j, $\{V(A_{n,k}^{(j:i)}) | 1 \leqslant i \leqslant n\}$ forms a partition of $V(A_{n,k})$. Let $A_{n,k}^{(j:i)}$ denote the subgraph of $A_{n,k}$ induced by $V(A_{n,k}^{(j:i)})$. It is easy to see that each $A_{n,k}^{(j:i)}$ is isomorphic to $A_{n-1,k-1}$. Thus, $A_{n,k}$ can be recursively constructed from n copies of $A_{n-1,k-1}$. Each $A_{n,k}^{(j:i)}$ represents a *subcomponent* of $A_{n,k}$, and we say that $A_{n,k}$ is decomposed into subcomponents according to the jth position. Let I be a subset of $\{1, 2, \ldots, n\}$. We use $A_{n,k}^{(j:I)}$ to denote the subgraph of $A_{n,k}$ induced by $\bigcup_{i \in I} V(A_{n,k}^{(j:i)})$. $A_{n,k}^{(j:I)}$ is called an *incomplete* arrangement graph if $|I| < n$. We observe that each $A_{n,k}^{(j:i)}$ can be recursively decomposed into its smaller subcomponents. For simplicity, if there is no ambiguity, we shall concentrate on the last position, and we use $A_{n,k}^{i}$ and $A_{n,k}^{I}$ to denote $A_{n,k}^{(k:i)}$ and $A_{n,k}^{(k:I)}$, respectively, where k is the last position, and $E^{i,j}$ to denote the set of edges between $A_{n,k}^{i}$ and $A_{n,k}^{j}$. Let F be a faulty set which may include faulty edges, faulty vertices, or both. The *good edge set* $GE^{i,j}(F)$ is the set of edges $(u, v) \in E^{i,j}$ such that $\{u, v, (u, v)\} \cap F = \emptyset$. We need some basic properties of the arrangement graph. The following proposition follows directly from the definition of the arrangement graphs.

Proposition 1. *Let n, k be two positive integers with $n, k \geqslant 2$, and let i and j be two distinct elements of $\langle n \rangle$. Suppose that H is one subcomponent of $A_{n,k}^{j}$ with the $(k-1)$th position being h and the kth position being j for some $h \in \langle n \rangle - \{j\}$. Then $|E^{i,j}| = \frac{(n-2)!}{(n-k-1)!}$, and the number of edges between $A_{n,k}^{i}$ and H is $\frac{(n-3)!}{(n-k-1)!}$. Moreover, if (u, v) and (u', v') are distinct edges in $E^{i,j}$, then $\{u, v\} \cap \{u', v'\} = \emptyset$, and $(u, u') \in E(A_{n,k}^{i})$ if and only if $(v, v') \in E(A_{n,k}^{j})$.*

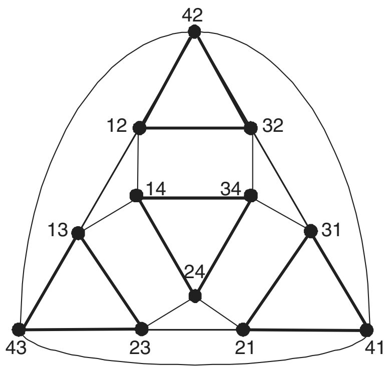

Fig. 1. The arrangement graph $A_{4,2}$.

Let $u \in V(A^i_{n,k})$ for some $i \in \langle n \rangle$. We say that v is a *neighbor* of u if v is adjacent to u. Let I be a subset of $\{1,2,\ldots,n\}$, and we use $N^I(u)$ to denote the set of all neighbors of u which are in $A^I_{n,k}$. Particularly, we use $N^*(u)$ and $N^i(u)$ as an abbreviation of $N^{\langle n \rangle - \{i\}}(u)$ and $N^{\{i\}}(u)$, respectively. We call vertices in $N^*(u)$ the *outer neighbors* of u. It follows from the definitions, $|N^i(u)| = (k-1)(n-k)$ and $|N^*(u)| = (n-k)$. We say that vertex u is adjacent to subcomponent $A^j_{n,k}$ if u has an outer neighbor in $A^j_{n,k}$. Then, we define the *adjacent subcomponent* $AS(u)$ of u as$\{j \mid u$ is adjacent to $A^j_{n,k}\}$. We have the following proposition:

Proposition 2. *Suppose that $k \geqslant 2$, $n-k \geqslant 2$, and $i \in \langle n \rangle$. Let u and v be two distinct vertices in $A^i_{n,k}$.*

(a) *If $d(u,v) = 1$, then $|AS(u) \cap AS(v)| = n-k-1$.*
(b) *If $d(u,v) \leqslant 2$, then $AS(u) \neq AS(v)$.*

Proof. Let $u = u_1u_2\ldots u_k$, $v = v_1v_2\ldots v_k$, and $u_k = v_k = i$. If $d(u,v) = 1$, we have $u_s \neq v_s$ for some $s \in \langle k-1 \rangle$, and $u_t = v_t$ for all $t \neq s$. Then, $AS(u) = \langle n \rangle - \{u_1,u_2,\ldots,u_s,\ldots,u_k\}$ and $AS(v) = \langle n \rangle - \{v_1,v_2,\ldots,v_s,\ldots,v_k\}$. Thus $AS(u) \cap AS(v) = \langle n \rangle - \{u_1,u_2,\ldots,u_s,\ldots,u_k,v_s\}$ and $|AS(u) \cap AS(v)| = n-(k+1) = n-k-1$. Since $u_s \neq v_s$, $v_s \in AS(u)$ but $v_s \notin AS(v)$.

If $d(u,v) = 2$, there exists a vertex $w \in V(A^i_{n,k})$ such that $d(u,w) = d(w,v) = 1$. Let $w = w_1,w_2,\ldots,w_k$. And, let s' and t' be two indices such that $w_{s'} \neq u_{s'}$ and $v_{t'} \neq w_{t'}$. Clearly, $s' \neq t'$ or $d(u,v) = 1$. Hence $w_{s'}$ is not in $\{u_1,u_2,\ldots,u_k\}$ but in $\{v_1,v_2,\ldots,v_k\}$. Thus $w_{s'} \in AS(u)$ but $w_{s'} \notin AS(v)$. Hence, the statement follows. □

Hsu et al. studied the fault hamiltonicity and fault hamiltonian connectivity of the arrangement graphs in [13]. Some results are listed as follows.

Theorem 1 [13]. *Let n and k be two positive integers with $n-k \geqslant 2$. Then $A_{n,k}$ is $k(n-k)-2$ fault tolerant hamiltonian and $k(n-k)-3$ fault tolerant hamiltonian connected.*

The above theorem states that with up to $k(n-k)-2$ faulty edges and faulty vertices $A_{n,k}$ still has a hamiltonian cycle, and with up to $k(n-k)-3$ faulty edges and faulty vertices $A_{n,k}$ is still hamiltonian connected.

Lemma 1 [13]. *Suppose that*

1. *$k \geqslant 3$ and $n-k \geqslant 2$,*
2. *t is a fixed position with $1 \leqslant t \leqslant k$,*
3. *$I \subseteq \langle n \rangle$ with $|I| \geqslant 2$,*
4. *$F \subseteq V(A_{n,k}) \cup E(A_{n,k})$, and*
5. *$A^{(t:l)}_{n,k} - F$ is hamiltonian connected for each $l \in I$ and $|F(A^{(t:I)}_{n,k})| \leqslant k(n-k)-3$.*

Then, for any $x \in V(A^{(t:i)}_{n,k})$ and $y \in V(A^{(t:j)}_{n,k})$ with $i \neq j \in I$, there is a hamiltonian path of $A^{(t:I)}_{n,k} - F$ joining x and y.

The following lemma considers the hamiltonian connectivity of the incomplete arrangement graphs $A_{n,2}$. The lemma states that for any two vertices x and y in different subcomponents of the incomplete arrangement graphs $A_{n,2}$, there exists a hamiltonian path joining them if $n \geqslant 5$. The result holds even when there is one faulty vertex or one faulty edge if $n \geqslant 6$.

Lemma 2. *Suppose that $n \geqslant 5$, t is a fixed position with $1 \leqslant t \leqslant 2$, $F \subseteq V(A_{n,2})$, and $I \subseteq \langle n \rangle$ with $|I| \geqslant 2$.*

(a) *If $n \geqslant 5$, then for any $x \in V(A^{(t:i)}_{n,2})$ and $y \in V(A^{(t:j)}_{n,2})$ with $i \neq j \in I$, there is a hamiltonian path of $A^{(t:I)}_{n,2}$ joining x and y.*
(b) *If $n \geqslant 6$ and $|F| \leqslant 1$, then for any $x \in V(A^{(t:i)}_{n,2})$ and $y \in V(A^{(t:j)}_{n,2})$ with $i \neq j \in I$, there is a hamiltonian path of $A^{(t:I)}_{n,2} - F$ joining x and y.*

Proof. Because of the symmetric property of $A_{n,2}$, without loss of generality, we may assume that $t = 2$. By Proposition 1, $|E^{i,j}| = \frac{(n-2)!}{(n-2-1)!} = n-2 \geqslant 3$ if $n \geqslant 5$, and $n-2 \geqslant 4$ if $n \geqslant 6$ for every $i,j \in I$, and $\{u,v\} \bigcap \{u',v'\} = \emptyset$ if (u,v) and (u',v') are distinct edges in $E^{i,j}$. Hence the number of good edge $|GE^{i,j}| \geqslant 3$

if $n \geqslant 5$, or $n \geqslant 6$ with $|F| \leqslant 1$. We then prove this lemma by induction on $|I|$. Suppose that $|I| = 2$, and $I = \{i,j\}$ for some i, j. Since $|GE^{i,j}| \geqslant 3$, there exists an edge $(u,v) \in GE^{i,j}$ such that $u \neq x \in V(A^i_{n,2})$ and $v \neq y \in V(A^j_{n,2})$. By Theorem 1, for each $l \in I$, $A^l_{n,2} - F$ is hamiltonian connected if $|F| \leqslant 1$. There is a hamiltonian path P_1 of $A^i_{n,2} - F$ from x to u and a hamiltonian path P_2 of $A^j_{n,2} - F$ from v to y. Thus $\langle x, P_1, u, v, P_2, y\rangle$ forms a hamiltonian path of $A^I_{n,2} - F$ from x to y.

Assume that the statement is true for all I' with $2 \leqslant |I'| < |I|$. There exists an $i' \in I$ with $i' \neq i,j$. Since $|GE^{i',j}| \geqslant 3$, we can find an edge $(u,v) \in GE^{i',j}$ with$u \in V(A^{i'}_{n,2})$ and $v \neq y \in V(A^j_{n,2})$. Then there is a hamiltonian path P_1 of $A^{I-\{j\}}_{n,2} - F$ from x to u and a hamiltonian path P_2 of $A^j_{n,2} - F$ from v to y. Thus $\langle x, P_1, u, v, P_2, y\rangle$ forms a hamiltonian path of $A^I_{n,2} - F$ from x to y. Hence the lemma follows. □

3. Panpositionable hamiltonicity of $A_{n,1}$ and $A_{n,2}$

We shall prove that the arrangement graph $A_{n,k}$ is panpositionable hamiltonian for all $k \geqslant 1$ and $n - k \geqslant 2$. The basic idea is to study $A_{n,1}$ and $A_{n,2}$ first, and then to prove the general case by induction on k.

Lemma 3. *The arrangement graph $A_{n,1}$ is panconnected and panpositionable hamiltonian for all $n \geqslant 3$.*

Proof. Since $A_{n,1}$ is isomorphic to the complete graph K_n, the lemma follows trivially. □

Lemma 4. *The arrangement graph $A_{n,2}$ is panpositionable hamiltonian for all $n \geqslant 4$.*

Proof. Chiang and Chen [22] showed that the $A_{n,n-2}$ is isomorphic to the n-alternating group graph AG_n, and the panpositionable hamiltonian property of AG_n, which $n \geqslant 4$, has been shown in [21]. Hence the result holds for $n = 4$. Alternatively, we can verify this case, $A_{4,2}$, by brute force. Suppose that $n \geqslant 5$, and s and t are two distinct vertices of $A_{n,2}$. Then for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{|V(A_{n,2})|}{2}\}$, we shall find a hamiltonian cycle of $A_{n,2}$ such that the distance between s and t on the cycle is l.

We would like to make a remark here. Throughout the paper, the proof idea of the panpositionable hamiltonian property of the arrangement graph is essentially similar to Case 1 described below except for some minor adjustments.

Case 1

Suppose that s and t belong to the same subcomponent $A^i_{n,2}$. See Fig. 2. We assume that $s,t \in V(A^i_{n,2})$ for some $i \in \langle n\rangle$. Since $A^i_{n,2}$ is isomorphic to the complete graph K_{n-1}, we have $d(s,t) = 1$. For each $l_0 \in \{1,2,3,\ldots,n-2\}$, we can construct a hamiltonian cycle HC_i of $A^i_{n,2}$ such that the distance between s and t on the cycle is l_0. Node t has two distinct neighbors on cycle HC_i. Let u and v be two neighbors of t on HC_i. Let $\mathrm{HC}_i = \langle s, LP, u, t, v, RP, s\rangle$ and $P_0 = \langle s, LP, u, t\rangle$. Without loss of generality, let $L(P_0) = l_0$. Since $|N^*(t)| = n-2 \geqslant 3$ for $n \geqslant 5$, we can find a subcomponent $A^{h_t}_{n,2}$ different from $A^i_{n,2}$, and a vertex $t' \in V(A^{h_t}_{n,2})$ such that $(t,t') \in E^{i,h_t}$ for some $h_t \in \langle n\rangle - \{i\}$. By Proposition 2, $d(t,u) = 1$, hence we have $|AS(t) \cap AS(u)| = n-3 \geqslant 2$ for $n \geqslant 5$. It means that we can find a subcomponent $A^{j_1}_{n,2}$ which $j_1 \in \langle n\rangle - \{i, h_t\}$, such that there exist two disjoint edges (u,p_1) and (t,q_1) in E^{i,j_1}. By Proposition 1, $(p_1,q_1) \in E(A^{j_1}_{n,2})$. Since $|N^*(v)| = n-2 \geqslant 3$ for $n \geqslant 5$, we can find a subcomponent $A^{h_v}_{n,2}$, and a vertex $v' \in V(A^{h_v}_{n,2})$ such that $(v,v') \in E^{i,h_v}$ for some $h_v \in \langle n\rangle - \{i, h_t, j_1\}$. By Lemma 2(a), there exists a hamiltonian path HP of $A^{\langle n\rangle-\{i\}}_{n,2}$ joining t' and v'. Thus $\langle s, P_0, t, t', \mathrm{HP}, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{1,2,3,\ldots,n-2\}$, the distance between s and t on the cycle is l_0.

Now we present an algorithm to expand the path $P_0 = \langle s, LP, u, t\rangle$ between s and t to various lengths. The idea is to expand the path by inserting the vertices of $A^{j_1}_{n,2}$ into P_0. We now describe the details.

If we want to insert p_1 and q_1 into P_0, let $P_1 = \langle s, LP, u, p_1, q_1, t\rangle$. See Fig. 3a for an illustration. Thus we have $L(P_1) = l_0 + 2$. We can expand the path P_1 to a longer path as follows. By Theorem 1, there is a hamiltonian path HP_1 from p_1 to q_1 in $A^{j_1}_{n,2}$. So we can join all the vertices of $A^{j_1}_{n,k}$ to P_1, let $P_1^* = \langle s, LP, u, p_1, \mathrm{HP}_1, q_1, t\rangle$. Hence $L(P_1^*) = l_0 + n - 1$. Since $1 \leqslant l_0 \leqslant n-2$, we have $3 \leqslant L(P_1) \leqslant n$ and $n \leqslant L(P_1^*) \leqslant 2n-3$. Therefore, for each $l_1 \in \{1,2,3,\ldots,2n-3\}$, we can construct a path $PP_1 \in \{P_0, P_1, P_1^*\}$ from s to t such that the distance between s and t on the path is l_1.

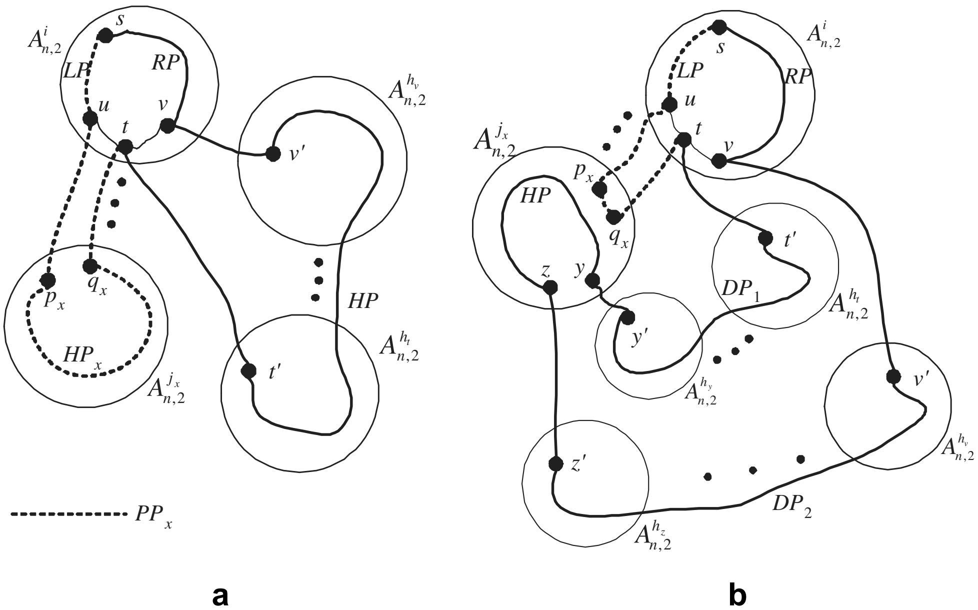

Fig. 2. Lemma 4, Case 1.

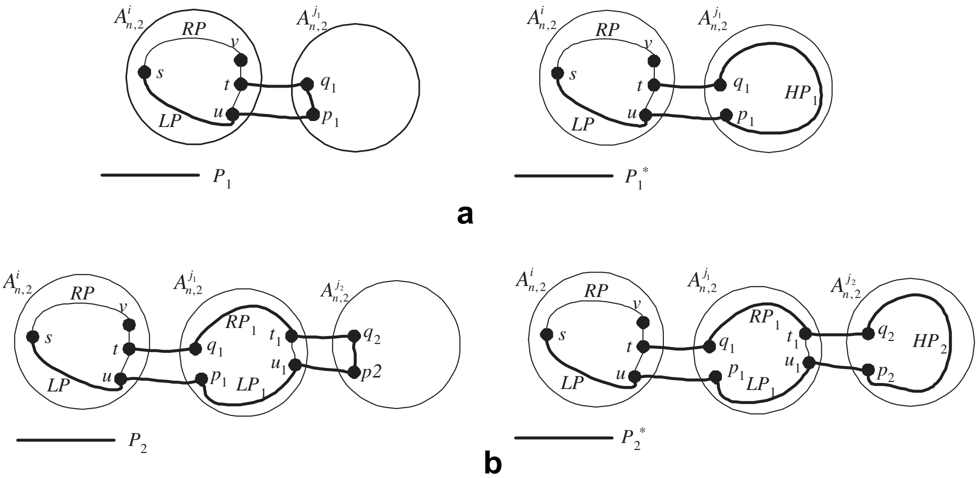

Fig. 3. The paths P_1, P_1^*, P_2, and P_2^*.

Using the same idea, we can expand the path HP_1. Let u_1 and t_1 be two adjacent vertices on HP_1. That is, $\mathrm{HP}_1 = \langle p_1, LP_1, u_1, t_1, RP_1, q_1\rangle$. By Propositions 1 and 2, there exist two distinct edges (u_1, p_2) and (t_1, q_2) in E^{j_1,j_2} for some $j_2 \in \langle n\rangle - \{i, h_t, h_v, j_1\}$ such that $(p_2, q_2) \in E(A_{n,2}^{j_2})$. See Fig. 3b for an illustration. Let $P_2 = \langle s, LP, u, p_1, LP_1, u_1, p_2, q_2, t_1, RP_1, q_1, t\rangle$. Thus we have $L(P_2) = l_0 + n + 1$. By Theorem 1, there is a hamiltonian path HP_2 from p_2 to q_2 in $A_{n,2}^{j_2}$. Let $P_2^* = \langle s, LP, u, p_1, LP_1, u_1, p_2, \mathrm{HP}_2, q_2, t_1, RP_1, q_1, t\rangle$. Hence we have $L(P_2^*) = l_0 + 2n - 2$. Since $1 \leqslant l_0 \leqslant n - 2$, we have $n + 2 \leqslant L(P_2) \leqslant 2n - 1$ and $2n - 1 \leqslant L(P_2^*) \leqslant 3n - 4$. Therefore, for each $l_2 \in \{1, 2, 3, \ldots, 3n - 4\}$, we can construct a path $PP_2 \in \{P_0, P_1, P_1^*, P_2, P_2^*\}$ from s to t such that the distance between s and t on the path is l_2 if $n \geqslant 5$. The maximal value of l_2 is $3n - 4$. If $n = 5$, then we have $3n - 4 \geqslant \frac{|V(A_{n,2})|}{2} = \frac{n(n-1)}{2}$.

We can use the algorithm repeatedly for $n \geqslant 6$. For each $3 \leqslant x \leqslant \lfloor \frac{n}{2} \rfloor$, let u_{x-1} and t_{x-1} be the two adjacent vertices on HP_{x-1}. That is, $\mathrm{HP}_{x-1} = \langle p_{x-1}, LP_{x-1}, u_{x-1}, t_{x-1}, RP_{x-1}, q_{x-1}\rangle$. By Propositions 1 and 2, there exist two distinct edges (u_{x-1}, p_x) and (t_{x-1}, q_x) in E^{j_{x-1}, j_x} for some $j_x \in \langle n \rangle - \{i, h_t, h_v, j_1, \ldots, j_{x-1}\}$ such that $(p_x, q_x) \in E(A^{j_x}_{n,2})$. Let $P_x = \langle s, LP, u, p_1, LP_1, \ u_1, \ldots, u_{x-1}, p_x, \ q_x, t_{x-1}, \ldots, t_1, RP_1, q_1, t \rangle$. Thus we have $L(P_x) = l_0 + (x-1)(n-1) + 2$. By Theorem 1, there is a hamiltonian path HP_x from p_x to q_x in $A^{j_x}_{n,2}$. Let $P^*_x = \langle s, LP, u, p_1, LP_1, u_1, \ldots, u_x - 1, p_x, \mathrm{HP}_x, q_x, \ \ t_{x-1}, \ldots, t_1, RP_1, q_1, t \rangle$. Hence we have $L(P^*_x) = l_0 + (x-1)(n-1) + n - 1$. Since $1 \leqslant l_0 \leqslant n-2$, we have $(x-1)(n-1) + 3 \leqslant L(P_x) \leqslant (x-1)(n-1) + n$ and $(x-1)(n-1) + n \leqslant L(P^*_x) \leqslant (x-1)(n-1) + 2n - 3$. Therefore, for each $l_x \in \{1, 2, 3, \ldots, (x-1)(n-1) + 2n - 3\}$, we can construct a path $PP_x \in \{P_0, P_1, P^*_1, \ldots, P_x, P^*_x\}$ from s to t such that the distance between s and t on the path is l_x if $n \geqslant 6$. The maximal value of l_x is $(\lfloor \frac{n}{2} \rfloor - 1)(n-1) + 2n - 3$, and $(\lfloor \frac{n}{2} \rfloor - 1)(n-1) + 2n - 3 \geqslant \frac{|V(A_{n,2})|}{2} = \frac{n(n-1)}{2}$. To construct a hamiltonian cycle, we consider the following two subcases.

Subcase 1.1: Consider the case $PP_x \in \{P_0, P^*_1, \ldots, P^*_x\}$ for each $1 \leqslant x \leqslant \lfloor \frac{n}{2} \rfloor$. See Fig. 2a for an illustration. By Lemma 2(a), there exists a hamiltonian path HP of $A^{\langle n \rangle - \{i, j_1, \ldots, j_x\}}_{n,2}$ joining t' and v' which $t' \in V(A^{h_t}_{n,2})$ and $v' \in V(A^{h_v}_{n,2})$. Thus $\langle s, PP_x, t, t', \mathrm{HP}, v', v, RP, s \rangle$ forms a hamiltonian cycle, and for each $l \in \{1, 2, 3, \ldots, \frac{|V(A_{n,2})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 1.2: Consider the case $PP_x \in \{P_1, \ldots, P_x\}$ for each $1 \leqslant x \leqslant \lfloor \frac{n}{2} \rfloor$. See Fig. 2b for an illustration. Assume that $H_1, H_2 \in \langle n \rangle - \{i, j_1, \ldots, j_x\}$ and $H_1 \cap H_2 = \emptyset$. Let $h_t, h_y \in H_1$ and $h_v, h_z \in H_2$. Let $F \subseteq V(A^{j_x}_{n,2})$ and $F = \{p_x, q_x\}$. Let y, z be two distinct vertices in $A^{j_x}_{n,2} - F$. Since $|N^*(y)| = |N^*(z)| = n - 2 \geqslant \lceil \frac{n}{2} \rceil$ for $n \geqslant 5$, there exist two distinct edges $(y, y') \in E^{j_x, h_y}$ and $(z, z') \in E^{j_x, h_z}$ such that $y' \neq t' \in V(A^{h_y}_{n,2})$ and $z' \neq v' \in V(A^{h_z}_{n,2})$, respectively. $A^{j_x}_{n,2} - F$ is isomorphic to K_{n-3}, hence there is a hamiltonian path HP from y to z in $A^{j_x}_{n,2} - F$. By Theorem 1 and Lemma 2(a), there exist a hamiltonian path DP_1 from t' to y' in $A^{H_1}_{n,2}$ and a hamiltonian path DP_2 from v' to z' in $A^{H_2}_{n,2}$. Thus $\langle s, PP_x, t, t', DP_1, y', y, \mathrm{HP}, z, z', DP_2, v', v, RP, s \rangle$ forms a hamiltonian cycle, and for each $l \in \{1, 2, 3, \ldots, \frac{|V(A_{n,2})|}{2}\}$, the distance between s and t on the cycle is l.

Case 2

Suppose that s and t belong to different subcomponents of $A_{n,2}$. We assume that $s \in V(A^i_{n,2})$ and $t \in V(A^{h_t}_{n,2})$ for $i \neq h_t \in \langle n \rangle$. Each subcomponent of $A_{n,2}$ is isomorphic to the complete graph K_{n-1}, and $|E^{i,h_t}| > 0$, we have $d(s,t) = 1$, $d(s,t) = 2$ or $d(s,t) = 3$. In the case of $d(s,t) = 1$, suppose that $s = s_1 s_2 \ldots s_{k-1} i$ and $t = t_1 t_2 \ldots t_{k-1} h_t$ are adjacent, and $s_x = t_x$ for each $1 \leqslant x \leqslant k-1$. We may decompose $A_{n,2}$ into subcomponents according to the first position such that s and t belong to the same subcomponent. Hence the case for $d(s,t) = 1$ is the same as Case 1. In the following, we discuss the other two cases.

Subcase 2.1: Suppose that $d(s,t) = 2$. See Fig. 4 for an illustration. Without loss of generality, let (t'', t) be an edge in E^{i,h_t} such that $t' \in V(A^i_{n,2})$ and $t' \in N^*(t)$. Since $A^i_{n,2}$ is isomorphic to complete graph K_{n-1}, we have $d(s,t') = 1$. For each $l_0 \in \{1, 2, 3, \ldots, n-2\}$, we can construct a hamiltonian cycle HC_i of $A^i_{n,2}$ such that the

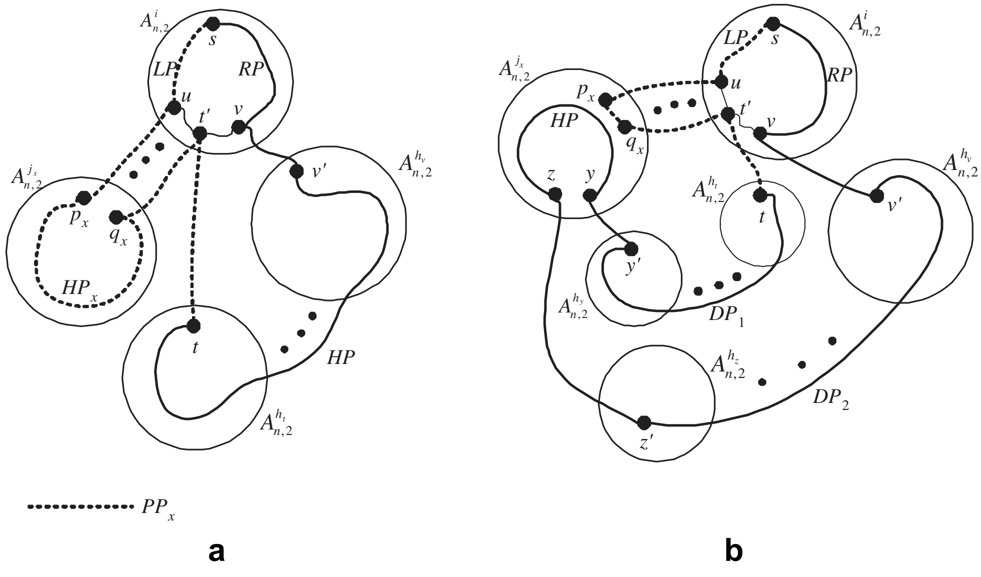

Fig. 4. Lemma 4, Case 2.1.

distance between s and t' on the cycle is l_0. Let u and v be two neighbors of t' on HC_i, and $\mathrm{HC}_i = \langle s, LP, u, t', v, RP, s\rangle$. Let $P_0 = \langle s, LP, u, t', t\rangle$. Without loss of generality, we may assume that $L(P_0) = l_0 + 1$.

By Proposition 2, $d(t',u) = 1$, so we have $|AS(t') \cap AS(u)| = n-3 \geqslant 2$ if $n \geqslant 5$. This means that we can find an index $j_1 \in \langle n\rangle - \{i, h_t\}$, such that there exist two disjoint edges (u, p_1) and (t', q_1) in E^{i,j_1}. By Proposition 1, $(p_1, q_1) \in E(A^{j_1}_{n,2})$. Since $|N^*(v)| = n-2 \geqslant 3$ if $n \geqslant 5$, we can find a vertex $v\prime \in V(A^{h_v}_{n,2})$ such that $(v, v') \in E^{i,h_v}$ for some $h_v \in \langle n\rangle - \{i, h_t, j_1\}$. If we want to join p_1 and q_1 to P_0, let $P_1 = \langle s, LP, u, p_1, q_1, t', t\rangle$. Then we have $L(P_1) = l_0 + 3$. By Theorem 1, there is a hamiltonian path HP_1 from p_1 to q_1 in $A^{j_1}_{n,2}$. Let $P^*_1 = \langle s, LP, u, p_1, \mathrm{HP}_1, q_1, t', t\rangle$. Hence we have $L(P^*_1) = l_0 + n$. Since $1 \leqslant l_0 \leqslant n-2$, we have $4 \leqslant L(P_1) \leqslant n+1$ and $n+1 \leqslant L(P^*_1) \leqslant 2n-2$. Therefore, for each $l_1 \in \{2,3,4,\ldots,2n-2\}$, we can construct a path $PP_1 \in \{P_0, P_1, P^*_1\}$ from s to t such that the distance between s and t on the path is l_1.

Recursively, for each $2 \leqslant x \leqslant \lfloor \frac{n}{2} \rfloor$, let u_{x-1} and t_{x-1} be two adjacent vertices on HP_{x-1}. That is, $\mathrm{HP}_{x-1} = \langle p_{x-1}, LP_{x-1}, u_{x-1}, t_{x-1}, RP_{x-1}, q_{x-1}\rangle$. By Propositions 1 and 2, there exist two distinct edges (u_{x-1}, p_x) and (t_{x-1}, q_x) in E^{j_{x-1},j_x} for some $j_x \in \langle n\rangle - \{i, h_t, h_v, j_1, \ldots, j_{x-1}\}$. And, $(p_x, q_x) \in E(A^{j_x}_{n,2})$. Let $P_x = \langle s, LP, u, p_1, LP_1, u_1, \ldots, u_{x-1}, p_x, q_x, t_{x-1}, \ldots, t_1, RP_1, q_1, t\prime, t\rangle$. Thus we have $L(P_x) = l_0 + (x-1)(n-1) + 3$. By Theorem 1, there is a hamiltonian path HP_x from p_x to q_x in $A^{j_x}_{n,2}$. Let $P^*_x = \langle s, LP, u, p_1, LP_1, u_1, \ldots, u_{x-1}, p_x, \mathrm{HP}_x, q_x, t_{x-1}, \ldots, t_1, RP_1, q_1, t', t\rangle$. Hence we have $L(P^*_x) = l_0 + (x-1)(n-1) + n$. Since $1 \leqslant l_0 \leqslant n-2$, we have $(x-1)(n-1) + 4 \leqslant L(P_x) \leqslant (x-1)(n-1) + n + 1$ and $(x-1)(n-1) + n + 1 \leqslant L(P^*_x) \leqslant (x-1)(n-1) + 2n - 2$. Therefore, for each $l_x \in \{2,3,4,\ldots,(x-1)(n-1)+2n-2\}$, we can construct a path $PP_x \in \{P_0, P_1, P^*_1, \ldots, P_x, P^*_x\}$ from s to t such that the distance between s and t on the path is l_x if $n \geqslant 5$. The maximal value of l_x is $(\lfloor \frac{n}{2} \rfloor - 1)(n-1) + 2n - 2$, and $(\lfloor \frac{n}{2} \rfloor - 1)(n-1) + 2n - 2 \geqslant \frac{|V(A_{n,2})|}{2} = \frac{n(n-1)}{2}$. To construct a hamiltonian cycle, we consider the following two subcases:

Subcase 2.1.1: Consider the case $PP_x \in \{P_0, P^*_1, \ldots, P^*_x\}$ for each $1 \leqslant x \leqslant \lfloor \frac{n}{2} \rfloor$. See Fig. 4a for an illustration. By Lemma 2(a), there exists a hamiltonian path HP of $A^{\langle n\rangle - \{i,j_1,\ldots,j_x\}}_{n,2}$ joining t and v'. Thus $\langle s, PP_x, t', t, \mathrm{HP}, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{2,3,4,\ldots,\frac{|V(A_{n,2})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 2.1.2: Consider the case $PP_x \in \{P_1, \ldots, P_x\}$ for each $1 \leqslant x \leqslant \lfloor \frac{n}{2} \rfloor$. See Fig. 4b for an illustration. Assume that $H_1, H_2 \subseteq \langle n\rangle - \{i, j_1, \ldots, j_x\}$ and $H_1 \cap H_2 = \emptyset$. Let $h_t, h_y \in H_1$ and $h_v, h_z \in H_2$. Let $F \subseteq V(A^{j_x}_{n,2})$ and $F = \{p_x, q_x\}$. Let y and z be two distinct vertices in $A^{j_x}_{n,2} - F$. Since $|N^*(y)| = |N^*(z)| = n-2 \geqslant \lceil \frac{n}{2} \rceil$ for $n \geqslant 5$, there exist two distinct edges $(y, y') \in E^{j_x,h_y}$ and $(z, z') \in E^{j_x,h_z}$ such that $y' \neq t \in V(A^{h_y}_{n,2})$ and $z' \neq v' \in V(A^{h_z}_{n,2})$, respectively. Since $A^{j_x}_{n,2} - F$ is isomorphic to K_{n-3}, there is a hamiltonian path HP from y to z in $A^{j_x}_{n,2} - F$. By Theorem 1 and Lemma 2(a), there exist a hamiltonian path DP_1 from t to y' in $A^{H_1}_{n,2}$ and a hamiltonian path DP_2 from v' to z' in $A^{H_2}_{n,2}$. Thus $\langle s, PP_x, t', t, DP_1, y', y, \mathrm{HP}, z, z', DP_2, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{2,3,4,\ldots,\frac{|V(A_{n,2})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 2.2: Suppose that $d(s,t) = 3$ and $n \geqslant 6$. See Fig. 5 for an illustration. We shall discuss the subcase $d(s,t) = 3$ and $n = 5$ later in Subcase 2.3. Let (t', t'') be an edge in E^{i,h_t} such that $t' \in V(A^i_{n,2})$, $t'' \in V(A^{h_t}_{n,2})$, $t'' \in N(t)$, and $t'' \in N^*(t')$. Since $A^i_{n,2}$ is isomorphic to complete graph K_{n-1}, we have $d(s,t') = 1$. For each $l_0 \in \{1,2,3,\ldots,n-2\}$, we can construct a hamiltonian cycle HC_i of $A^i_{n,2}$ such that the distance between s and t' on the cycle is l_0. Suppose that u and v are two distinct vertices in $V(A^i_{n,2})$, and u and v are two neighbors of t' on HC_i. Let $\mathrm{HC}_i = \langle s, LP, u, t', v, RP, s\rangle$. Let $P_0 = \langle s, LP, u, t', t'', t\rangle$. Hence, without loss of generality, we have $L(P_0) = l_0 + 2$.

By Proposition 2, $d(t',u) = 1$, so we have $|AS(t') \cap AS(u)| = n-3 \geqslant 2$ if $n \geqslant 6$. It means that we can find an index $j_1 \in \langle n\rangle - \{i, h_t\}$, such that there exist two disjoint edges (u, p_1) and (t', q_1) in E^{i,j_1}. By Proposition 1, $(p_1, q_1) \in E(A^{j_1}_{n,2})$. Since $|N^*(v)| = n-2 \geqslant 3$ if $n \geqslant 5$, we can find a vertex $v' \in V(A^{h_v}_{n,2})$ such that $(v, v') \in E^{i,h_v}$ for some $h_v \in \langle n\rangle - \{i, h_t, j_1\}$. If we want to join p_1 and q_1 to P_0, let $P_1 = \langle s, LP, u, p_1, q_1, t', t'', t\rangle$. Thus we have $L(P_1) = l_0 + 4$. By Theorem 1, there is a hamiltonian path HP_1 from p_1 to q_1 in $A^{j_1}_{n,2}$. Let $P^*_1 = \langle s, LP, u, p_1, \mathrm{HP}_1, q_1, t', t'', t\rangle$. Hence we have $L(P^*_1) = l_0 + n + 1$. Since $1 \leqslant l_0 \leqslant n-2$, we have $5 \leqslant L(P_1) \leqslant n+2$ and $n+2 \leqslant L(P^*_1) \leqslant 2n-1$. Therefore, for each $l_1 \in \{3,4,5,\ldots,2n-1\}$, we can construct a path $PP_1 \in \{P_0, P_1, P^*_1\}$ from s to t such that the distance between s and t on the path is l_1.

Similarly, for each $2 \leqslant x \leqslant \lfloor \frac{n}{2} \rfloor$, let u_{x-1} and t_{x-1} be the two adjacent vertices on HP_{x-1}. That is, $\mathrm{HP}_{x-1} = \langle p_{x-1}, LP_{x-1}, u_{x-1}, t_{x-1}, RP_{x-1}, q_{x-1}\rangle$. By Propositions 1 and 2, there exist two distinct edges (u_{x-1}, p_x)

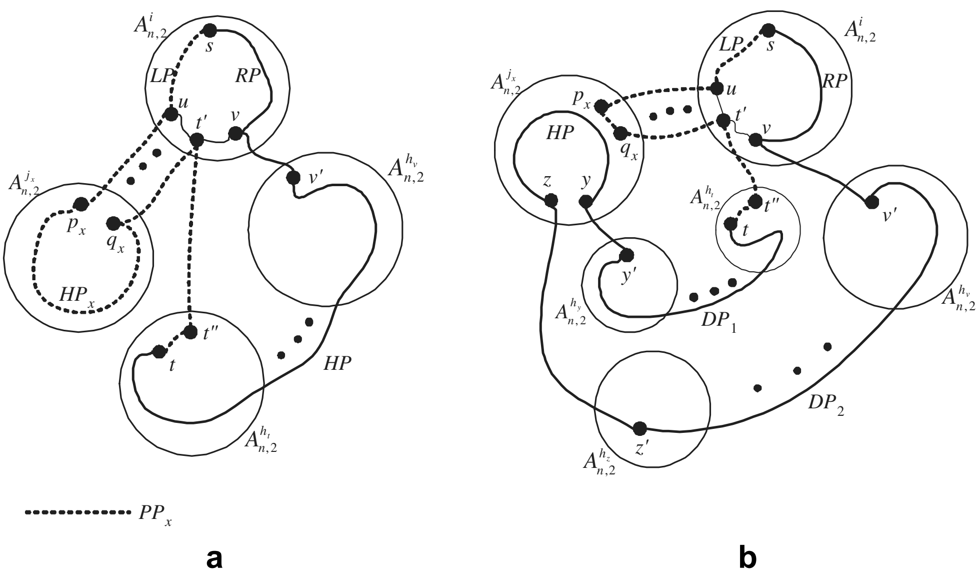

Fig. 5. Lemma 4, Case 2.2.

and (t_{x-1}, q_x) in E^{j_{x-1},j_x} for some $j_x \in \langle n\rangle - \{i, h_t, h_v, j_1, \ldots, j_{x-1}\}$. And, $(p_x, q_x) \in E(A^{j_x}_{n,2})$. Let $P_x = \langle s, LP, u, p_1, LP_1, u_1, \ldots, u_{x-1}, p_x, q_x, t_{x-1}, \ldots, t_1, RP_1, q_1, t', t'', t\rangle$. Thus we have $L(P_x) = l_0 + (x-1)(n-1) + 4$. By Lemma 1, there is a hamiltonian path HP_x from p_x to q_x in $A^{j_x}_{n,2}$. Let $P^*_x = \langle s, LP, u, p_1, LP_1, u_1, \ldots, u_{x-1}, p_x, \text{HP}_x, q_x, t_{x-1}, \ldots, t_1, RP_1, q_1, t', t'', t\rangle$. Hence we have $L(P^*_x) = l_0 + (x-1)(n-1) + n + 1$. Since $1 \leqslant l_0 \leqslant n-2$, we have $(x-1)(n-1) + 5 \leqslant L(P_x) \leqslant (x-1)(n-1) + n + 2$ and $(x-1)(n-1) + n + 2 \leqslant L(P^*_x) \leqslant (x-1)(n-1) + 2n - 1$. Therefore, for each $l_x \in \{3, 4, 5, \ldots, (x-1)(n-1) + 2n - 1\}$, we can construct a path $PP_x \in \{P_0, P_1, P^*_1, \ldots, P_x, P^*_x\}$ from s to t such that the distance between s and t on the path is l_x if $n \geqslant 5$. The maximal value of l_x is $(\lfloor \frac{n}{2} \rfloor - 1)(n-1) + 2n - 1$, and $(\lfloor \frac{n}{2} \rfloor - 1)(n-1) + 2n - 1 \geqslant \frac{|V(A_{n,2})|}{2} = \frac{n(n-1)}{2}$. To construct a hamiltonian cycle, we consider the following two subcases:

Subcase 2.2.1: Consider the case $PP_x \in \{P_0, P^*_1, \ldots, P^*_x\}$ for each $1 \leqslant x \leqslant \lfloor \frac{n}{2} \rfloor$. See Fig. 5a for an illustration. Let $F_t \subseteq V(A^{h_t}_{n,2})$ and $F_t = \{t'\}$. By Lemma 2(b), there exists a hamiltonian path HP of $A^{\langle n\rangle - \{i, j_1, \ldots, j_x\}}_{n,2} - F_t$ joining t and v'. Thus $\langle s, PP_x, t', t'', t, \text{HP}, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{3, 4, 5, \ldots, \frac{|V(A_{n,2})|}{2}\}$, the distance between s and t on the cycle is l.

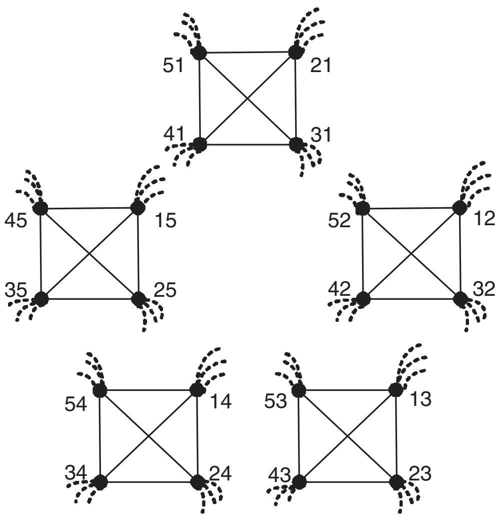

Fig. 6. The arrangement graph $A_{5,2}$.

Subcase 2.2.2: Consider the case $PP_x \in \{P_1, \ldots, P_x\}$ for each $1 \leqslant x \leqslant \lfloor \frac{n}{2} \rfloor$. See Fig. 5b for an illustration. Assume that $H_1, H_2 \in \langle n \rangle - \{i, j_1, \ldots, j_x\}$ and $H_1 \cap H_2 = \emptyset$. Let $h_t, h_y \in H_1$ and $h_v, h_z \in H_2$. Let $F_j \subseteq V(A^{j_x}_{n,2})$ and $F_j = \{p_x, q_x\}$. Let y and z be two distinct vertices in $A^{j_x}_{n,2} - F_j$. Since $|N^*(y)| = |N^*(z)| = n - 2 \geqslant \lceil \frac{n}{2} \rceil$ for $n \geqslant 5$, there exist two distinct edges $(y, y') \in E^{j_x, h_y}$ and $(z, z') \in E^{j_x, h_z}$ such that $y' \neq t, t'' \in V(A^{h_y}_{n,2})$ and $z' \neq v' \in V(A^{h_z}_{n,2})$, respectively. $A^{j_x}_{n,2} - F_j$ is isomorphic to K_{n-3}, hence there is a hamiltonian path HP from y to z in $A^{j_x}_{n,2} - F_j$. By Theorem 1 and Lemma 2(b), there exist a hamiltonian path DP_1 from t to y' in $A^{H_1}_{n,2} - F_t$ and a hamiltonian path DP_2 from v' to z' in $A^{H_2}_{n,2}$. Thus $\langle s, PP_x, t', t'', t, DP_1, y', y, \mathrm{HP}, z, z', DP_2, v', v, RP, s \rangle$ forms a hamiltonian cycle, and for each $l \in \{3, 4, 5, \ldots, \frac{|V(A_{n,2})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 2.3: Suppose that $d(s,t) = 3$ and $n = 5$. Let s and t be two distinct vertices of $A_{5,2}$ in Fig. 6. By the vertex and edge symmetric properties, we may assume that $s = 12$ and $t = 21$ for $d(s,t) = 3$. The corresponding hamiltonian cycle HC in $A_{5,2}$ are listed below.

$d_{\mathrm{HC}}(s,t)$	The cycle HC
3	$\langle 21, 23, 13, 12, 15, 25, 35, 45, 43, 53, 54, 14, 24, 34, 32, 42, 52, 51, 41, 31, 21 \rangle$
4	$\langle 21, 31, 32, 42, 12, 52, 53, 13, 23, 43, 41, 51, 54, 14, 24, 34, 35, 45, 15, 25, 21 \rangle$
5	$\langle 21, 31, 32, 42, 52, 12, 13, 53, 23, 43, 41, 51, 54, 14, 24, 34, 35, 45, 15, 25, 21 \rangle$
6	$\langle 21, 31, 41, 42, 32, 52, 12, 13, 23, 43, 53, 51, 54, 14, 24, 34, 35, 45, 15, 25, 21 \rangle$
7	$\langle 21, 31, 41, 51, 52, 42, 32, 12, 13, 23, 43, 53, 54, 14, 24, 34, 35, 45, 15, 25, 21 \rangle$
8	$\langle 21, 31, 41, 51, 53, 52, 42, 32, 12, 13, 43, 23, 24, 14, 54, 34, 35, 45, 15, 25, 21 \rangle$
9	$\langle 21, 31, 41, 51, 53, 43, 42, 32, 52, 12, 13, 23, 24, 14, 54, 34, 35, 45, 15, 25, 21 \rangle$
10	$\langle 21, 31, 41, 51, 53, 13, 43, 42, 32, 52, 12, 15, 45, 35, 34, 54, 14, 24, 23, 25, 21 \rangle$

Hence the lemma follows. □

4. Panpositionable hamiltonicity and panconnectivity of $A_{n,k}$

In this section, we show that the arrangement graph $A_{n,k}$ is panpositionable hamiltonian for $n - k \geqslant 2$ and $k \geqslant 3$. We need some known results on $A_{n,k}$. It is known that the $A_{n,n-2}$ is isomorphic to the n-alternating group graph AG_n [22], and AG_n is known to be panpositionable hamiltonian for all $n \geqslant 3$ in [21]. Therefore, we have the following result.

Lemma 5. *$A_{n,k}$ is panpositionable hamiltonian if $k \geqslant 1$ and $n - k = 2$.*

Day and Tripathi [2] presented a shortest path routing algorithm for the arrangement graph, and gave some characterizations of the minimum length path between two arbitrary vertices in $A_{n,k}$. We can derive the following lemma directly from their routing algorithm.

Lemma 6. *Let $u = u_1, u_2, \ldots, u_k$ and $v = v_1, v_2, \ldots, v_k$ be two vertices in $A_{n,k}$. There exists a way of decomposing $A_{n,k}$ into subcomponents such that one of the following three cases holds.*

(a) *If $u_x = v_x = i$ for some position $x \in \langle k \rangle$ and $i \in \langle n \rangle$, we decompose $A_{n,k}$ into subcomponents according to the xth position. Then u and v belong to the same subcomponent and $u, v \in V(A^{(x:i)}_{n,k})$. Moreover, a shortest path from u to v in $A_{n,k}$ is completely contained in $A^{(x:i)}_{n,k}$.*

(b) *If $u_x \neq v_x$ for every $x \in \langle k \rangle$ and $\{u_1, u_2, \ldots, u_k\} \neq \{v_1, v_2, \ldots, v_k\}$, there exists a position $u_y \notin \{v_1, v_2, \ldots, v_k\}$ for some $y \in \langle k \rangle$, say the yth position. We decompose $A_{n,k}$ into subcomponents according to the yth position, then u and v belong to different subcomponents, say $u \in V(A^{(y:i)}_{n,k})$ and $v \in V(A^{(y:j)}_{n,k})$ for some $i \neq j \in \langle n \rangle$. Moreover, a minimum length path connecting u and v has the form $\langle u, P, u', v \rangle$, in which $u' \in V(A^{(y:i)}_{n,k})$, and P is a path completely contained in $A^{(y:i)}_{n,k}$.*

(c) *If $u_x \neq v_x$ for every $x \in \langle k \rangle$ and $\{u_1, u_2, \ldots, u_k\} = \{v_1, v_2, \ldots, v_k\}$, decomposing $A_{n,k}$ into subcomponents according to any position, say yth position, $y \in \langle k \rangle$, then u and v belong to different subcomponents, say $u \in V(A^{(y:i)}_{n,k})$ and $v \in V(A^{(y:j)}_{n,k})$ for some $i \neq j \in \langle n \rangle$. Moreover, a minimum length path connecting u and v has the form $\langle u, P, u', v', v \rangle$, in which $u' \in V(A^{(y:i)}_{n,k})$, $v' \in V(A^{(y:j)}_{n,k})$, and P is a path completely contained in $A^{(y:i)}_{n,k}$.*

Example. Suppose that u and v are two vertices in $A_{7,5}$. If $u = 12345$ and $v = 13452$, then $u, v \in V(A_{7,5}^{(1:1)})$. A minimum length path connecting u and v is $\langle 12345, 12\underline{6}45, 13\underline{6}45, 1364\underline{2}, 136\underline{5}2, 13\underline{4}52\rangle$ which is completely contained in $A_{7,5}^{(1:1)}$, and case (a) holds. If $u = 12345$ and $v = 26453$, then $u \in V(A_{7,5}^{(1:1)})$ and $v \in V(A_{7,5}^{(1:2)})$. A minimum length path connecting u and v is $\langle 12345, 1234\underline{6}, 123\underline{5}6, 12\underline{4}56, 1245\underline{3}, 1\underline{6}453, \underline{2}6453\rangle$, and case (b) holds. If $u = 12345$ and $v = 23451$, then $u \in V(A_{7,5}^{(1:1)})$ and $v \in V(A_{7,5}^{(1:2)})$. A minimum length path connecting u and v is $\langle 12345, 1234\underline{6}, 123\underline{5}6, 12\underline{4}56, 1\underline{3}456, \underline{2}3456, 2345\underline{1}\rangle$, and case (c) holds.

We need the following lemma later in our main theorem. One may skip the proof temporarily, and come back to it later.

Lemma 7. *Suppose that*

1. $k \geqslant 3$, $n - k \geqslant 2$,
2. $I \subseteq \langle n\rangle$ *with* $|I| \geqslant 2$,
3. $F \subseteq V(A_{n,k}^I)$ *with* $|F| \leqslant 1$, *and*
4. $x_1 \in V(A_{n,k}^{i_1}) - F$ *and* $x_2 \in V(A_{n,k}^{i_2}) - F$ *with* $i_1 \neq i_2 \in I$.

Then, for any pair of distinct vertices $\{y_1, y_2\}$ *in* $V(A_{n,k}^I) - F$, *there exist two disjoint paths, one joining* x_1 *and* y_i *for some* $i \in \{1, 2\}$, *and the other joining* x_2 *and* y_j *with* $i \neq j$, *such that these two paths span all the vertices in* $A_{n,k}^I - F$.

Proof. Let $i_1, i_2, \ldots, i_{|I|}$ be $|I|$ distinct indices of $\langle n\rangle$. We prove this lemma by finding two disjoint paths P_1 and P_2 in $A_{n,k}^I - F$ such that P_1 joins x_1 and y_i, and P_2 joins x_2 and y_j with $i \neq j$. Moreover, P_1 and P_2 span all the vertices in $A_{n,k}^I - F$. According to the location of y_1 and y_2, we have the following cases:

Case 1

Suppose that y_1 and y_2 are located in different subcomponents.

Subcase 1.1: Suppose that x_1, x_2, y_i and y_j are located in four different subcomponents. $y_i \in V(A_{n,k}^{i_3})$ and $y_j \in V(A_{n,k}^{i_4})$ with $|I| \geqslant 4$. See Fig. 7a for an illustration. By Lemma 1, we can find a hamiltonian path P_1 from x_1 to y_i in $A_{n,k}^{\{i_1,i_3\}} - F$. Similarly, we can find a hamiltonian path P_2 from x_2 to y_j in $A_{n,k}^{I-\{i_1,i_3\}} - F$.Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $A_{n,k}^I - F$.

Subcase 1.2: Suppose that one of y_1, y_2 and one of x_1, x_2 are located in the same subcomponent. Without loss of generality, we may assume that x_1 and y_i are located in the same subcomponent, and x_2 and y_j are located in different subcomponents. $y_i \in V(A_{n,k}^{i_1})$ and $y_j \in V(A_{n,k}^{i_3})$ with $|I| \geqslant 3$. See Fig. 7b for an illustration. By Theorem 1, since $A_{n,k}^{i_1} - F$ is hamiltonian connected, we can find a hamiltonian path P_1 from x_1 to y_i in $A_{n,k}^{i_1} - F$. By Lemma 1, we can find a hamiltonian path P_2 from x_2 to y_j in $A_{n,k}^{I-\{i_1\}} - F$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $A_{n,k}^I - F$.

Subcase 1.3: Suppose that x_1 and y_i are located in the same subcomponent for some $i \in \{1, 2\}$, and x_2 and y_j are located in the same subcomponent with $i \neq j$. $y_i \in V(A_{n,k}^{i_1})$ and $y_j \in V(A_{n,k}^{i_2})$ with $|I| \geqslant 2$. See Fig. 7c for an illustration.Without loss of generality, we may assume that $i = 1$ and $j = 2$. By Theorem 1, since $A_{n,k}^{i_1} - F$ is hamiltonian connected, we can find a hamiltonian path P_1 from y_1 to x_1 in $A_{n,k}^{i_1} - F$. If $|I| \geqslant 3$, since $|N^*(y_2)| > 2$, we can find an edge $(y_2, y_2') \in E^{i_2,j}$ such that $y_2' \in V(A_{n,k}^j)$ for some $j \in I - \{i_1, i_2\}$. By Lemma 1, we can find a hamiltonian path P_2' from y_2' to x_2 in $A_{n,k}^{I-\{i_1\}} - \{y_2\} \cup F$. Let $P_2 = \langle y_2, y_2', P_2', x_2\rangle$. If $|I| = 2$, by Theorem 1, there is a hamiltonian path P_2' from y_2 to b_2 in $A_{n,k}^{i_2} - F$. Let $P_2 = \langle y_2, P_2', x_2\rangle$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $A_{n,k}^I - F$.

Case 2

Suppose that y_i and y_j are located in the same subcomponent.

Subcase 2.1: Suppose that $y_1, y_2 \in V(A_{n,k}^{i_1})$ or $y_1, y_2 \in V(A_{n,k}^{i_2})$ with $|I| \geqslant 2$. See Fig. 7d for an illustration.Without loss of generality, we consider the former case and assume that $i = 1$ and $j = 2$. By Theorem 1, $A_{n,k}^{i_1} - (\{y_2\} \cup F)$ is hamiltonian connected, hence we can find a hamiltonian path P_1 from y_1 to x_1 in $A_{n,k}^{i_1} - \{y_2\} \cup F$. If $|I| \geqslant 3$, since $|N^*(y_2)| > 2$, we can find an edge $(y_2, y_2') \in E^{i_1,j}$ such that $y_2' \in V(A_{n,k}^j)$ for some $j \in I - \{i_1, i_2\}$. By Lemma 1, we can find a hamiltonian path P_2' from y_2' to x_2 in $A_{n,k}^{I-\{i_1\}} - F$. If $|I| = 2$, there exists an edge $(y_2, y_2') \in E^{i_1,i_2}$ such that $y_2' \in V(A_{n,k}^{i_2})$. By Theorem 1, there is a hamiltonian path P_2' from

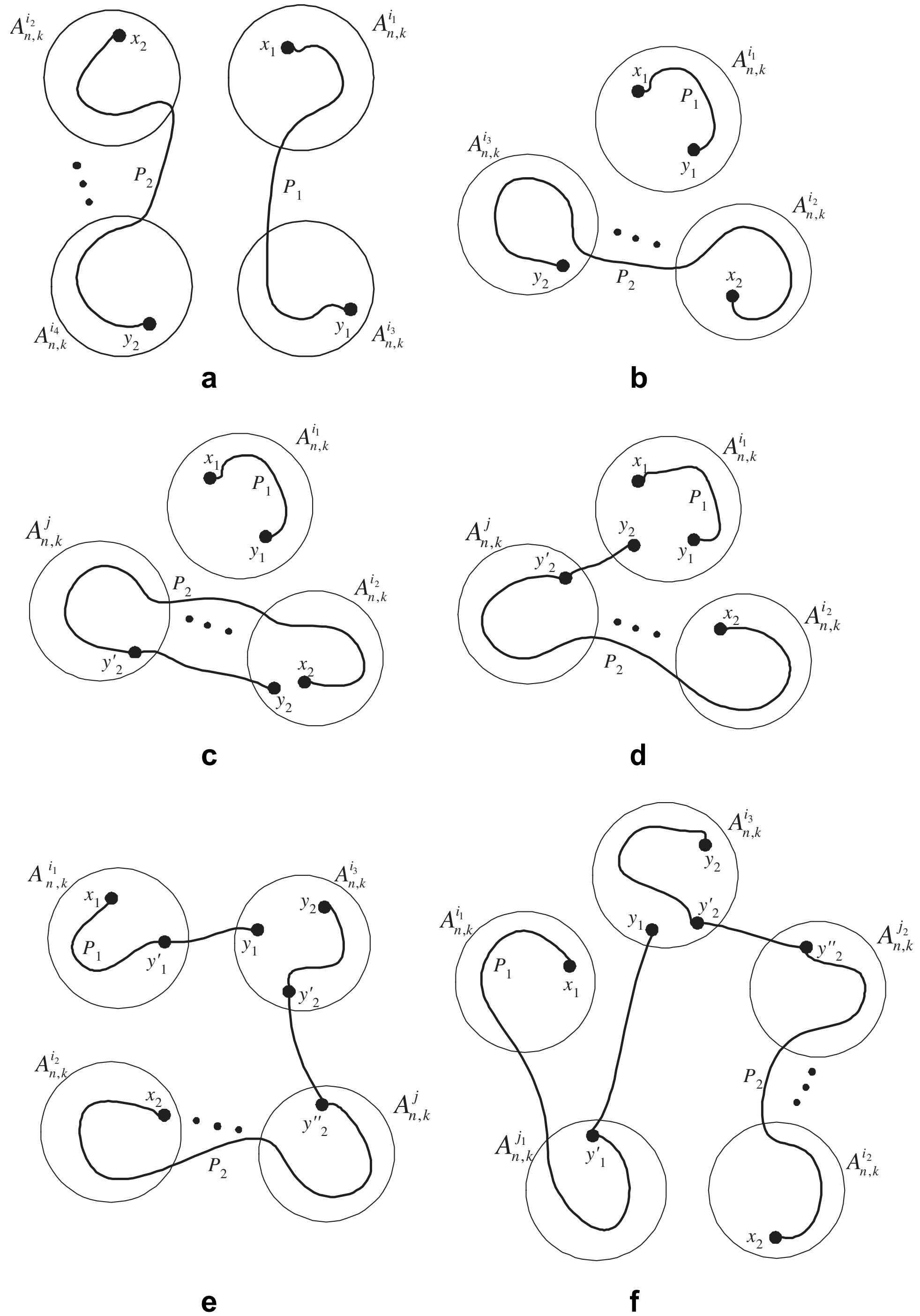

Fig. 7. Illustrations for Lemma 7. Notice that $|F| \leqslant 1$ in each $A_{n,k}^{l}$.

y'_2 to x_2 in $A_{n,k}^{i_2} - F$. Let $P_2 = \langle y_2, y'_2, P'_2, x_2 \rangle$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $A_{n,k}^{I} - F$.

Subcase 2.2: Suppose that $y_1, y_2 \in V(A_{n,k}^{i_3})$. Without loss of generality, we consider the following two subcases:

Subcase 2.2.1: Suppose that there exists some $i_1 \in AS(y_1)$ for $i \in \{1,2\}$ with $|I| \geqslant 3$.Without loss of generality, we may assume that $i = 1$. See Fig. 7e for an illustration. Since $x_1 \in AS(y_1)$, we can find an edge $(y_1, y_1') \in E^{i_1,i_3}$ such that $y_1' \in V(A^{i_1}_{n,k})$ and $x_1 \neq y_1'$. By Theorem 1, we can find a hamiltonian path P_1' from y_1' to x_1 in $A^{i_1}_{n,k} - F$. Let $P_1 = \langle y_1, y_1', P_1', x_1\rangle$. Let $y_2' \neq y_1 \in V(A^{i_3}_{n,k})$. By Theorem 1, since $A^{i_3}_{n,k} - \{y_1\} \cup F$ is hamiltonian connected, we can find a hamiltonian path P_2'' from y_2 to y_2' in $A^{i_3}_{n,k} - \{y_1\} \cup F$. If $|I| \geqslant 4$, since $|N^*(y_2')| > 2$, we can find an edge $(y_2', y_2'') \in E^{i_3,j}$ such that $y_2'' \in V(A^{j}_{n,k})$ for some $j \in I - \{i_1, i_2, i_3\}$. By Lemma 1, we can find a hamiltonian path P_2' from y_2'' to x_2 in $A^{I-\{i_1,i_3\}}_{n,k} - F$. If $|I| = 3$, there exists an edge $(y_2', y_2'') \in E^{i_3,i_2}$ such that $y_2'' \in V(A^{i_2}_{n,k})$.By Theorem 1, there is a hamiltonian path P_2' from y_2'' to x_2 in $A^{i_2}_{n,k} - F$. Let $P_2 = \langle y_2, P_2'', y_2', y_2'', P_2', x_2\rangle$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $A^{I}_{n,k} - F$.

Subcase 2.2.2: Suppose that $\{i_1, i_2\} \bigcap \{AS(y_1) \cup AS(y_2)\} = \emptyset$ with $|I| \geqslant 4$. See Fig. 7(f) for an illustration. Since $|N^*(y_1)| > 2$, we can find an edge $(y_1, y_1') \in E^{i_1,j_1}$ such that $y_1' \in V(A^{j_1}_{n,k})$ for some $j_1 \in I - \{i_1, i_2, i_3\}$. By Lemma 1, we can find a hamiltonian path P_1' from y_1' to x_1 in $A^{\{i_1,j_1\}}_{n,k} - F$. Let $P_1 = \langle y_1, y_1', P_1', x_1\rangle$. Let $y_2' \in V(A^{i_3}_{n,k})$ and $y_2' \in N^{i_3}(y_1)$. By Proposition 2, we have $AS(y_1) \neq AS(y_2')$.By Theorem 1, since $A^{i_3}_{n,k} - \{y_1\} \cup F$ is hamiltonian connected, we can find a hamiltonian path P_2'' from y_2 to y_2' in $A^{i_3}_{n,k} - \{y_1\} \cup F$. If $|I| \geqslant 5$, since $|N^*(y_2')| > 2$, we can find an edge $(y_2', y_2'') \in E^{i_3,j_2}$ such that $y_2'' \in V(A^{j_2}_{n,k})$ for some $j_2 \in I - \{i_1, i_2, i_3, j_1\}$. By Lemma 1, we can find a hamiltonian path P_2' from y_2'' to x_2 in $A^{I-\{i_1,i_3,j_1\}}_{n,k} - F$. If $|I| = 4$, since $|N^*(y_2')| > 2$, we can find an edge $(y_2', y_2'') \in E^{i_3,i_2}$ such that $y_2'' \in V(A^{i_2}_{n,k})$. Since $A^{i_2}_{n,k} - F$ is hamiltonian connected, there is a hamiltonian path P_2' from y_2'' to x_2 in $A^{i_2}_{n,k} - F$. Let $P_2 = \langle y_2, P_2'', y_2', y_2'', P_2', x_2\rangle$. Therefore, P_1 and P_2 are two disjoint paths spanning all the vertices in $A^{I}_{n,k} - F$.

Thus the lemma follows. □

We now prove our main result.

Theorem 2. *The arrangement graph $A_{n,k}$ is panpositionable hamiltonian if $k \geqslant 1$ and $n - k \geqslant 2$.*

Proof. By Lemma 5, $A_{n,k}$ is panpositionable hamiltonian if $k \geqslant 1$ and $n - k = 2$. Hence we consider the case that $n - k > 2$ in our proof. We prove this theorem by induction on k. By Lemma 3, $A_{n,1}$ is panpositionable hamiltonian for all $n > 3$. By Lemma 4, $A_{n,2}$ is panpositionable hamiltonian for all $n > 4$. Suppose that the result holds for $A_{n,k-1}$ for some $k \geqslant 3$ and for all $n - (k-1) > 2$. Consider $A_{n,k}$ for $n - k > 2$, we observe that $A_{n,k}$ can be recursively constructed from n copies of $A_{n-1,k-1}$, and each $A_{n-1,k-1}$ is panpositionable hamiltonian by the inductive hypothesis, since $(n-1) - (k-1) > 2$. Let s and t be two distinct vertices of $A_{n,k}$. For each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{|V(A_{n,k})|}{2}\}$, we shall find a hamiltonian cycle of $A_{n,k}$ such that the distance between s and t on the cycle is l. The basic idea of our construction is similar to that presented in Lemma 4.

Case 1

Suppose that s and t belong to the same subcomponent $A^{i}_{n,k}$. See Fig. 8 for an illustration. We assume that $s, t \in V(A^{i}_{n,k})$ for some $i \in \langle n\rangle$. Since $A^{i}_{n,k}$ is isomorphic to $A_{n-1,k-1}$, by the inductive hypothesis, for each $l_0 \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, |V(A^{i}_{n,k})| - d(s,t)\}$, we can construct a hamiltonian cycle HC_i of $A^{i}_{n,k}$ such that the distance between s and t on the cycle is l_0. Let u and v be the two neighbors of t on HC_i. Let $\mathrm{HC}_i = \langle s, LP, u, t, v, RP, s\rangle$, and let $P_0 = \langle s, LP, u, t\rangle$. Without loss of generality, let $L(P_0) = l_0$. By Proposition 2, $d(t,u) = 1$, we have $|AS(t) \cap AS(u)| = n - k - 1 > 1$ if $n - k > 2$. It means that we can find a subcomponent $A^{j_1}_{n,k}$ which $j_1 \in \langle n\rangle - \{i\}$, such that there exist two disjoint edges (u, p_1) and (t, q_1) in E^{i,j_1}. By Proposition 1, $(p_1, q_1) \in E(A^{j_1}_{n,k})$. Since $|N^*(t)| = n - k > 2$, we can find a subcomponent $A^{h_t}_{n,k}$ different from $A^{i}_{n,k}$ and $A^{j_1}_{n,k}$, and a vertex $t\prime \in V(A^{h_t}_{n,k})$ such that $(t, t') \in E^{i,h_t}$ for some $h_t \in \langle n\rangle - \{i, j_1\}$. By Proposition 2, $d(t,v) \leqslant 2$ hence $AS(t) \supseteq \{j_1, h_t\}$ and $AS(t) \neq AS(v)$, and $|N^*(v)| = n - k > 2$, we can find another subcomponent $A^{h_v}_{n,k}$, and a vertex $v\prime \in V(A^{h_v}_{n,k})$ such that $(v, v\prime) \in E^{i,h_v}$ for some $h_v \in \langle n\rangle - \{i, j_1, h_t\}$. By Lemma 1, there exists a hamiltonian path HP of $A^{\langle n\rangle - \{i\}}_{n,k}$ joining t' and v'. Thus $\langle s, P_0, t, t', \mathrm{HP}, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, |V(A^{i}_{n,k})| - d(s,t)\}$, the distance between s and t on the cycle is l_0.

Now we present an algorithm called *st-expansion* to expand the path P_0 between s and t to various lengths. We describe the details as follows.

We can insert one subcomponent of $A^{j_1}_{n,k}$ into P_0 as follows. See Fig. 9a for an illustration. Because p_1 and q_1 are adjacent, and $A_{n-1,k-1}$ is edge symmetric, we may regard them as in the same subcomponent of $A^{j_1}_{n,k}$, say C.

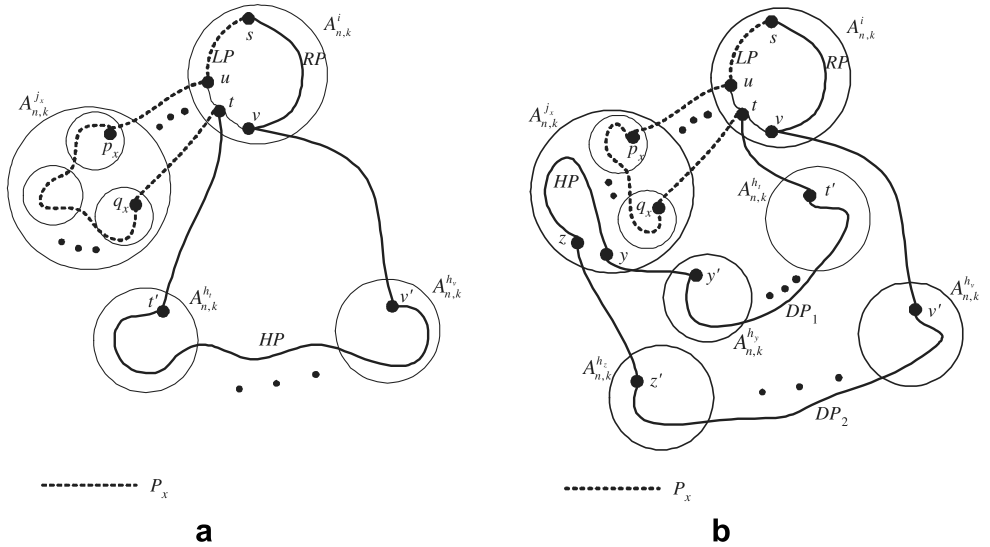

Fig. 8. Theorem 2, Case 1.

C is isomorphic to $A_{n-2,k-2}$. By Theorem 1, there is a hamiltonian path HP_1 of C joining p_1 and q_1 with $L(\mathrm{HP}_1) = |V(A_{n-2,k-2})| - 1$. We can insert more than one subcomponent of $A^{j_1}_{n,k}$ into P_0 as following. See Fig. 9b for an illustration. We regard p_1 and q_1 as in different subcomponents of $A^{j_1}_{n,k}$. By Lemma 1, there is a hamiltonian path HP_1 joining p_1 and q_1 with $L(\mathrm{HP}_1) = m|V(A_{n-2,\,k-2})|-1$, where m is the number of subcomponents of $A^{j_1}_{n,k}$ we wanted to insert. Thus we can construct a path HP_1 between p_1 and q_1 such that $L(\mathrm{HP}_1) = I_1|V(A_{n-2,k-2})| - 1$ for each integer I_1 with $1 \leqslant I_1 \leqslant n-1$. Let $P_1 = \langle s, LP, u, p_1, \mathrm{HP}_1, q_1, t\rangle$. Thus we have $L(P_1) = l_0 + I_1|V(A_{n-2,k-2})| = l_0 + \frac{I_1(n-2)!}{(n-k)!}$. Since $d(s,t) \leqslant l_0 \leqslant |V(A^i_{n,k})| - d(s,t)$, we have $\frac{I_1(n-2)!}{(n-k)!} + d(s,t) \leqslant L(P_1) \leqslant \frac{I_1(n-2)!}{(n-k)!} + \frac{(n-1)!}{(n-k)!} - d(s,t)$. For each $1 \leqslant I_1 \leqslant n-1$, $\frac{(I_1-1)(n-2)!}{(n-k)!} + \frac{(n-1)!}{(n-k)!} - d(s,t) \geqslant \frac{I_1(n-2)!}{(n-k)!} + d(s,t)$ if $n \geqslant 5$. Therefore, for each $l_1 \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{2(n-1)!}{(n-k)!} - d(s,t)\}$, we can construct a path P_1 from s to t such that the distance between s and t on the path is l_1.

Similar as above, we can expand the path between s and t more. For each $2 \leqslant x \leqslant \lfloor \frac{n}{2} \rfloor$, let u_{x-1} and t_{x-1} be two adjacent vertices on HP_{x-1}, where HP_{x-1} is a hamiltonian path of $A^{j_{x-1}}_{n,k}$ joining p_{x-1} and q_{x-1}. By

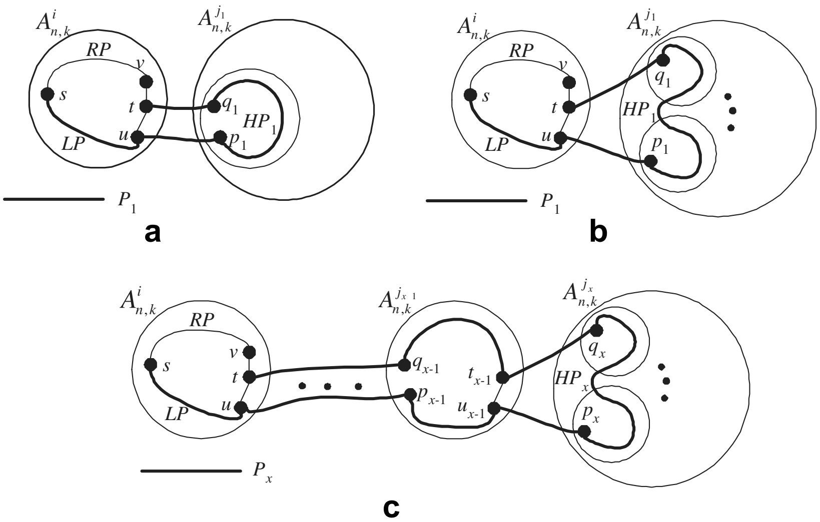

Fig. 9. *st*-Expansion.

Propositions 1 and 2, there exist two distinct edges (u_{x-1}, p_x) and (t_{x-1}, q_x) in $E^{j_{x-1}j_x}$ for some $j_x \in \langle n\rangle - \{i, h_t, h_v, j_1, \ldots, j_{x-1}\}$ such that $(p_x, q_x) \in E(A^{j_x}_{n,k})$. See Fig. 9c for an illustration. We can insert one subcomponent of $A^{j_x}_{n,k}$ into P_0 as follows. Because p_x and q_x are adjacent, and $A_{n-1,k-1}$ is edge symmetric, we may regard them as in the same subcomponent of $A^{j_x}_{n,k}$, say C. C is isomorphic to $A_{n-2,k-2}$. By Theorem 1, there is a hamiltonian path HP_x of C joining p_x and q_x with $L(\mathrm{HP}_x) = |V(A_{n-2,k-2})| - 1$. We can insert more than one subcomponent of $A^{j_x}_{n,k}$ into P_0 as follows. We regard p_x and q_x as in different subcomponents of $A^{j_x}_{n,k}$. By Lemma 1, there is a hamiltonian path HP_x joining p_x and q_x with $L(\mathrm{HP}_x) = m|V(A_{n-2,k-2})| - 1$, where m is the number of subcomponents of $A^{j_x}_{n,k}$ we wanted to insert. Thus we can construct a path HP_x between p_x and q_x such that $L(\mathrm{HP}_x) = I_x|V(A_{n-2,k-2})| - 1$ for each integer I_x with $1 \leqslant I_x \leqslant n-1$. Let $P_x = \langle s, LP, u, p_1, \ldots, p_x, \mathrm{HP}_x, q_x, \ldots, q_1, t\rangle$. Thus we have $L(P_x) = l_0 + (x-1)|V(A_{n-1,k-1})| + I_x|V(A_{n-2,k-2})| = l_0 + \frac{(x-1)(n-1)!}{(n-k)!} + \frac{I_x(n-2)!}{(n-k)!}$. Since $d(s,t) \leqslant l_0 \leqslant |V(A^i_{n,k})| - d(s,t)$, we have $\frac{(x-1)(n-1)!}{(n-k)!} + \frac{I_x(n-2)!}{(n-k)!} + d(s,t) \leqslant L(P_x) \leqslant \frac{I_x(n-2)!}{(n-k)!} + \frac{x(n-1)!}{(n-k)!} - d(s,t)$. For each $1 \leqslant I_x \leqslant n-1$, $\frac{(I_x-1)(n-2)!}{(n-k)!} + \frac{x(n-1)!}{(n-k)!} - d(s,t) \geqslant \frac{I_x(n-2)!}{(n-k)!} + \frac{(x-1)(n-1)!}{(n-k)!} + d(s,t)$ if $n \geqslant 5$. Therefore, for each $l_x \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{(x+1)(n-1)!}{(n-k)!} - d(s,t)\}$, we can construct a path P_x from s to t such that the distance between s and t on the path is l_x by using st-expansion. Notice that the maximal value of l_x is $\frac{(\lfloor\frac{n}{2}\rfloor+1)(n-1)!}{(n-k)!} - d(s,t)$, which is greater than $\frac{n!}{2(n-k)!}$, and $\frac{|V(A_{n,k})|}{2} = \frac{n!}{2(n-k)!}$. Hence for any integer l with $d(s,t) \leqslant l \leqslant \frac{|V(A_{n,k})|}{2}$, we can construct a path joining s and t with the length of the path being l. We will use st-expansion for the remaining cases of the proof.

To complete the construction of a hamiltonian cycle, we consider the following two subcases:

Subcase 1.1: All the vertices of $A^{\{j_1,\ldots,j_x\}}_{n,k}$ are on the path P_x for some $1 \leqslant x \leqslant \lfloor\frac{n}{2}\rfloor$. See Fig. 8a for an illustration. By Lemma 1, there is a hamiltonian path HP of $A^{\langle n\rangle-\{i,j_1,\ldots,j_x\}}_{n,k}$ joining t' and v' in which $t' \in V(A^{h_t}_{n,k})$ and $v' \in V(A^{h_v}_{n,k})$. Thus $\langle s, P_x, t, t', \mathrm{HP}, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{|V(A_{n,k})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 1.2: Not all the vertices of $A^{\{j_1,\ldots,j_x\}}_{n,k}$ are on the path P_x for some $1 \leqslant x \leqslant \lfloor\frac{n}{2}\rfloor$. See Fig. 8b for an illustration. Then we can find two adjacent vertices y and z in $A^{j_x}_{n,k}$ which are not on the path P_x. Let $F \subseteq V(P_x)$. By Propositions 1 and 2, there exist two distinct edges $(y, y') \in E^{j_x,h_y}$ and $(z, z') \in E^{j_x,h_z}$ such that $y' \neq t' \in V(A^{h_y}_{n,k})$ and $z' \neq v' \in V(A^{h_z}_{n,k})$, respectively. If $A^{j_x}_{n,k} - F$ is isomorphic to $A_{n-2,k-2}$, by Theorem 1, there is a hamiltonian path HP from y to z in $A^{j_x}_{n,k} - F$. If $A^{j_x}_{n,k} - F$ contains more than one subcomponent of $A^{j_x}_{n,k}$, by Lemma 1, there is a hamiltonian path HP from y to z in $A^{j_x}_{n,k} - F$. By Lemma 7, there exist two disjoint paths DP_1 and DP_2, such that DP_1 joins t' and y', and DP_2 joins v' and z'. Moreover, the two paths span all of the vertices in $A^{\langle n\rangle-\{i,j_1,\ldots,j_x\}}_{n,k}$. Thus $\langle s, P_x, t, t', DP_1, y', y, \mathrm{HP}, z, z', DP_2, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{|V(A_{n,k})|}{2}\}$, the distance between s and t on the cycle is l.

Case 2

Suppose that s and t belong to different subcomponents of $A_{n,k}$. We assume that $s \in V(A^i_{n,k})$ and $t \in V(A^j_{n,k})$ for any $i \neq j \in \langle n\rangle$. By Lemma 6, there exists a minimum length path connecting s and t with the form $\langle s, MP, t'', t\rangle$ or $\langle s, MP, t'', t', t\rangle$, where MP is a path in $A^i_{n,k}$, $t'' \in V(A^i_{n,k})$, and $t' \in V(A^j_{n,k})$. Hence we have the following two subcases:

Subcase 2.1: The minimum length path connecting s and t has the form $\langle s, MP, t'', t\rangle$. Then $d(s,t) = d(s,t'') + 1$. See Fig. 10a for an illustration. Since $A^i_{n,k}$ is isomorphic to $A_{n-1,k-1}$, by the inductive hypothesis, for each $l_0 \in \{d(s,t''), d(s,t'')+1, d(s,t'')+2, \ldots, |V(A^i_{n,k})| - d(s,t'')\}$, we can construct a hamiltonian cycle HC_i of $A^i_{n,k}$ such that the distance between s and t'' on the cycle is l_0. Let u and v be the two neighbors of t'' on HC_i. Let $\mathrm{HC}_i = \langle s, LP, u, t'', v, RP, s\rangle$, and let $P_0 = \langle s, LP, u, t'', t\rangle$. Without loss of generality, let $L(P_0) = l_0 + 1$. By Proposition 2, $d(t'', u) = 1$, so we have $|AS(t'') \cap AS(u)| = n-k-1 > 1$ if $n-k > 2$. This means that we can find a subcomponent $A^{j_1}_{n,k}$ in which $j_1 \in \langle n\rangle - \{i,j\}$, such that there exist two disjoint edges (u, p_1) and (t'', q_1) in E^{i,j_1}. By Proposition 1, $(p_1, q_1) \in E(A^{j_1}_{n,k})$. By Proposition 2, $d(t'', v) \leqslant 2$ hence $AS(t'') \supseteq \{j, j_1\}$, and $AS(t'') \neq AS(v)$, and $|N^*(v)| = n-k > 2$, we can find a subcomponent $A^{h_v}_{n,k}$, and a vertex $v' \in V(A^{h_v}_{n,k})$ such that $(v, v') \in E^{i,h_v}$ for some $h_v \in \langle n\rangle - \{i, j, j_1\}$. By Lemma 1, there exists a hamiltonian path HP of $A^{\langle n\rangle-\{i\}}_{n,k}$ joining t and v'. Thus $\langle s, P_0, t, \mathrm{HP}, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, |V(A^i_{n,k})| - d(s,t)+1\}$, the distance between s and t on the cycle is l_0.

Similar to Case 1, by using st''-expansion, for any integer l'' with $d(s,t'') \leqslant l'' \leqslant \frac{|V(A_{n,k})|}{2}$, we can construct a path joining s and t'' with the length of the path being l''. Since $d(s,t'') = d(s,t) - 1$, for any

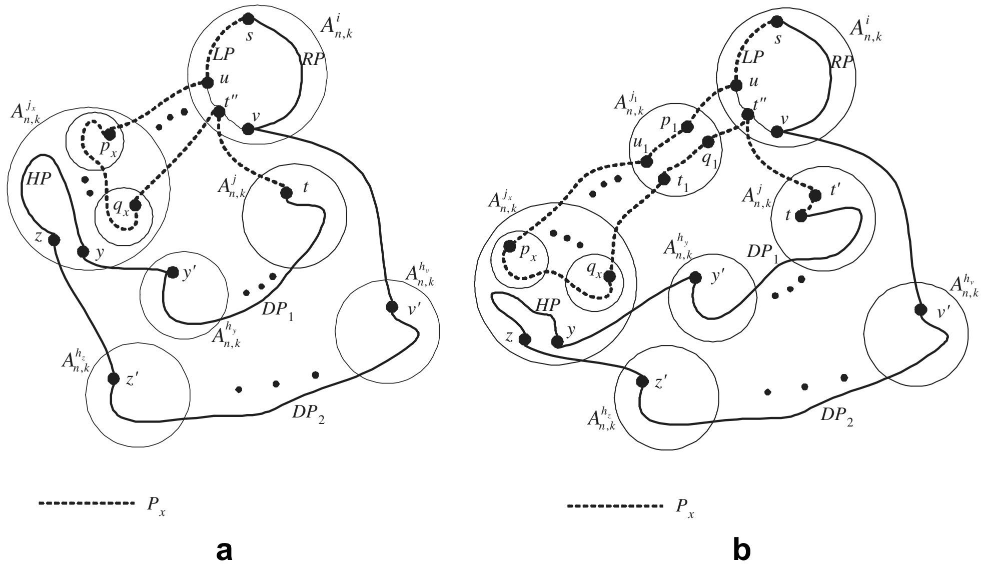

Fig. 10. Theorem 2, Subcase 2.1 and Subcase 2.2.

integer l with $d(s,t) \leqslant l \leqslant \frac{|V(A_{n,k})|}{2}$, we can construct a path joining s and t with the length of the path being l.

To complete the construction of a hamiltonian cycle, the proof is the same as that given in Subcase 1.1 and Subcase 1.2 by replacing vertices t and t' in Case 1 with vertices t'' and t in this case, respectively.

Subcase 2.2: The minimum length path connecting s and t has the form $\langle s, MP, t'', t', t\rangle$. Then $d(s,t) = d(s,t'') + 2$. See Fig. 10b for an illustration. Since $A^i_{n,k}$ is isomorphic to $A_{n-1,k-1}$, by the inductive hypothesis, for each $l_0 \in \{d(s,t''), d(s,t'')+1, d(s,t'')+2, \ldots, |V(A^i_{n,k})| - d(s,t'')\}$, we can construct a hamiltonian cycle HC_i of $A^i_{n,k}$ such that the distance between s and t'' on the cycle is l_0. Let u and v be the two neighbors of t'' on HC_i. Let $\mathrm{HC}_i = \langle s, LP, u, t'', v, RP, s\rangle$, and let $P_0 = \langle s, LP, u, t'', t', t\rangle$. Without loss of generality, let $L(P_0) = l_0 + 2$. By Proposition 2, $d(t'', u) = 1$, so we have $|AS(t'') \cap AS(u)| = n-k-1 > 1$ if $n-k > 2$. This means that we can find a subcomponent $A^{j_1}_{n,k}$ in which $j_1 \in \langle n\rangle - \{i,j\}$, such that there exist two disjoint edges (u, p_1) and (t'', q_1) in E^{i,j_1}. By Proposition 1, $(p_1, q_1) \in E(A^{j_1}_{n,k})$. By Proposition 2, $d(t'', v) \leqslant 2$ hence $AS(t'') \supseteq \{j, j_1\}$, and $AS(t'') \neq AS(v)$, and $|N^*(v)| = n-k > 2$, we can find a subcomponent $A^{h_v}_{n,k}$, and a vertex $v' \in V(A^{h_v}_{n,k})$ such that $(v, v') \in E^{i,h_v}$ for some $h_v \in \langle n\rangle - \{i, j, j_1\}$. Let $F \subseteq V(A_{n,k})$ and $F' = \{t'\}$. By Lemma 1, there exists a hamiltonian path HP of $A^{\langle n\rangle - \{i\}}_{n,k} - F'$ joining t and v'. Thus $\langle s, P_0, t, \mathrm{HP}, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l_0 \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, |V(A^i_{n,k})| - d(s,t)+2\}$, the distance between s and t on the cycle is l_0.

By using st''-expansion, for any integer l'' with $d(s,t'') \leqslant l'' \leqslant \frac{|V(A_{n,k})|}{2}$, we can construct a path joining s and t'' with the length of the path being l''. Since $d(s,t'') = d(s,t) - 2$, for any integer l with $d(s,t) \leqslant l \leqslant \frac{|V(A_{n,k})|}{2}$, we can construct a path joining s and t with the length of the path being l.

To construct a hamiltonian cycle, we consider the following two subcases:

Subcase 2.2.1: All the vertices of $A^{\{j_1,\ldots,j_x\}}_{n,k}$ are on the path P_x for some $1 \leqslant x \leqslant \lfloor \frac{n}{2} \rfloor$. By Lemma 1, there is a hamiltonian path HP of $A^{\langle n\rangle - \{i, j_1, \ldots, j_x\}}_{n,k} - F'$ joining t and v' which $F' = \{t'\}$, $t \in V(A^j_{n,k})$ and $v' \in V(A^{h_v}_{n,k})$. Thus $\langle s, P_x, t'', t', t, \mathrm{HP}, v', v, RP, s\rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{|V(A_{n,k})|}{2}\}$, the distance between s and t on the cycle is l.

Subcase 2.2.2: Not all the vertices of $A^{\{j_1,\ldots,j_x\}}_{n,k}$ are on the path P_x for some $1 \leqslant x \leqslant \lfloor \frac{n}{2} \rfloor$. See Fig. 10b for an illustration. Then we can find two adjacent vertices y and z in $A^{j_x}_{n,k}$ which are not on the path P_x. Let $F \subseteq V(P_x)$. By Propositions 1 and 2, there exist two distinct edges $(y, y') \in E^{j_x,h_y}$ and $(z, z') \in E^{j_x,h_z}$ such that $y' \neq t' \in V(A^{h_y}_{n,k})$ and $z' \neq v' \in V(A^{h_z}_{n,k})$, respectively. If $A^{j_x}_{n,k} - F$ is isomorphic to $A_{n-2,k-2}$, by Theorem 1, there is a hamiltonian path HP from y to z in $A^{j_x}_{n,k} - F$. If $A^{j_x}_{n,k} - F$ contains more than one subcomponents of

$A^{j_x}_{n,k}$, by Lemma 1, there is a hamiltonian path HP from y to z in $A^{j_x}_{n,k} - F$. By Lemma 7, there exist two disjoint paths DP_1 and DP_2, such that DP_1 joins t and y', and DP_2 joins v' and z'. Moreover, the two paths span all the vertices in $A^{\langle n \rangle - \{i, j_1, \ldots, j_x\}}_{n,k} - F'$ which $F' = \{t'\}$. Thus $\langle s, P_x, t'', t', t, DP_1, y', y, \text{HP}, z, z', DP_2, v', v, RP, s \rangle$ forms a hamiltonian cycle, and for each $l \in \{d(s,t), d(s,t)+1, d(s,t)+2, \ldots, \frac{|V(A_{n,k})|}{2}\}$, the distance between s and t on the cycle is l.

Hence the theorem is proved. □

Applying the above theorem we can prove that $A_{n,k}$ is panconnected for all $n \geqslant 3$ and $n - k \geqslant 2$.

Theorem 3. *The arrangement graph $A_{n,k}$ is panconnected for all $n \geqslant 3$ and $n - k \geqslant 2$.*

Proof. For $k = 1$, by Lemma 3, $A_{n,1}$ is panconnected for all $n \geqslant 3$. Chiang and Chen [22] showed that the $A_{n,n-2}$ is isomorphic to the n-alternating group graph AG_n, and Chang et al. [18] proved that AG_n is panconnected for all $n \geqslant 4$. Hence the result holds for $n \geqslant 4$ and $k = n - 2$. Now we prove that $A_{n,k}$ is panconnected for all $n \geqslant 5$ and $n - k > 2$. Suppose that u and v are any two distinct vertices in $A_{n,k}$. By Theorem 2, $A_{n,k}$ is panpositionable hamiltonian. That is, for each integer l with $d(u,v) \leqslant l \leqslant |V(A_{n,k})| - d(u,v)$, we can construct a path P of length l joining u and v.

For each integer l with $|V(A_{n,k})| - d(u,v) + 1 \leqslant l \leqslant |V(A_{n,k})| - 1$,we can construct a path P of length l joining u and v as following. The diameter of $A_{n,k}$ is $\lfloor \frac{3k}{2} \rfloor$, and we have $d(u,v) \leqslant \lfloor \frac{3k}{2} \rfloor$. By Theorem 1, $A_{n,k}$ is $k(n-k) - 3$ fault tolerant hamiltonian connected. For $n \geqslant 5$ and $n - k > 2$, we have $k(n-k) - 3 \geqslant \lfloor \frac{3k}{2} \rfloor - 1$. That means that for each integer l with $|V(A_{n,k})| - d(u,v) + 1 \leqslant l \leqslant |V(A_{n,k})| - 1$, we can construct a path P of length l joining u and v by regarding the vertices not in P as faulty vertices. Therefore, for each integer l with $d(u,v) \leqslant l \leqslant |V(A_{n,k})| - 1$, there is a path of length l joining u and v in $A_{n,k}$. The theorem is proved. □

For example, there are 60 vertices in $A_{5,3}$, and the diameter of $A_{5,3}$ is 4. Let u and v be two vertices in $A_{5,3}$ with $d(u,v) = 4$. By the panpositionable hamiltonian property, we can find a path joining u and v with length $l \in \{4, 5, 6, \ldots, 56\}$. Let $F \subseteq V(A_{5,3}) - \{u, v\}$. We can find three paths of length 57, 58, and 59 joining u and v with $|F| = 2$, $|F| = 1$, and $|F| = 0$, respectively. By choosing two adjacent vertices u and v and applying the above theorem, we can obtain the following corollary immediately.

Corollary 1. *The arrangement graph $A_{n,k}$ is pancyclic for all $n \geqslant 3$ and $n - k \geqslant 2$.*

5. Concluding remarks

In this paper, we have proposed a new concept called panpositionable hamiltonicity. We have showed that the arrangement graph $A_{n,k}$ is panpositionable hamiltonian if $k \geqslant 1$ and $n - k \geqslant 2$. Applying this result we can prove that $A_{n,k}$ is panconnected and pancyclic if $k \geqslant 1$ and $n - k \geqslant 2$. We now explain some relationship between the panpositionable hamiltonian property and the panconnected property. We give an example to show that a panconnected graph G is not necessarily panpositionable hamiltonian. Consider the circulant graph, let $n, s_1, s_2, \ldots, s_r$ be integers with $1 \leqslant s_1 < s_2 < \cdots < s_r$. A circulant graph $C(n; s_1, s_2, \ldots, s_r)$ is the graph with vertex set $\{0, 1, \ldots, n-1\}$. Two vertices i and j are adjacent if and only if $i - j = \pm s_k (\text{mod}\, n)$ for some k where $1 \leqslant k \leqslant r$. We can check that $C(n; 1, 2)$ is panconnected by brute force for $n \in \{5, 6, 7, 8, 9, 10\}$. However, $C(10; 1, 2)$ is not panpositionable hamiltonian. In fact, the circulant graph $C(n; 1, 2)$ is panconnected for every $n \geqslant 5$, but it is not panpositionable hamiltonian for some values of n. Therefore, the panpositionable hamiltonian property is a stronger property for an interconnection network.

Another important issue in the design of an interconnection network is connectivity. It is a widely used measurement for evaluating the reliability of an interconnection network. The *connectivity* of G, $\kappa(G)$ is the minimum number of nodes whose removal leaves the remaining graph disconnected or trivial. Let G be a graph with connectivity $\kappa(G) = \kappa$. It follows from Menger's Theorem [23] that there are l *internally vertex-disjoint* (abbreviated as disjoint) paths joining any two vertices u and v when $l \leqslant \kappa(G)$. A *container* $C(u,v)$ between two distinct nodes u and v in G is a set of internally disjoint paths $P_1, P_2, \ldots, P_r$ between u and v. The *width* of $C(u,v)$ is r. A *w-container* is a container of width w. The *length* of a $C(u,v)$, written as $l(C(u,v))$, is the length of the longest path in $C(u,v)$. A w-container $C(u,v)$ is a w^*-*container* if every vertex of G is incident with a path in $C(u,v)$. A graph G is w^*-*connected* if there exists a w^*-container between any two distinct vertices u and v. Obviously, a graph G is 2^*-connected if it is hamiltonian. We also define w^*-*distance* between any two vertices u and v, $d_w^{s_L}(u,v)$, to be $\min\{l(C(u,v))|C(u,v)$ is w^*-container$\}$. The $w^{*_L}-$ spanning diameter of G, denoted by $D_w^{s_L}(G)$, as the maximum number of $d_w^{s_L}(u,v)$. The spanning diameter is used to measure the performance of multipath communication in networks [24,25].

By the panpositionable hamiltonian property of the arrangement graph $A_{n,k}$, for any two different vertices x and y in $A_{n,k}$ and for any integer l satisfying $d(x,y) \leqslant l \leqslant |V(A_{n,k})| - d(x,y)$, there exists a hamiltonian cycle of $A_{n,k}$ such that the relative distance between x and y on the cycle is l. Since the diameter of $A_{n,k}$ is $\lfloor \frac{3k}{2} \rfloor$, $d(x,y) \leqslant \lfloor \frac{3k}{2} \rfloor$. Then $\lfloor \frac{3k}{2} \rfloor \leqslant \frac{|V(A_{n,k})|}{2} \leqslant |V(A_{n,k})| - \lfloor \frac{3k}{2} \rfloor$. Let $l = \frac{|V(A_{n,k})|}{2}$, we can find a hamiltonian cycle $C = \langle x, P_1, y, P_2, x \rangle$ of $A_{n,k}$ such that the distance between x and y on C is $\frac{|V(A_{n,k})|}{2}$. Obviously, P_1 and P_2 forms a 2^*-container. Moreover, $L(P_1) = \frac{|V(A_{n,k})|}{2} = \frac{n!}{2(n-k)!}$, and $P_2 = \frac{|V(A_{n,k})|}{2} = \frac{n!}{2(n-k)!}$. Hence the following corollary holds.

Corollary 2. *Suppose that* $k \geqslant 2$ *and* $n-k \geqslant 2$. *Then* $d_2^{s_L}(x,y) = \frac{|V(A_{n,k})|}{2} = \frac{n!}{2(n-k)!}$ *for any two vertices x and y in the arrangement graph* $A_{n,k}$. *That is, the* 2^{*_L}*-diameter* $D_2^{s_L}(A_{n,k}) = \frac{|V(A_{n,k})|}{2} = \frac{n!}{2(n-k)!}$.

For a graph G with even vertices, $D_2^{s_L}(G) \geqslant \frac{|V(G)|}{2}$. The arrangement graph $A_{n,k}$ with $k \geqslant 2$ has even vertices, thus our result about the 2^{*_L}-diameter of $A_{n,k}$ is optimal.

Future work will try to find the panpositionable hamiltonicity of other interconnection networks. It would be interesting to study some relationship between these specific properties, such as panpositionable hamiltonicity, panconnectivity and pancyclicity, and the other criteria for measuring the performance of a network.

References

[1] S.B. Akers, D. Harel, B. Krishnamurthy, The star graph: an attractive alternative to the *n*-cube, in: Proceedings of the International Conference on Parallel Processing, 1986, pp. 216–223.
[2] K. Day, A. Tripathi, Arrangement graphs: a class of generalized star graphs, Information Processing Letters 42 (5) (1992) 235–241.
[3] K. Efe, The crossed cube architecture for parallel computing, IEEE Transactions on Parallel and Distributed Systems 3 (5) (1992) 513–524.
[4] P.A.J. Hilbers, M.R.J. Koopman, J.L.A. van de Snepscheut, The twisted cube, Parallel Architectures and Languages Europe, Lecture Notes in Computer Science (1987) 152–159.
[5] Y. Saad, M.H. Schultz, Topological properties of hypercubes, IEEE Transactions on Computers 37 (7) (1988) 867–872.
[6] I. Stojmenovic, Honeycomb networks: topological properties and communication algorithms, IEEE Transactions on Parallel and Distributed Systems 8 (10) (1997) 1036–1042.
[7] S.B. Akers, B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, IEEE Transactions on Computers 38 (4) (1989) 555–566.
[8] F.T. Leighton, Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes, Morgan Kaufman Publishers, San Mateo, CA, 1992.
[9] K. Day, A. Tripathi, Characterization of Node Disjoint Paths in Arrangement Graphs, Technical Report TR 91-43, Computer Science Department, University of Minnesota, 1991.
[10] K. Day, A. Tripathi, Embedding of cycles in arrangement graphs, IEEE Transactions on Computers 42 (8) (1993) 1002–1006.
[11] K. Day, A. Tripathi, Embedding grids, hypercubes, and trees in arrangement graphs, in: Proceedings of the International Conference on Parallel Processing, 1993, pp. III-65–III-72.
[12] S.Y. Hsieh, G.H. Chen, C.W. Ho, Fault-free hamiltonian cycles in faulty arrangement graphs, IEEE Transactions on Parallel and Distributed Systems 10 (3) (1999) 223–237.
[13] H.C. Hsu, T.K. Li, J.M. Tan, L.H. Hsu, Fault hamiltonicity and fault hamiltonian connectivity of the arrangement graphs, IEEE Transactions on Computers 53 (1) (2004) 39–53.
[14] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1994.
[15] S. Latifi, N. Bagherzadeh, On embedding rings into a star-related network, Information Sciences 99 (1997) 1–35.
[16] J.A. Bondy, Pancyclic graphs, Journal of Combinatorial Theory Series B 11 (1971) 80–84.
[17] Y. Alavi, J.E. Williamson, Panconnected graphs, Studia Scientiarum Mathematicarum Hungarica 10 (1975) 19–22.
[18] J.M. Chang, J.S. Yang, Y.L. Wang, Y. Cheng, Panconnectivity, fault-Tolerant hamiltonicity and hamiltonian-connectivity in alternating group graphs, Networks 44 (2004) 302–310.
[19] S.Y. Hsieh, G.H. Chen, Pancyclicity on Möbius cubes with maximal edge faults, Parallel Computing 30 (2004) 407–421.
[20] M.C. Yang, T.K. Li, J.M. Tan, L.H. Hsu, On embedding cycle in faulty twisted cubes, Information Sciences 176 (2006) 676–690.
[21] Y.H. Teng, J.M. Tan, L.H. Hsu, Panpositionable hamiltonicity of the alternating group graphs, Networks 50 (2007) 146–156.

[22] W.K. Chiang, R.J. Chen, On the arrangement graph, Information Processing Letters 66 (4) (1998) 215–219.
[23] K. Menger, Zur allgemeinen Kurventheorie, Fundamenta Mathematicae 10 (1927) 95–115.
[24] D. Frank Hsu, On container width and length in graphs, groups and networks, IEICE Transactions on Fundamentals E77-A (4) (1994) 668–680.
[25] C.K. Lin, H.M. Huang, D.F. Hsu, L.H. Hsu, On the spanning *w*-wide diameter of the star graph, Networks 48 (2006) 235–249.