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改良型的 Cpmk，具有非對稱規格區間的製程能力指標

On the Generalizations of the Capability Index Cpmk for Asymmetric Tolerances

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摘要

Boyles (1994) 針對數個使用於非對稱規格區間的製程能力指標，廣泛的加以研究。並就測度製程能力方面，比較他所提出的指標與已存在的指標的精確度。但是，這一些指標沒有一個能夠精確地反應製程能力。事實上，對於給定製程標準差的情況下，這些指標的最大值並沒有發生在 $\mu = T$ 的製程。最近，Vännman (1997) 也提出新的製程能力指標來處理具非對稱規格區間的製程能力評估問題。但很不幸的是此一新指標某些情形下會有低估製程能力的情況發生，其他情形下指標的最大值並沒有發生在 $\mu = T$ 的製程。因此，Vännman (1997)所提出的新指標亦不適用於具非對稱規格區間的製程能力評估問題。

在這篇文章裡面，我們將提出一個 C_{pmk} 的改良版，此一新指標，我們將它稱為 C''_{pmk}。我們證明此一新指標 C''_{pmk} 優於現存的其他指標。我們也探討了新指標的自然估計式在常態假設下的一些統計性質。

關鍵詞：製程能力指標；目標值；製程良率；製程集中量；製程損失；非對稱規格區間；偏離率

abstract

Boyles (1994) presented a comprehensive study of several proposed indices for processes with asymmetric tolerances, and provided a comparison between the proposed and the existing indices on the accuracy in measuring process potential and performance. However, none of those proposed indices reflect process capability accurately. In fact, for fixed process standard deviation σ, those indices obtain their maximal values not at $\mu = T$, but at μ^* which is between the target value T and the center of the specification interval. Recently, Vännman (1997) considered several generalizations of the existing indices for processes with asymmetric tolerances. But, those generalizations either understates the process capability, or obtain their maximal values not at $\mu = T$. Therefore, Vännman's generalizations are inappropriate for cases with asymmetric tolerances. In this paper, we consider a new generalization of C_{pmk} which we refer to as C''_{pmk}. We show that the new generalization C''_{pmk} is superior to the existing generalizations. We also investigate the statistical properties of the natural estimator of C''_{pmk} assuming the process is normally distributed.

Keywords: Process capability index; Target value; Process yield; Process centering; Process loss; Asymmetric Tolerances; Departure ratio.

1. Introduction

Process capability indices (PCIs), which provide numerical measures on whether a manufacturing process met the capability requirement preset in the factory, have recently been the research focus in quality assurance literature. Pearn, Kotz, and Johnson (1992) proposed the index called C_{pmk}, which combines the advantages of C_{pk} and C_{pm}. The index C_{pmk} has been defined as:

$$C_{pmk} = \frac{\min\{USL-\mu,\ \mu-LSL\}}{3\sqrt{\sigma^2+(\mu-T)^2}},$$

where USL is the upper specification limit, LSL is the lower specification limit, T is the target value, μ is the process mean, and σ is the process standard deviation.

Pearn and Chen (1998) considered a generalization of C_{pk} for processes with asymmetric tolerances. The generalization takes into account the asymmetry of the tolerance, which reflects the process capability more accurately than the original C_{pk}. In this paper, we take the same approach and consider a new generalization of C_{pmk}, to handle processes with asymmetric tolerances. The new generalization, which we refer to as C''_{pmk}, may be defined as:

$$C''_{pmk} = \frac{d^*-A^*}{3\sqrt{\sigma^2+A^2}},$$

where $A = \max\{d(\mu - T)/D_u,\ d(T - \mu)/D_l\}$, $A^* = \max\{d^*(\mu - T)/D_u,\ d^*(T - \mu)/D_l\}$, $D_u = USL - T$, $D_l = T - LSL$, and $d^* = \min\{D_u, D_l\}$. Obviously, if $T = m$ (symmetric case), then $A = A^* = |\mu - T|$ and C''_{pmk} reduces to the original index C_{pmk}. The factors A and A^* ensure that the new generalization C''_{pmk} obtains its maximal value at $\mu = T$ (process is on-target) regardless of whether the tolerances are symmetric ($T = m$) or asymmetric (T m). It is easy to verify that if the process is on the specification limits (μ = LSL, or μ = USL), then $C''_{pmk} = 0$. On the other hand, if LSL < μ < USL, then $C''_{pmk} > 0$. We note that for fixed σ the value of C''_{pmk} decreases when μ shifts away from T. In fact, the value of C''_{pmk} decreases faster when μ moves away from T to the closer specification limit, and decreases slower when μ moves away from T to the farther specification limit.

Since Yield $\geq 2\Phi(3C_{pk}) - 1$ and $C''_{pmk} \leq C_{pmk} \leq C_{pk}$, then given value of C''_{pmk}, we can calculate the lower bound of process yield as $2\Phi(3C''_{pmk}) - 1$ (see Boyles (1991)). For example, given a process with capability $C''_{pmk} = 1$, the process yield is guaranteed to be no less than $2\Phi(3) - 1 = 99.73\%$. On the other hand, the upper bound on the departure ratio can be calculated as $(A/d) \leq d/(3cd^* + d)$ for $C''_{pmk} \geq c$. The result indicates that for large value of C''_{pmk} the process departure is small. We note that the departure ratio, $A/d = (\mu - T)/D_u$ if $\mu \geq T$, and $A/d = (T - \mu)/D_l$ if $\mu < T$.

2. Estimation of C''_{pmk}

The natural estimator $\hat{C}''_{pmk}$ is obtained by replacing the process mean μ and the process variance σ^2 by their conventional estimators $\bar{X}$ and S_n^2, which may be obtained from a stable process.

$$\hat{C}''_{pmk} = \frac{d^*-\hat{A}^*}{3\sqrt{S_n^2+\hat{A}^2}},$$

where $d^* = \min\{D_u, D_l\}$, $\hat{A} = \max\{d(\bar{X} - T)/D_u,\ d(T - \bar{X})/D_l\}$, $\hat{A}^* = \max\{d^*(\bar{X} - T)/D_u,\ d^*(T - \bar{X})/D_l\}$, $\bar{X} = (\sum_{i=1}^{n} X_i)/n$, and $S_n^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/n$. If the manufacturing tolerance is symmetric, then $d^* = d$, $\hat{A} = \hat{A}^* = |\bar{X} - T|$, and the estimator $\hat{C}''_{pmk}$ reduces to $\hat{C}_{pmk} = (d - |\bar{X} - m|)/3[S_n^2 + (\bar{X} - T)^2]^{1/2}$, the natural estimator of C_{pmk} (symmetric case) considered by Pearn, Kotz, and Johnson (1992).

3. Distributions and Moments

We now define $D^* = n^{1/2}(d^*/\sigma)$, $D = n^{1/2}(d/\sigma)$, $K = nS_n^2/\sigma^2$, $Z = n^{1/2}(\bar{X} - T)/\sigma$, $Y = [\max\{(d/D_u)Z,\ -(d/D_l)Z)\}]^2$, $\delta = n^{1/2}(\mu - T)/\sigma$, and $\lambda = \delta^2$. Under the assumption of normality, the probability density function of $\hat{C}''_{pmk}$ can be derived as:

$$f_{\hat{C}''_{pmk}}(x) = \begin{cases} B^* \times \dfrac{e^{-\lambda/2}}{2\sqrt{\pi}} \displaystyle\sum_{j=0}^{\infty} \dfrac{(D^*\delta)^j}{j!} \left\{ \sum_{i=0}^{1} \int_0^{\infty} (-1)^{ij} I_j^*(u_i, v_i, x, z)dz \right\}, & -\dfrac{d^*}{3d} < x < 0, \\ B^* \times \dfrac{e^{-\lambda/2}}{2\sqrt{\pi}} \displaystyle\sum_{j=0}^{\infty} \dfrac{(D^*\delta)^j}{j!} \left\{ \sum_{i=0}^{1} \int_0^{1/x} (-1)^{ij} I_j^*(u_i, v_i, x, z)dz \right\}, & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha_0 = [(D^* \times D_u)/(d^* + 3xd)]^2$, $\alpha_1 = [(D^* \times D_l)/(d^* + 3xd)]^2$, $\beta_0 = [(D^* \times D_u)/d^*]^2$, $\beta_1 = [(D^* \times D_l)/d^*]^2$, F_K is the cumulative distribution function of K, $G_0(y) = F_K((3x)^{-2}[D^* - (d^*/D_u)y^{1/2}]^2 - (d/D_u)^2 y)$, $G_1(y) = F_K((3x)^{-2}[D^* - (d^*/D_l)y^{1/2}]^2 - (d/D_l)^2 y)$, $B^* = 12(n^{1/2}d^*)^n / \{(18\sigma^2)^{n/2}\Gamma[(n-1)/2]\}$, $u_0 = d^*/D_u$, $u_1 = d^*/D_l$, $v_0 = (d/D_u)^2$, $v_1 = (d/D_l)^2$, and

$$I_j^*(u_i, v_i, x, z) = \frac{(1-xz)^j (u_i z - 3\sqrt{v_i})^2 z_i^{(n-3)/2} [u_i z + 3\sqrt{v_i}(2 - xz)]^{(n-3)/2}}{(u_i + 3x\sqrt{v_i})^{j+(n+3)/2}}$$

$$\times \exp\left\{ -\frac{(D^*)^2 [(u_i z + 3\sqrt{v_i})^2 + 9(1 - v_i)(1 - xz)^2]}{18(u_i + 3x\sqrt{v_i})^2} \right\}.$$

In general, the r-th moment of $\hat{C}''_{pmk}$ can be obtained as:

$$E(\hat{C}''_{pmk})^r = \frac{1}{3^r} \sum_{i=0}^{r} \left\{ (-1)^i \binom{r}{i} \left(\frac{D^*}{\sqrt{2}} \right)^{r-i} \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \right.$$

$$\times \sum_{j=0}^{\infty} \left[\Gamma\left(\frac{n-r+i+j}{2} \right) \Gamma\left(\frac{1+i+j}{2} \right) \Big/ \left(\Gamma\left(\frac{1+j}{2} \right) \Gamma\left(\frac{n+i+j}{2} \right) \right) \right]$$

$$\left. \times g_j \times \delta^j \times \left[\left(\frac{d^*}{D_u} \right)^i {}_2F_1(a, b; c; z_u) + (-1)^j \left(\frac{d^*}{D_l} \right)^i {}_2F_1(a, b; c; z_l) \right] \right\}, \qquad (4)$$

where ${}_2F_1(a, b; c; z)$ is the Gaussian hypergeometric function (Abramowitz and Stegun (1970)) with parameters $a = r/2$, $b = (1+i+j)/2$, $c = (n+i+j)/2$, $z_u = 1 - (d/D_u)^2$, and $z_l = 1 - (d/D_l)^2$.

4. Conclusions

In this paper, we propose a new generalization of C_{pmk}, to handle processes with asymmetric tolerances. The new generalization, which we refer to as C''_{pmk}, incorporates the asymmetry of the manufacturing tolerance, thus reflects process performance more accurately. The new generalization guarantees the yield which is greater than a certain level for given index value of C''_{pmk}. In fact, given value of C''_{pmk}, we can calculate the lower bound of process yield as $2\Phi(3C''_{pmk}) - 1$. Further, we obtain the upper bound on the departure ratio $A/d \le d/(3cd^* + d)$ for all $C''_{pmk} \ge c$. We investigated the statistical properties of the natural estimator of C''_{pmk} assuming that the process is normally distributed. We obtained the exact distribution of the natural estimator. We also obtained the r-th moment, expected value, and the variance of the natural estimator. In addition, we derived the cumulative distribution function and the probability density function of the natural estimator.

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