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超對稱可積分系統的研究
A Study of Supersymmetric Integrable System

Hamiltonian structures of generalized Manin–Radul super-KdV and constrained super KP hierarchies

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A study of Hamiltonian structures associated with supersymmetric Lax operators is presented. Following a constructive approach, the Hamiltonian structures of Inami–Kanno super-KdV hierarchy and constrained modified super-KP hierarchy are investigated from the reduced supersymmetric Gelfand–Dickey brackets. By applying a gauge transformation on the Hamiltonian structures associated with these two nonstandard super-Lax hierarchies, we obtain the Hamiltonian structures of generalized Manin–Radul super-KdV and constrained super-KP hierarchies. We also work out a few examples and compare them with the known results. © *1999 American Institute of Physics.* [S0022-2488(99)02206-9]

I. INTRODUCTION

In the past decade and more, the supersymmetric integrable systems have received much attention in the literature (for recent reviews, see Refs. 1–3 and references therein), especially in the explorations of the relationship to the supersymmetric conformal field theories and string theories. On the one hand, in superconformal/superstring theories, correlation functions are governed by supersymmetric extensions of the Korteweg–de Vries (KdV) [or Kadomtsev–Petviashvili (KP)] systems. On the other hand, the knowledge of super-KdV/KP systems have motivated people to study nonperturbative properties of superstrings. These superintegrable systems share many features in common: they have supersymmetric Lax representations, infinitely many conserved quantities and soliton solutions, etc. Furthermore, it is a common belief that they also possess bi-Hamiltonian structures that define the dynamical flows on the corresponding Poisson supermanifolds. In particular, for the super-KdV-type systems, the Poisson brackets relative to their associated second Hamiltonian structures provide extended superconformal algebras (W superalgebras) whose quantum versions serve as the highest weight representations of some infinite-dimensional symmetries in string theories.

The main purpose of this paper is to construct the Hamiltonian structures of the generalized Manin–Radul super-KdV (MR sKdV) and constrained super-KP (csKP) hierarchies (for the definitions of these hierarchies, see Sec. IV) using the method of gauge transformation. Although the Hamiltonian structures for the simplest cases have been obtained in Refs. 4 and 5, however, to our knowledge, those for the general cases are still unexplored. Our motivation comes from the fact that, for two gauge-equivalent integrable systems, the gauge transformation between them transforms not only the Lax formulations but also the Hamiltonian structures of the corresponding hierarchies. Hence, the preparation of suitable superintegrable systems that are gauge equivalent to the generalized MR sKdV and csKP hierarchies is the key in this approach. Our strategy is the following: First, for an odd-order super-Lax operator $\hat{L}$, we consider its associated supersymmet-

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ric Gelfand–Dickey (GD) bracket[6] defined by the Hamiltonian map J. We then consider a usual reduction that modifies the Hamiltonian map J to J_c. Second, we construct out two nonstandard superhierarchies from $(\hat{L}, J_c)$ that have super-Lax operators defined by $K_A = \hat{L}D$ and $K_B = D^{-1}\hat{L}$, respectively. The former is referred to the Inami–Kanno sKdV (IK sKdV) hierarchy,[7] whereas the latter to the constrained modified sKP (cmsKP) hierarchy.[8–10] The Hamiltonian structures associated with K_i can also be constructed from J_c and are denoted by $\Omega^{(i)}$ ($i = A, B$). Finally, we perform a gauge transformation on the systems $(K_i, \Omega^{(i)})$ and denote the resulting systems by $(\tilde{L}_i, \Theta^{(i)})$, which describe the Lax operators and the Hamiltonian structures of the generalized MR sKdV and csKP hierarchies.

In summary, we shall follow the following steps to achieve the goal:

$$(\hat{L}, J_c) \rightarrow (K_i, \Omega^{(i)}) \rightarrow (\tilde{L}_i, \Theta^{(i)}). \tag{1.1}$$

It will be shown below that each step described above automatically guarantees the requirement that the associated Hamiltonian structures should obey the super-Jacobi identity.

We organize this paper as follows: In Sec. II, we recall some basic facts concerning superpseudodifferential operators (SPDOs). We then introduce the second supersymmetric GD bracket and its reduction from a Miura transformation viewpoint. In Sec. III, the IK sKdV and the cmsKP hierarchies are defined. We give a detailed construction of their associated Hamiltonian structures from the reduced supersymmetric GD bracket. We find that, up to a sign, the Poisson brackets defined by their corresponding Lax operators have the same form. In Sec. IV, we define the generalized MR sKdV and csKP hierarchies by applying a gauge transformation to the IK sKdV and cmsKP hierarchies, respectively. We also show that this gauge transformation enables us to obtain the Hamiltonian structures associated with the generalized MR sKdV and csKP hierarchies. In Sec. V, we give several examples to compare them with the known results. We present our concluding remarks in Sec. VI.

II. SUPERSYMMETRIC GELFAND–DICKEY BRACKETS

To begin with, we consider the supersymmetric Lax operator of the form

$$L = D^n + U_{n-1}D^{n-1} + \cdots + U_0, \tag{2.1}$$

where the supercovariant derivative $D \equiv \partial_\theta + \theta \partial$ ($\partial \equiv \partial/\partial x$) satisfies $D^2 = \partial$, θ is the Grassmann variable ($\theta^2 = 0$), which together with the even variable $x \equiv t_1$ defines the $(1|1)$ superspace with coordinate (x, θ). The coefficients U_i are superfields that depend on the variables θ, t_i and can be represented by $U_i = u_i(t) + \theta v_i(t)$. The parity of a superfield U is denoted by $|U|$, which is zero for U being even and one for U being odd. Since L is assumed to be homogeneous under Z_2 grading, thus $|U_i| = n - i \pmod 2$. We will introduce the Poisson bracket associated with L on functionals of the form

$$F(U) = \int_B f(U), \tag{2.2}$$

where $f(U)$ is a homogeneous differential polynomial of U_i and $\int_B \equiv \int dx\, d\theta$ is the Berezin integral, such that if $f(U) = a(u, v) + \theta b(u, v)$, then $\int_B f(U) = \int b$. The supercovariant derivative D satisfies the supersymmetric version of the Leibniz rule;[11]

$$D^i U = \sum_{k=0}^{\infty} (-1)^{|U|(i-k)} \begin{bmatrix} i \\ k \end{bmatrix} U^{[k]} D^{i-k}, \tag{2.3}$$

where $U^{[k]} \equiv (D^k U)$ and the superbinomial coefficients $\begin{bmatrix} i \\ k \end{bmatrix}$ are defined by

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$$\begin{bmatrix} i \\ k \end{bmatrix} = \begin{cases} \begin{pmatrix} [i/2] \\ [k/2] \end{pmatrix} & \text{for } 0 \leqslant k \leqslant i \quad \text{and} \quad (i,k) \neq (0,1) \bmod 2, \\ (-1)^{[k/2]} \begin{bmatrix} -i+k-1 \\ k \end{bmatrix}, & \text{for } i < 0, \\ 0, & \text{otherwise.} \end{cases}$$ (2.4)

For a given SPDO $P = \Sigma p_i D^i$, it is convenient to separate P into a direct sum of two linear spaces: $P = P_{\geqslant k} \oplus P_{<k}$ with $P_{\geqslant k} = \Sigma_{i \geqslant k} p_i D^i$ and $P_{<k} = \Sigma_{i < k} p_i D^i$. In particular, we denote $P_- = \Sigma_{i \geqslant 0} p_i D^i$, $P_- = \Sigma_{i < 0} p_i D^i$, and $(P)_0 = p_0$. We also define its super-residue as sres $P = p_{-1}$ and its supertrace as Str $P = \int_B$ sres P. It can be shown that, for any two SPDOs P and Q, Str$[P,Q]$ $= 0$ for $[P,Q] \equiv PQ - (-1)^{|P| |Q|} QP$, and hence Str $PQ = (-1)^{|P| |Q|}$ Str QP. Given a functional $F(U) = \int_B f(U)$, we define its gradient as

$$d_L F = \sum_{k=0}^{n-1} (-1)^k D^{-k-1} \frac{\delta f}{\delta U_k},$$ (2.5)

and its variation as

$$\delta F = (-1)^{|F| + |L| + 1} \text{Str}(\delta L d_L F),$$ (2.6)

where the variational derivative is defined by

$$\frac{\delta f}{\delta U_k} = \sum_{i=0}^{\infty} (-1)^{|U_k| i + i(i+1)/2} \left(\frac{\partial f}{\partial U_k^{(i)}} \right)^{[i]}.$$ (2.7)

The supersymmetric second GD bracket associated with L is given by[6,12,13]

$$\{F, G\}(L) = (-1)^{|F| + |G| + |L| + 1} \text{Str}[J(d_L F) d_L G],$$ (2.8)

where the Hamiltonian map J is defined by

$$J(X) = (LX)_+ L - L(XL)_+,$$ (2.9)

where $X = \Sigma_k X_k D^k$. It has been shown[6,13] that (2.8) indeed defines a Hamiltonian structure, namely, it is antisymmetric and satisfies the super-Jacobi identity.

If we factorize $L = (D - \Phi_n)(D - \Phi_{n-1}) \cdots (D - \Phi_1)$, which defines a supersymmetric Miura transformation between the coefficient functions U_i and the Miura fields Φ_i, then the second GD bracket (2.8) becomes

$$\{F, G\}(L) = \int_B \sum_{i=1}^{n} (-1)^i \left(D \frac{\delta f}{\delta \Phi_i} \right) \frac{\delta g}{\delta \Phi_i},$$ (2.10)

which implies that the fundamental brackets of the Miura fields Φ_i are given by[6,12]

$$\{\Phi_i(X), \Phi_j(Y)\} = (-1)^i \delta_{ij} D \delta(X - Y),$$ (2.11)

where $X = (x, \theta)$, $Y = (y, \omega)$ and $\delta(X - Y) \equiv \delta(x - y)(\theta - \omega)$. This result is what we called the supersymmetric Kupershmidt–Wilson theorem. Equation (2.11) enables us to write down the fundamental brackets of U_k through the super-Miura transformation.

Next, let us consider the case when the constraint $U_{n-1} = 0$ is imposed in (2.1). It can be easily shown that such constraint for odd n is second class, which will modify the Hamiltonian structure J. On the other hand, for even n, the constraint is first class and hence the induced

Poisson brackets can not be well defined. Therefore, for the odd-order operator $\hat{L} = D^{2k+1} + U_{2k-1}D^{2k-1} + \cdots + U_0$, we shall consider the factorization $\hat{L} = (D - \Phi_{2k-1})(D - \Phi_{2k}) \cdots (D - \Phi_1)$. Then the modified Poisson bracket defined by $\hat{L}$ becomes

$$\{F, G\}_c = (-1)^{|F| - |G|} \operatorname{Str}(J_c(\partial F)\partial G), \tag{2.12}$$

where $\partial F \equiv d_L F = \sum_{i=0}^{2k-1} (-1)^i D^{-i-1}(\delta f/\delta U_i)$ and

$$J_c(\partial F) = J(\partial F) + \left[\hat{L}, \int^x D \operatorname{sres}[\hat{L}, \partial F] \right]. \tag{2.13}$$

We remark that the second term is called the third GD structure, which is compatible with the second structure. Equation (2.12) yields that the modified Poisson brackets for the Miura fields Φ_i are given by

$$\{\Phi_i(X), \Phi_j(Y)\}_c = [1 + (-1)^i \delta_{ij}] D \delta(X - Y), \tag{2.14}$$

which provide the free-field realizations of classical W superalgebras associated with the odd-order Lax operator $\hat{L}$.[12,14,15] Besides the usual reduction described above, there are other reductions that have been discussed in Refs. 13 and 16. Since the first Hamiltonian structure can be obtained from the second Hamiltonian structure by replacing L by $L + \lambda$, where λ is called the spectral parameter, we shall focus only on the second structure.

III. TWO NONSTANDARD SUPER-LAX HIERARCHIES

There are several superintegrable hierarchies whose Lax operators are related to the modifications or reductions of the supersymmetric Lax operator (2.1) in the literature. Here, for our purpose, we consider the following two Lax systems:

$$\frac{dK_i}{dt_k} = [(K_i^{k/n})_{\geq 1}, K_i] \quad (i = A, B), \tag{3.1}$$

with the Lax operators K_i defined by

$$K_A = D^{2n} + V_{2n-2}D^{2n-2} + \cdots + V_1 D, \tag{3.2}$$

$$K_B = D^{2n} + V_{2n-2}D^{2n-2} + \cdots + V_0 + D^{-1}V_{-1}. \tag{3.3}$$

The Lax equation for K_A is referred to the IK sKdV hierarchy.[7] The simplest example in this case is just the Laberge–Mathieu super KdV (LM sKdV) hierarchy ($n = 2$), which was constructed from a $N = 2$ sKdV hierarchy.[17] On the other hand, the Lax equation for K_B is the generalization of the super two-boson hierarchy (sTB) ($n = 1$),[18] which we call the cmsKP hierarchy. In particular, from (3.1) it is easy to show that the coefficient function V_{-1} obeys the evolution equation

$$\frac{dV_{-1}}{dt_k} = -((K_B^{k/n})^*_{\geq 1} V_{-1}), \tag{3.4}$$

which implies that V_{-1} is an adjoint eigenfunction associated with the Lax operator K_B.

In general, the second Poisson brackets associated with the Lax operators K_i can be written as

$$\{F, G\}^{(i)}(K_i) = (-1)^{|F| - |G| + 1} \operatorname{Str}(\Omega^{(i)}(d_i F)d_i G), \tag{3.5}$$

where $d_i F \equiv d_{K_i} F$, and the Hamiltonian maps $\Omega^{(i)}$ are defined by

$$\Omega^{(A)}(d_A F) = (K_A d_A F)_- K_A - K_A (d_A F K_A)_+ + [K_A, (d_A F K_A)_0]$$

$$+ (-1)^{|F|} \left[\int^x D \ \mathrm{sres}[d_A F, K_A], K_A \right] + (-1)^{|F|} K_A D^{-1} \ \mathrm{sres}[d_A F, K_A], \quad (3.6)$$

$$\Omega^{(B)}(d_B F) = (K_B d_B F)_+ K_B - K_B (d_B F K_B)_+ + [K_B, (K_B d_B F)_0]$$

$$+ (-1)^{|F|} \left[K_B, \int^x D \ \mathrm{sres}[d_B F, K_B] \right] + (-1)^{|F|} D^{-1} \ \mathrm{sres}[d_B F, K_B] K_B. \quad (3.7)$$

Notice that the map $\Omega^{(A)}$, in operator form, is similar to but different from $\Omega^{(B)}$. Instead of giving $\Omega^{(i)}$ by other methods,[8–10,19] we will follow a constructive approach, analogous to that of the supersymmetric GD structure,[6] to verify the Hamiltonian maps $\Omega^{(i)}$ from a supersymmetric Miura transformation point of view. To show that the maps $\Omega^{(i)}$ are indeed Hamiltonian, we have to check that the Poisson brackets defined in (3.5) are antisymmetric and obey the super-Jacobi identity. For antisymmetry, by direct computation, it can be easily shown that

$$\{F, G\}^{(i)} = -(-1)^{|F \cdot G|} \{G, F\}^{(i)}. \quad (3.8)$$

For the super-Jacobi identity, instead of direct computation, we rewrite the Lax operator K_i as

$$K_A = \hat{L}_A D, \quad K_B = D^{-1} \hat{L}_B, \quad (3.9)$$

where $\hat{L}_A$ and $\hat{L}_B$ are superdifferential operators with order $2n-1$ and $2n+1$, respectively. Furthermore, from the relation

$$\delta F = (-1)^{|F|+1} \ \mathrm{Str}(\delta K_i d_i F) = (-1)^{|F|} \ \mathrm{Str}(\delta \hat{L}_i \hat{\partial}_i F), \quad (3.10)$$

where $\hat{\partial}_i \equiv d_{\hat{L}_i}$, we have

$$\hat{\partial}_A F = -D d_A F, \quad \hat{\partial}_B F = (-1)^{|F|} d_B F D^{-1}. \quad (3.11)$$

Substituting (3.9) and (3.11) into (3.6) and (3.7), we find

$$\Omega^{(A)}(d_A F) = -J_c(\hat{\partial}_A F) D, \quad \Omega^{(B)}(d_B F) = (-1)^{|F|} D^{-1} J_c(\hat{\partial}_B F), \quad (3.12)$$

which imply that the Poisson brackets defined by K_i can be transformed to those defined by $\hat{L}_i$ as follows:

$$\{F, G\}^{(i)}(K_i) = \eta_i \{F, G\}_c(\hat{L}_i), \quad (3.13)$$

where $\eta_A = -1$ and $\eta_B = +1$. Hence, the super-Jacobi identity associated with the maps $\Omega^{(i)}$ is automatically satisfied due to the fact that the reduced supersymmetric GD brackets defined by $\hat{L}_i$ admit Miura representations (2.14).

Therefore the maps $\Omega^{(i)}$ provide the Hamiltonian formulation for the Lax equations (3.1):

$$\frac{dK_i}{dt_k} = \{H_k^{(i)}, K_i\}^{(i)} = \Omega^{(i)}(d_i H_k^{(i)}). \quad (3.14)$$

where the Hamiltonian functionals $H_k^{(i)}$ are given by

$$H_k^{(i)} = -\frac{n}{k} \mathrm{Str}(K_i^{k/n}). \quad (3.15)$$

Notice that the relative signs in the Hamiltonian maps $\Omega^{(i)}$ are crucial. It is this choice so that $\Omega^{(i)}(d_t H_k^{(i)})$ are differential operators of order less than $2n-2$, and Eq. (3.14) makes sense.

Before ending this section, two remarks are in order. First, we note that both Piosson brackets defined by K_i, up to a sign, are mapped to the same reduced supersymmetric GD bracket defined by $\tilde{L}_i$, which is different from the situation in the bosonic case, where type A is mapped to the *difference* of the second and the third GD structures,[20] whereas type B is the *sum* of the second and the third ones.[20,21] Second, both Lax operators K_A and K_B can be factorized into multiplicative forms, i.e.,

$$K_A = (D - \Phi_{2n-1})(D - \Phi_{2n-2}) \cdots (D - \Phi_1)D,$$

$$K_B = D^{-1}(D - \Phi_{2n-1})(D - \Phi_{2n}) \cdots (D - \Phi_1),$$

(3.16)

where the Miura fields Φ_i obey the Poisson brackets,

$$\{\Phi_j(X), \Phi_k(Y)\}^{(i)} = \eta_i[1 + (-1)^j \delta_{jk}]D\delta(X-Y).$$

(3.17)

IV. GENERALIZED MR SKDV AND CONSTRAINED SKP HIERARCHIES

Having constructed the Hamiltonian structures of two nonstandard super-Lax hierarchies in the previous section, we are now ready to discuss gauge equivalences related to these two non-standard hierarchies. Based on the fact that gauge transformations are canonical transformations, we can use them to obtain new integrable Hamiltonian systems from the known ones. In the following, we will show that the second Hamiltonian structures of the generalized MR sKdV and csKP hierarchies are just the ones that can be obtained in this way.

Let us perform the following gauge transformation to the Lax operators K_i:

$$\tilde{L}_i = T^{-1}K_i T \quad (i = A, B),$$

(4.1)

where the gauge operator T is defined by $T = \exp(-\int^x V_{2n-2}/n)$, and hence the next leading term of K_i can be gauged away. The resulting differential operators $\tilde{L}_i$ are thus given by

$$\tilde{L}_A = D^{2n} + U_{2n-3}D^{2n-3} + \cdots + U_0,$$

$$\tilde{L}_B = D^{2n} + U_{2n-3}D^{2n-3} + \cdots + U_0 + \phi D^{-1}\psi,$$

(4.2)

where $\phi \equiv T^{-1}$ and $\psi \equiv V_{-1}T$. It can be proved that T^{-1} is an even eigenfunction associated with the operator $\tilde{L}_i$, i.e., $\partial T^{-1}/\partial t_k = ((\tilde{L}_i^{k/n})_+ T^{-1})_0$, and the nonstandard Lax equations in (3.1) are then transformed to the standard ones,

$$\frac{d\tilde{L}_i}{dt_k} = [(\tilde{L}_i^{k/n})_+, \tilde{L}_i].$$

(4.3)

Therefore the gauge transformation (4.1) provides a connection between K_i and $\tilde{L}_i$ in the Lax formulation. For $\tilde{L}_A$, the Lax equation (4.3) gives the generalization of the MR sKdV hierarchy ($n=2$), which was originally constructed from the MR sKP hierarchy by reduction.[11] On the other hand, the Lax equation (4.3) for $\tilde{L}_B$ describes the csKP hierarchy that contains the sAKNS hierarchy ($n=1$)[5,22] as the simplest example. It can be easily shown that the Lax equation (4.3) for $\tilde{L}_B$ is consistent with the following equations:

$$\frac{\partial \phi}{\partial t_k} = ((\tilde{L}_B^{k/n})_+ \phi)_0, \quad \frac{\partial \psi}{\partial t_k} = -((\tilde{L}_B^{k/n})_+^* \psi)_0,$$

(4.4)

and thus ϕ and ψ are an even eigenfunction and an odd adjoint eigenfunction of the csKP hierarchy, respectively.

Moreover, since the hierarchy flows associated with K_i have Hamiltonian descriptions, it is quite natural to ask whether we can use such gauge equivalence to obtain the second Hamiltonian structures of the generalized MR sKdV and csKP hierarchies. The answer is yes. To see this, consider an infinitesimal gauge transformation $K_i \rightarrow K_i + Q$, where Q is a homogeneous superdifferential operator of order, at most, $2n-2$. Then, in view of (4.1), we can read off the linearized map T' and its transposed map $T'^{\dagger}$ as

$$T': Q \rightarrow T^{-1}QT + \frac{1}{n}\left[\int^x q_{2n-2}, \tilde{L}_i\right], \tag{4.5}$$

$$T'^{\dagger}: P \rightarrow TPT^{-1} + \frac{(-1)^{|P|+1}}{n}\int^x \mathrm{sres}[P, \tilde{L}_i], \tag{4.6}$$

where P is an arbitrary SPDO, $q_{2n-2} \equiv \mathrm{sres}(QD^{-2n+1})$, and the adjoint of an operator R is defined by $\mathrm{Str}(PRQ) = (-1)^{|R||P|}\,\mathrm{Str}(R^{\dagger}PQ)$. Using T' and $T'^{\dagger}$, a straightforward but tedius calculation (see Appendix A) shows that

$$T'\Theta^{(i)}T'^{\dagger}(P) = (\tilde{L}_iP)_+\tilde{L}_i - \tilde{L}_i(P\tilde{L}_i)_+ + \frac{1}{n}\left[\int^x \mathrm{res}[P,\tilde{L}_i], \tilde{L}_i\right] + \frac{1}{n}\left[\left(\int^x \mathrm{sres}[P,\tilde{L}_i]\right)D, \tilde{L}_i\right]$$

$$- \frac{2}{n^2}\left[\int^x\left(\left(\int^{x'} \mathrm{sres}[P,\tilde{L}_i]\right)U_{2n-3}\right), \tilde{L}_i\right] = \Theta^{(i)}(P). \tag{4.7}$$

That means the Hamiltonian maps $\Theta^{(A)}$ and $\Theta^{(B)}$, in terms of their own Lax operators, have the same form. Since $\Theta^{(i)}$ are canonical equivalent to the Hamiltonian map $\Omega^{(i)}$, the Poisson brackets defined by $\Theta^{(i)}$ are also antisymmetric and obey the super-Jacobi identity. As a result, $\Theta^{(A)}(\Theta^{(B)})$ can be defined as the Hamiltonian map of the generalized MR sKdV (csKP) hierarchy. A further consistent check shows that $\Theta^{(i)}$ map the Hamiltonian one-forms $\tilde{d}_i\tilde{H}_k^{(i)}$ to (pseudo-) superdifferential operators of order, at most, $2n-3$. Now we can write down the Hamiltonian flows associated with the Lax operators $\tilde{L}_i$ as

$$\frac{d\tilde{L}_i}{dt_k} = \{\tilde{H}_k^{(i)}, \tilde{L}_i\} = \Theta^{(i)}(\tilde{d}_i\tilde{H}_k^{(i)}), \tag{4.8}$$

where the Hamiltonian functionals, in view of (3.15) and (4.1), are defined by

$$\tilde{H}_k^{(i)} = -\frac{n}{k}\mathrm{Str}\,\tilde{L}_i^{k/n}. \tag{4.9}$$

From the Hamiltonian flows (4.8) we can read off the Poisson brackets for the coefficient functions of $\tilde{L}_i$.

In fact, for $\tilde{L}_B$, we can express the associated Poisson brackets for U_i, ϕ, and ψ more precisely. Let us rewrite $\tilde{L}_B = 1 + \phi D^{-1}\psi$ and denote $H = \int_B h$ as one of the Hamiltonian functionals $\tilde{H}_k^{(B)}$. Then the Hamiltonian one-form can be expressed as

$$\tilde{d}_B H = d_i H + X, \tag{4.10}$$

where X is a superdifferential operator and

$$d_i H = \sum_{k=0}^{2n-3}(-1)^k D^{-k-1}\frac{\delta h}{\delta U_k}. \tag{4.11}$$

Then, from the relation

$$\delta H = -\mathrm{Str}((\delta l + \delta\phi D^{-1}\psi + \phi D^{-1}\delta\psi)(d_i H + X)) = -\mathrm{Str}(\delta l d_i H) + \int_B \left(\delta\phi\frac{\delta h}{\delta\phi} + \delta\psi\frac{\delta h}{\delta\psi}\right),$$

(4.12)

we have the following identifications:

$$\frac{\delta h}{\delta\phi} = (X^*\psi)_0, \quad \frac{\delta h}{\delta\psi} = (X\phi)_0.$$

(4.13)

Inserting (4.10) with X satisfying (4.13) into the Hamiltonian map $\Theta^{(B)}$ gives

$$\frac{dl}{dt} = (l d_i H)_+ l - l(d_i H l)_+ + ((l d_i H)_+ \phi D^{-1}\psi)_+ - (\phi D^{-1}\psi(d_i H l)_+)_+ + \left(l\frac{\delta h}{\delta\psi}D^{-1}\psi\right)_-$$

$$- \left(\phi D^{-1}\frac{\delta h}{\delta\phi}l\right)_+ + \frac{1}{n}\left[\int^x \mathrm{res}[\tilde{d}_B H, \tilde{L}_B], l\right] - \frac{2}{n}\phi\psi\int^x \mathrm{sres}[\tilde{d}_B H, \tilde{L}_B]$$

$$+ \frac{1}{n}\left[\int^x_{\sim} \mathrm{sres}[\tilde{d}_B H, \tilde{L}_B], l\right] + \frac{2}{n^2}\left[\int^x\left(U_{2n-3}\int^{x'}\mathrm{sres}[\tilde{d}_B H, \tilde{L}_B]\right), l\right],$$

$$\frac{d\phi}{dt} = ((l d_i H)_+ \phi)_0 + \left(l\frac{\delta h}{\delta\psi}\right)_0 + \phi\left[\int^x\left(D\psi\frac{\delta h}{\delta\psi}\right) - \int^x\left(D\phi\frac{\delta h}{\delta\phi}\right)\right] + \frac{1}{n}\phi\int^x \mathrm{res}[\tilde{d}_B H, \tilde{L}_B]$$

$$- \frac{1}{n}(D\phi)\int^x \mathrm{sres}[\tilde{d}_B H, \tilde{L}_B] + \frac{2}{n^2}\phi\int^x\left(U_{2n-3}\int^{x'}\mathrm{sres}[\tilde{d}_B H, \tilde{L}_B]\right),$$

$$\frac{d\psi}{dt} = -((l^*(d_i H)^*)_+ \psi)_0 - \left(l^*\frac{\delta h}{\delta\phi}\right)_0 + \psi\left[\int^x\left(D\phi\frac{\delta h}{\delta\phi}\right) - \int^x\left(D\psi\frac{\delta h}{\delta\psi}\right)\right] - \frac{1}{n}\psi\int^x \mathrm{res}[\tilde{d}_B H, \tilde{L}_B]$$

$$+ \frac{1}{n}\left(D\psi\int^x \mathrm{sres}[\tilde{d}_B H, \tilde{L}_B]\right) - \frac{2}{n^2}\psi\int^x\left(U_{2n-3}\int^{x'}\mathrm{sres}[\tilde{d}_B H, \tilde{L}_B]\right),$$

(4.14)

where

$$\mathrm{res}[\tilde{d}_B H, \tilde{L}_B] = \mathrm{res}[d_i H, l] + (D\psi)\frac{\delta h}{\delta\psi} - \phi\left(D\frac{\delta h}{\delta\phi}\right) - \mathrm{sres}(d_i H\phi\psi) - \phi(D(d_i H)^*\psi),$$

$$\mathrm{sres}[\tilde{d}_B H, \tilde{L}_B] = \mathrm{sres}[d_i H, l] - \psi\frac{\delta h}{\delta\psi} + \phi\frac{\delta h}{\delta\phi}.$$

(4.15)

Equation (4.14) can be regarded as the supersymmetric generalization of the second Hamiltonian structures of constrained KP hierarchy derived by Oevel and Strampp.[23]

V. EXAMPLES

In this section we work out a number of examples to illustrate the previous results explicitly. We write down the Poisson brackets for these systems according to the formulas given above and compare them with the known results.

A. Laberge–Mathieu super-KdV hierarchy

For $K_A = \partial^2 + v_2 \partial + v_1 D$, the first equations in (3.1) are given by

$$\frac{d}{dt_0}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_{1x} \\ v_{2x} \end{pmatrix},$$

$$\frac{d}{dt_1}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} (v_{1xx}+3v_1(Dv_1)-\frac{3}{2}v_1 v_2^2 - 3v_1 v_{2x})_x \\ (v_{2xx}-\frac{1}{2}v_2^3 + 3v_1(Dv_2))_x \end{pmatrix}, \qquad (5.1)$$

which represents the first equations of the LM sKdV hierarchy. The Hamiltonian formulation for these equations is given by (3.14), where the second Poisson structure can be obtained by substituting $d_A H_k^{(A)} = -D^{-2}(\delta h_k^{(A)}/\delta v_1) + D^{-3}(\delta h_k^{(A)}/\delta v_2)$ into (3.6). We find

$$\frac{d}{dt_k}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2v_1\partial - v_{1x} & -\partial^2 - v_2\partial + v_1 D - (Dv_1) \\ \partial^2 - v_2\partial + v_1 D - v_{2x} & -2D^3 + (Dv_2) - 2v_1 \end{pmatrix}\begin{pmatrix} \dfrac{\delta h_k^{(A)}}{\delta v_1} \\ \dfrac{\delta h_k^{(A)}}{\delta v_2} \end{pmatrix}, \qquad (5.2)$$

where the first Hamiltonian functionals are given by

$$H_0^{(A)} = -2\,\text{Str}\,K_A^{1/2} = -\int_B v_1,$$

$$H_1^{(A)} = -\frac{2}{3}\,\text{Str}\,K_A^{3/2} = -\frac{3}{8}\int_B\left[\frac{1}{2}v_1 v_2^2 + v_1 v_{2x} - v_1(Dv_1)\right]. \qquad (5.3)$$

To compare with the known result, we consider the change of variables as follows:

$$(v_1, v_2) \rightarrow (-(Du)-\tau, -2u), \qquad (5.4)$$

then the Poisson structure in (5.2) becomes

$$\frac{1}{2}\begin{pmatrix} -D\partial + \tau & 2u\partial - (Du)D + 2u_x \\ 2u\partial - (Du)D + u_x & -D\partial^2 + 3\tau\partial + (D\tau)D + 2\tau_x \end{pmatrix}, \qquad (5.5)$$

which is just the form presented in Ref. 24.

B. Super-two-boson hierarchy

For $K_B = \partial + v_0 + D^{-1}v_{-1}$ the first Lax equations in (3.1) are given by

$$\frac{d}{dt_1}\begin{pmatrix} v_0 \\ v_{-1} \end{pmatrix} = \begin{pmatrix} v_{0x} \\ v_{-1x} \end{pmatrix},$$

$$\frac{d}{dt_2}\begin{pmatrix} v_0 \\ v_{-1} \end{pmatrix} = \begin{pmatrix} v_{0xx} + 2(Dv_{-1})_x + (v_0^2)_x \\ -v_{-1xx} + 2(v_0 v_{-1})_x \end{pmatrix}, \qquad (5.6)$$

which represents the first equations of the sTB hierarchy. The Hamiltonian description for these equations are given by (3.14), where the second Poisson structure can be obtained by substituting $d_B H_k^{(B)} = D^{-1}(\delta h_k^{(B)}/\delta v_0) + (\delta h_k^{(B)}/\delta v_{-1})$ into (3.7). It turns out that

$$\frac{d}{dt_k}\begin{pmatrix} v_0 \\ v_{-1} \end{pmatrix} = \begin{pmatrix} 2D^3+(Dv_0)+2v_{-1} & \partial^2+v_0\partial+v_{-1}D+v_{0x} \\ -\partial^2+v_0\partial+v_{-1}D-(Dv_{-1}) & 2v_{-1}\partial+v_{-1x} \end{pmatrix}\begin{pmatrix} \dfrac{\delta h_k^{(B)}}{\delta v_0} \\ \dfrac{\delta h_k^{(B)}}{\delta v_{-1}} \end{pmatrix}, \quad (5.7)$$

where the first Hamiltonian functionals are given by

$$H_1^{(B)} = -\operatorname{Str} K_B = -\int_B v_{-1},$$

$$H_2^{(B)} = -\frac{1}{2}\operatorname{Str} K_B^2 = \int_B v_0 v_{-1}. \tag{5.8}$$

Equation (5.7) provides the second Hamiltonian formulation of the sTB hierarchy.
If we make the following identification:

$$(v_0, v_{-1}) \rightarrow (-(DJ_0), J_1), \tag{5.9}$$

then the second Poisson structure in (5.7) becomes

$$\begin{pmatrix} 2D+2D^{-1}J_1D^{-1}-D^{-1}J_{0x}D^{-1} & -D^3+D(DJ_0)-D^{-1}J_1D \\ D^3+(DJ_0)D+DJ_1D^{-1} & J_1D^2+D^2J_1 \end{pmatrix}. \tag{5.10}$$

which is the form of the second Poisson structure discussed in Ref. 18.

C. Manin–Radul super-KdV hierarchy

For $\tilde{L}_A = \partial^2 - \varphi D + a$, the first Lax equations in (4.3) are given by

$$\frac{d}{dt_0}\begin{pmatrix} a \\ \varphi \end{pmatrix} = \begin{pmatrix} a_x \\ \varphi_x \end{pmatrix},$$

$$\frac{d}{dt_1}\begin{pmatrix} a \\ \varphi \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \varphi_{xxx}-3(\varphi(D\varphi))_x+6(a\varphi)_x \\ a_{xxx}-3(\varphi(Da))_x+3(a^2)_x \end{pmatrix}, \tag{5.11}$$

which represents the first equations of the MP sKdV hierarchy. The Hamiltonian formulation of these equations are given by (4.8), in which the first Hamiltonian functionals are given by

$$\tilde{H}_0^{(A)} = -2\operatorname{Str}\tilde{L}_A^{1/2} = \int_B \varphi,$$

$$\tilde{H}_1^{(A)} = -\frac{2}{3}\operatorname{Str}\tilde{L}_A^{3/2} = -\frac{1}{4}\int_B [\varphi(D\varphi)-2\varphi a], \tag{5.12}$$

and the second Poisson structure can be obtained by substituting $\tilde{d}_A \tilde{H}_k^{(A)} = D^{-1}(\delta\tilde{h}_k^{(A)}/\delta a) + D^{-2}(\delta\tilde{h}_k^{(A)}/\delta\varphi)$ into (4.7). It turns out that

$$\frac{d}{dt_k}\begin{pmatrix} a \\ \varphi \end{pmatrix} = \begin{pmatrix} P_{aa} & P_{a\varphi} \\ P_{\varphi a} & P_{\varphi\varphi} \end{pmatrix}\begin{pmatrix} \dfrac{\delta\tilde{h}_k^{(A)}}{\delta a} \\ \dfrac{\delta\tilde{h}_k^{(A)}}{\delta\varphi} \end{pmatrix}, \tag{5.13}$$

where the second Poisson matrix is given by

$$P_{aa} = \tfrac{1}{2}[D\partial^3 - 3\varphi\partial^2 + 4aD\partial + (2(Da) - 3\varphi_x)\partial + 2a_xD + 3\varphi(D\varphi) + (D^3a) - 4a\varphi - \varphi_{xx}$$

$$+ \varphi D^{-1}(Da) - (Da)D^{-1}\varphi - \varphi D^{-1}\varphi D^{-1}\varphi - \varphi D^{-1}\varphi_x + \varphi_x D^{-1}\varphi],$$

$$P_{a\varphi} = \tfrac{1}{2}[\partial^3 - 2\varphi D\partial + 4a\partial - \varphi_x D + 2a_x + \varphi D^{-1}(D\varphi)],$$

$$P_{\varphi a} = \tfrac{1}{2}[\partial^3 + 2\varphi D\partial + (4a - 2(D\varphi))\partial + \varphi_x D + 2a_x - (D^3\varphi) + (D\varphi)D^{-1}\varphi],$$

$$P_{\varphi\varphi} = \tfrac{1}{2}[4\varphi\partial + 2\varphi_x].$$

(5.14)

Equation (5.13) provides the second Hamiltonian formulation of the MR sKdV hierarchy reported in Ref. 4.

Starting from the Lax operator $K_A = \partial^2 + v_2\partial + v_1 D$ associated with the LM sKdV hierarchy, one can perform the gauge transformation $T = \exp(-\int^x v_2/2)^7$ on the Lax operator K_A as follows:

$$K_A \rightarrow \tilde{L}_A = e^{\int^x v_2/2} K_A e^{-\int^x v_2/2} = \partial^2 + v_1 D - \left(\frac{v_2^2}{4} + \frac{v_{2x}}{2} + \frac{v_1(D^{-1}v_2)}{2}\right). \tag{5.15}$$

Then the Lax operator $\tilde{L}_A = \partial^2 - \phi D + a$ associated with the MR sKdV hierarchy is related to the Lax operator K_A as

$$\phi = -v_1, \quad a = -\left(\frac{v_2^2}{4} + \frac{v_{2x}}{2} + \frac{v_1(D^{-1}v_2)}{2}\right), \tag{5.16}$$

which provides the gauge equivalence between the LM sKdV hierarchy (5.1) and the MR sKdV hierarchy (5.11). Moreover, it has been shown[24] that the second Hamiltonian structure (5.5) of the LM sKdV hierarchy can be transformed to the second Hamiltonian structure (5.14) of the MR sKdV hierarchy via this gauge transformation.

D. Super-AKNS hierarchy

For $\tilde{L}_B = \partial + \phi D^{-1}\psi$, the first equations in (4.3) are given by

$$\frac{d}{dt_1}\begin{pmatrix}\phi\\\psi\end{pmatrix} = \begin{pmatrix}\phi_x\\\psi_x\end{pmatrix},$$

$$\frac{d}{dt_2}\begin{pmatrix}\phi\\\psi\end{pmatrix} = \begin{pmatrix}\phi_{xx} + 2\phi(D\phi\psi)\\-\psi_{xx} - 2\psi(D\phi\psi)\end{pmatrix}, \tag{5.17}$$

which are the first equations in the sAKNS hierarchy. Hamiltonian formulations for these equations are given by (4.14), where the first Hamiltonian functions are given by

$$\tilde{H}_1^{(B)} = -\operatorname{Str}\tilde{L}_B = \int_B \phi\psi,$$

$$\tilde{H}_2^{(B)} = -\frac{1}{2}\operatorname{Str}\tilde{L}_B^2 = \int_B \phi_x\psi. \tag{5.18}$$

From (4.14), the Hamiltonian flow can be expressed as

$$\frac{d}{dt_k}\begin{pmatrix}\phi\\\psi\end{pmatrix}=\begin{pmatrix}P_{\phi\phi} & P_{\phi\psi}\\P_{\psi\phi} & P_{\psi\psi}\end{pmatrix}\begin{pmatrix}\dfrac{\delta\bar{h}_k^{(S)}}{\delta\phi}\\[2mm]\dfrac{\delta\bar{h}_k^{(S)}}{\delta\psi}\end{pmatrix},\tag{5.19}$$

where the Poisson brackets are given by

$$P_{\phi\phi}=-\phi D^{-1}\phi-\phi D^{-2}\phi D-(D\phi)D^{-2}\phi-2\phi D^{-2}\phi\psi D^{-2}\phi,$$

$$P_{\phi\psi}=D^2+\phi D^{-1}\psi+\phi D^{-2}(D\psi)+(D\phi)D^{-2}\psi+2\phi D^{-2}\phi\psi D^{-2}\psi,$$

$$P_{\psi\phi}=D^2+\psi D^{-2}\phi D+(D\psi)D^{-2}\phi+2\psi D^{-2}\phi\psi D^{-2}\phi,\tag{5.20}$$

$$P_{\psi\psi}=-(D\psi)D^{-2}\psi-\psi D^{-2}(D\psi)-2\psi D^{-2}\phi\psi D^{-2}\psi,$$

which is just the second Poisson structure obtained in Ref. 5. Equation (5.19) provides the second Hamiltonian formulation of the sAKNS hierarchy.

Starting from the Lax operator $K_B=\partial+v_0+D^{-1}v_{-1}$ associated with the sTB hierarchy, one can perform the gauge transformation $T=\exp(-\int^x v_0)$[22,25] to the Lax operator K_B as follows:

$$K_B\to\tilde{L}_B=e^{\int^x v_0}K_B e^{-\int^x v_0}=\partial+e^{\int^x v_0}D^{-1}e^{-\int^x v_0}.\tag{5.21}$$

Then the Lax operator $\tilde{L}_B=\partial+\phi D^{-1}\psi$ associated with the sAKNS hierarchy is related to the Lax operator K_B as

$$\phi=e^{\int^x v_0},\quad a=v_{-1}e^{-\int^x v_0},\tag{5.22}$$

which provides the gauge equivalence between the sTB hierarchy (5.6) and the sAKNS hierarchy (5.17). Moreover, it can be proved[25] that the second Hamiltonian structure (5.10) of the sTB hierarchy can be transformed to the second Hamiltonian structure (5.20) of the sAKNS hierarchy via this gauge transformation.

VI. CONCLUDING REMARKS

In this paper, we investigate the Hamiltonian structures associated with several supersymmetric extensions of the KdV hierarchy. Starting with the reduced super-GD bracket, the Hamiltonian structures of two nonstandard super-KdV hierarchies can be constructed via supersymmetric Miura transformations. We then perform a gauge transformation on these two nonstandard Lax hierarchies to obtain the Hamiltonian structures of the generalized MR sKdV hierarchy and constrained sKP hierarchy in a unified fashion. To compare the obtained Hamiltonian structures with the known results, we work out a few examples, including the LM sKdV, sTB, MR sKdV, and sAKNS hierarchies.

Our approach on the gauge transformation relies on the algebra of superpseudodifferential operators, which provides an effective method to achieve the goal. In fact, the gauge transformation (4.1) that maps $\Omega^{(i)}$ to $\Theta^{(i)}$ is by no means unique. There is another gauge transformation triggered by $S=D^{-1}T$[25,26] that also brings $\Omega^{(i)}$ to $\Theta^{(i)}$. Since the parity of S is odd, the gauge equivalence of the Hamiltonian maps given by (4.7) should be replaced by $S'\Omega^{(i)}S'^{+}=-\Theta^{(i)}$, where the minus sign will be compensated by that induced from the transformation of the Hamiltonians such that the hierarchy flows (3.14) are transformed to (4.8).

Finally, we would like to comment briefly on the algebraic structures associated with the Poisson brackets defined by the Hamiltonian maps $\Omega^{(i)}$ and $\Theta^{(i)}$. As we shows in Eq. (3.13), the Poisson brackets defined by $\Omega^{(i)}$ are encoded by the Poisson bracket defined by J_c. However, it has been shown[2,15] that in the space of the supersymmetric Lax operator of odd order, the reduced supersymmetric GD bracket (2.12) defines an infinite series of classical $N=2$ W superalgebras,

which contain $N=2$ super-Virasoro algebra as a subalgebra. Therefore, through the Miura transformation, the differential polynomials of the coefficient functions V_i of K_i can be identified as the $N=2$ supermultiplets, and Eq. (3.17) provides the free-field realizations of the corresponding W superalgebras. On the other hand, for the MR sKdV and csKP hierarchies, the Poisson algebras defined by $\Theta^{(i)}$ are not quite clear so far, even for the simplest cases. It seems not so obvious to construct the super-Virasoro generator by covariantizing the supersymmetric Lax operator $\tilde{L}_i$ due to the fact that $U_{2n-1}=U_{2n-2}=0$. Therefore, to explore the algebraic structures associated with $\Theta^{(i)}$, the decompositions of coefficient functions U_i into primary fields remain to be worked out. Work in this direction is still in progress.

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APPENDIX: PROOF FOR (4.7)

To prove (4.7), let P be an arbitrary superpseudodifferential operator; then

$$T'\Omega^{(A)}T'^\dagger P = T'Q, \tag{A1}$$

where

$$Q \equiv \Omega^{(A)}T'^\dagger P = (K_A T'^\dagger P)_- K_A^- - K_A (T'^\dagger P K_A)_- + [K_A, (T'^\dagger P K_A)_0]$$

$$+ (-1)^{|P|}\left[\int^x D\,\text{sres}[T'^\dagger P, K_A], K_A\right]$$

$$+ (-1)^{|P|} K_A D^{-1}\,\text{sres}[T'^\dagger P, K_A]. \tag{A2}$$

Using (4.6), each term in Q can be calculated as follows:

$$(1) = (TLPT^{-1})_- K_A + \frac{(-1)^{|P|+1}}{n} D\left(\int^x \text{sres}[P, L]\right) K_A,$$

$$(2) = -K_A (TPLT^{-1})_- + \frac{(-1)^{|P|}}{n} K_A\left(D\int^x \text{sres}[P, L]\right) - \frac{1}{n}\left(\int^x \text{sres}[P, L]\right) D,$$

$$(3) = [K_A, (TPLT^{-1})_0] + \frac{(-1)^{|P|+1}}{n}\left[K_A, \left(D\int^x \text{sres}[P, L]\right)\right],$$

$$(4) = (5) = 0,$$

which imply that

$$Q = (TLPT^{-1})_- K_A - K_A (TPLT^{-1})_- + [K_A, (TPLT^{-1})_0] + \frac{1}{n}\left[\left(\int^x \text{sres}[P, L]\right) D, K_A\right] \tag{A3}$$

and

$$\frac{1}{n}\int^{x}q_{2n-2}=\frac{1}{n}\int^{x}\text{sres}(QD^{-2n+1})$$

$$=(TPLT^{-1})_{0}+\frac{1}{n}\int^{x}\text{res}(T[P,L]T^{-1})$$

$$+\frac{1}{n}\int^{x}\left[\left(\int^{x'}\text{sres}[P,L]\right)\frac{(DV_{2n-2})}{n}\right]-\frac{2}{n^{2}}\int^{x}\left[\left(\int^{x'}\text{sres}[P,L]\right)V_{2n-3}\right].$$

$$\text{(A4)}$$

Substituting (A3) and (A4) into (4.5), we obtain the desired result (4.7).

Since the proof for K_B is parallel to the above one, we hence omit it here.

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