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Near automorphisms of cycles

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Abstract

Let f be a permutation of $V(G)$. Define $\delta_f(x, y) = |d_G(x, y) - d_G(f(x), f(y))|$ and $\delta_f(G) = \sum \delta_f(x, y)$ over all the unordered pairs $\{x, y\}$ of distinct vertices of G. Let $\pi(G)$ denote the smallest positive value of $\delta_f(G)$ among all the permutations f of $V(G)$. The permutation f with $\delta_f(G) = \pi(G)$ is called a near automorphism of G. In this paper, we study the near automorphisms of cycles C_n and we prove that $\pi(C_n) = 4\lfloor n/2 \rfloor - 4$, moreover, we obtain the set of near automorphisms of C_n.

Keywords: Near automorphism

1. Introduction

Let f denote a permutation of the n vertices of a connected graph G and the distance between two distinct vertices x and y in G be denoted by $d_G(x, y)$ (or $d(x, y)$ in short). Let $\delta_f(x, y) = |d(x, y) - d(f(x), f(y))|$ and $\delta_f(x) = \sum_y \delta_f(x, y)$. Then, it is easy to see that $\sum_x \delta_f(x) = 2\sum_{x \neq y} \delta_f(x, y)$ [1]. Now, let $\delta_f(G)$ be the sum of $\delta_f(x, y)$ over all the $\binom{n}{2}$ unordered pairs $\{x, y\}$ of distinct vertices of G. Clearly, a permutation f (of the vertices of G) is an automorphism of G if and only if $\delta_f(G) = 0$. Let $\pi(G)$ denote the smallest positive value of $\delta_f(G)$ among the $n!$ permutations f of the vertices of G. A permutation f for which $\delta_f(G) = \pi(G) > 0$ is called a *near automorphism* and $\pi(G)$ is the value of near automorphism. Chartrand et al. [2] observed that $\pi(G)$ is even and conjectured that $\pi(G) = 2n - 4$ when G is a path with n vertices. Later, Aitken [1] verified this conjecture and, among other things, characterized those permutations f for which $\pi(G) = \delta_f(G) = 2n - 4$ when G is a path with n vertices. Hence, the following result is established.

Theorem 1.1 (*Aitken [1]*). $\pi(P_n) = 2n - 4$.

On the study of near automorphism, Reid [5] provided the near automorphism values of complete multipartite graphs.

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Theorem 1.2. *For positive integers,* $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t$, *where* $t \geqslant 2$ *and* $n_t \geqslant 2$,

$$\pi(K_{n_1,n_2,\ldots,n_t}) = \begin{cases} 2n_{h+1}-2 & \text{if } 1=n_1=\cdots=n_h<n_{h+1}\leqslant\cdots\leqslant n_t, \\ & \quad \text{and } t\geqslant(h+1),\ \text{for some } h\geqslant 2, \\ 2n_{k_0} & \text{if } 1=n_1<n_2 \text{ or } n_1\geqslant 2, \\ & \quad n_{k+1}=n_k+1 \text{ for some } k,\ 1\leqslant k\leqslant t-1, \\ & \quad \text{and } 2+n_{k_0}\leqslant n_1+n_2, \\ 2(n_1+n_2-2) & \text{otherwise}, \end{cases}$$

where k_0 *is the smallest index for which* $n_{k_0+1}=n_{k_0}+1$.

It is worth of mentioning that we can also study the largest value $\pi^*(G)$ of $\delta_f(G)$ [3]. The permutations *f* with $\delta_f(G)=\pi^*(G)$ are called the chaotic mappings of *G*. Not much is known so far, see [4] for reference.

In this paper, we shall focus on the study of the near automorphisms of a cycle of length *n*, C_n. First, we show that $\pi(C_n)=4\lfloor n/2\rfloor-4$ and then obtain the set of near automorphisms of C_n. Finally, we explain how to derive Theorem 1.1 from $\pi(C_n)=4\lfloor n/2\rfloor-4$.

2. The main results

We start with several basic lemmas. For convenience, the *n*-cycle C_n we consider throughout this paper is denoted by $(v_{-\lfloor n/2\rfloor},\ldots,v_{-1},v_0,v_1,\ldots,v_{\lfloor n/2\rfloor})$ (for *n* even we let $v_{-\lfloor n/2\rfloor}=v_{\lfloor n/2\rfloor}$) and the vertices are distributed on the cycle evenly. Now, clearly the permutations $g(v_i)=v_{-i}$ and $h(v_i)=v_{i+j}$ for some *j* and for all *i* are automorphisms of C_n. The permutations *g* and *h* are the mirror reflection and rotation, respectively.

Lemma 2.1. *Let G be a connected graph which is not complete. Then* $2\leqslant\pi(G)\leqslant 2|V(G)|-4$.

Proof. Since *G* is not a complete graph, there exist three vertices x, y and z such that $xy, xz\in E(G)$ and $yz\notin E(G)$. Then, let *f* be the transposition (x,y). Clearly, *f* is not an automorphism and $\delta_f(G)=\sum\{\delta_f(u,v): |\{u,v\}\cap\{x,y\}|=1\}\leqslant 2|V(G)|-4$. Therefore, the proof follows by the fact that $\pi(G)$ is even and $\pi(G)\neq 0$. □

Lemma 2.2. *Let f be a permutation of* $V(G)$ *and* $f(v_0)=v_0$. *Then* $\delta_f(v_0)$ *is even.*

Proof. Since $\delta_f(v_0)=\sum_{v_i\neq v_0}\delta_f(v_i,v_0)=\sum_{v_i\neq v_0}|d(v_i,v_0)-d(f(v_i),v_0)|$, and $\delta_f(v_0)\equiv\sum_{v_i\neq v_0}\{d(v_i,v_0)-d(f(v_i),v_0)\}\equiv\sum_{v_i\neq v_0}d(v_i,v_0)-\sum_{v_i\neq v_0}d(f(v_i),v_0) \pmod 2$. By the fact that $\sum_{v_i\neq v_0}d(v_i,v_0)=\sum_{v_i\neq v_0}d(f(v_i),v_0)$. Hence, $\delta_f(v_0)\equiv 0 \pmod 2$. □

Lemma 2.3. *Let f be a permutation of* $V(G)$ *where G is a vertex-transitive graph. Then* $\delta_f(x)$ *is even for each* $x\in V(G)$.

Proof. Since *G* is vertex-transitive, for each vertex *x* there exists an automorphism *g* such that $g(y)=x$ where $y=f(x)$. Therefore, $(g\circ f)(x)=g(f(x))=g(y)=x$. By Lemma 2.2, $\delta_{g\circ f}(x)$ is even. But, $\delta_{g\circ f}(x)=\sum_{y\in V(G)\setminus\{x\}}|d(x,y)-d((g\circ f)(x),(g\circ f)(y))|=\sum_{y\in V(G)\setminus\{x\}}|d(x,y)-d(f(x),f(y))|=\delta_f(x)$. Hence $\delta_f(x)$ is even. □

Lemma 2.4. *Let f be a permutation of* $V(G)$ *and* $x\in V(G)$ *be a vertex satisfying* $f(x)=x$ *with* $\delta_f(x)\neq 0$. *Then there exist at least two distinct vertices u and v such that* $\delta_f(x,u)\neq 0$ *and* $\delta_f(x,v)\neq 0$.

Proof. Suppose not. There exists a unique $u\in V(G)$ such that $\delta_f(x)=\delta_f(x,u)\neq 0$. Then $\sum_{y\in V(G)\setminus\{x,u\}}|d(x,y)-d(f(x),f(y))|=0$. This implies that for each $y\in V(G)\setminus\{x,u\}$, $d(x,y)=d(f(x),f(y))=d(x,f(y))$. Now, since $f(u)\neq x$ and $f(u)\neq u$, $d(x,f(u))=d(x,f^2(u))=d(x,f^3(u))=\cdots$. By the fact that *f* is a permutation of finite order, there exists a $t \leqslant |V(G)|$ such that f^t is an identity and thus $d(x, f(u)) = d(x, f^t(u)) = d(x, u)$. This contradicts to $\delta_f(x, u) \neq 0$ and we have the proof. $\square$

Lemma 2.5. *If f is a permutation and g is an automorphism of a graph G, then* $\delta_{g\circ f}(G) = \delta_f(G) = \delta_{f\circ g}(G)$.

Proof. Since g is an automorphism, G is isomorphic to $g(G)$ and each vertex pair of $V(G)$ preserve their distance by g. Thus G and $g(G)$ have the same number of vertex pair of distance i and distance j by the same f and $d(g(f(x)), g(f(y))) = d(f(x), f(y))$. Then the equality follows:

$$\begin{aligned}
\delta_{gof}(G) &= \sum_{x,y\in V(G)} |d(x, y) - d(g(f(x)), g(f(y)))| \\
&= \sum_{x,y\in V(G)} |d(x, y) - d(f(x), f(y))| = \delta_f(G) \\
&= \sum_{j=1}\sum_{i=1} |\{x, y\} : x, y \in V(G), d(x, y) = i, d(f(x), f(y)) = j\}| \cdot |i - j| \\
&= \sum_{j=1}\sum_{i=1} |\{g(x), g(y)\} : g(x), g(y) \in V(G), d(g(x), g(y)) = i, \\
&\qquad d(f(g(x)), f(g(y))) = j\}| \cdot |i - j| \\
&= \delta_{fog}(G). \qquad \square
\end{aligned}$$

Now, we are ready for the proof of our main results.

Theorem 2.6. $\pi(C_n) = 4\lfloor n/2\rfloor - 4$.

Proof. Let the permutation be the transposition $(v_0 v_1)$. Then it is easy to check that $\pi(C_n) \leqslant 4\lfloor n/2\rfloor - 4$. For $n \leqslant 3$, all permutations of C_n are automorphisms. Therefore, we start our proof by showing that for each positive integer $n \geqslant 4$, $\delta_f(C_n) \geqslant 4\lfloor n/2\rfloor - 4$ for any non-automorphism f.

Since C_n is a vertex-transitive graph, by Lemma 2.5, we may assume that f is a non-automorphism of C_n such that $f(v_0) = v_0$ and $\delta_f(v_0) = \min\{\delta_f(v) : v \in V(C_n)\}$.

Clearly, if $\delta_f(v_0) \geqslant 4$, then $\delta_f(C_n) \geqslant 2n$ and the proof follows. So, we assume that $\min\{\delta_f(v) : v \in V(C_n)\}$ is equal to 0 or 2. Note that, by Lemma 2.3, $\delta_f(v_0)$ is even.

Case 1: $\delta_f(v_0) = 0$.

This implies that for each $v_i \in V(C_n)$, $i \neq 0$, $f(v_i) \in \{v_i, v_{-i}\}$. Let $A = \{k : f(v_k) = v_k, k = 1, 2, \ldots, \lceil n/2\rceil - 1\}$ and $B = \{h : f(v_h) = v_{-h}, h = 1, 2, \ldots, \lceil n/2\rceil - 1\}$. Since f is not an automorphism, then $|A| \neq 0$ and $|B| \neq 0$. Thus in this case, $n \geqslant 5$. Then, for each $k \in A$ and $h \in B$, $|d_{C_n}(v_k, v_h) - d_{C_n}(f(v_k), f(v_h))| \geqslant 1$ whenever n is odd and $|d_{C_n}(v_k, v_h) - d_{C_n}(f(v_k), f(v_h))| \geqslant 2$ whenever n is even. Now, let $A^- = \{-k : k \in A\}$ and $B^- = \{-h : h \in B\}$. The above inequalities also hold for $k \in A^-$ and $h \in B$ or $k \in A$ and $h \in B^-$ or $k \in A^-$ and $h \in B^-$ depending on n is odd and even, respectively. Thus, we conclude that $\delta_f(C_n) \geqslant 4|A||B|$or $8|A||B|$ depending on n is odd or even. Nevertheless, by the fact that $|A| + |B| = \lceil n/2\rceil - 1$, we have $\delta_f(C_n) \geqslant 4 \cdot 1 \cdot (\lceil n/2\rceil - 2)$ or $8 \cdot 1 \cdot (\lceil n/2\rceil - 2)$ with respect to n odd or even, respectively, and the equality holds only if $|h| = \lceil n/2\rceil - 1$ for all odd n and special for $n = 6$ since $8 \cdot (\lceil n/2\rceil - 2) = 4\lfloor n/2\rfloor - 4$.

Case 2: $\delta_f(v_0) = 2$.

By Lemma 2.4, there exist two distinct vertices v_h and v_k such that $\delta_f(v_0) = \delta_f(v_0, v_h) + \delta_f(v_0, v_k)$, where $h, k \in \{-\lfloor n/2\rfloor, \ldots, 0, \ldots, \lfloor n/2\rfloor\}$ (again, for n even we let $v_{-\lfloor n/2\rfloor} = v_{\lfloor n/2\rfloor}$) and $|h| = |k| + 1$. Hence, the near automorphism f satisfies one of the following four conditions: (a) $f(v_h) = v_k$ and $f(v_k) = v_h$, (b) $f(v_h) = v_k$ and $f(v_k) = v_{-h}$, (c) $f(v_h) = v_{-k}$ and $f(v_k) = v_h$, (d) $f(v_h) = v_{-k}$ and $f(v_k) = v_{-h}$. Since $\delta_f(v_0) = 2$, for each v_i, $i \neq h, k$, $f(v_i) = v_i$ or $f(v_i) = v_{-i}$. Obviously, if we compose an automorphism g to all the possible permutations in (a), then we can get all the possible permutations in (d), where g is a mirror reflection such that $g(v_0) = v_0$ and $g(v_i) = v_{-i}$ for all i, and so do (b) and (c). Thus, it suffices to find $\delta_f(C_n)$ for the f's satisfying (a) and (b) respectively. By considering the displacement of v_h and v_k, we have

$$\delta_f(C_n) \geqslant \sum_{i \neq h,k} \{\delta_f(v_h, v_i) + \delta_f(v_k, v_i)\}$$

$$= \sum_{i \neq h,k} \{|d(v_h, v_i) - d(f(v_h), f(v_i))| + |d(v_k, v_i) - d(f(v_k), f(v_i))|\}$$

Let $C = \{i : f(v_i) = v_i, i \neq h, k\}$ and $D = \{j : f(v_j) = v_{-j}, j \neq h, k\}$. Since $v_0 \in C$, $|C| \neq 0$, and $|C| + |D| = n - 2$. Then, for the f satisfying (a), we have

$$\delta_f(C_n) \geqslant \sum_{\substack{i \neq h,k \\ i \in C}} \{|d(v_h, v_i) - d(v_k, v_i)| + |d(v_k, v_i) - d(v_h, v_i)|\}$$

$$+ \sum_{\substack{i \neq h,k \\ i \in D}} \{|d(v_h, v_i) - d(v_k, v_{-i})| + |d(v_k, v_i) - d(v_h, v_{-i})|\}.$$

By the fact that $|d(v_h, v_i) - d(v_k, v_i)| \geqslant 1$ and $|d(v_h, v_i) - d(v_k, v_{-i})| \geqslant 1$ for each $i \neq h, k$ in the case n is even, we have $\delta_f(C_n) \geqslant 2(|C| + |D|) = 2(n - 2)$, as desired. On the other hand, if n is odd, exactly one vertex v_j, $j \neq h, k$ in $V(C_n)$ satisfying $d(v_h, v_j) = d(v_k, v_j)$. Thus, $\delta_f(C_n) \geqslant 2(n-3) = 4\lfloor n/2 \rfloor - 4$ in the case when n is odd. Since $|C| \neq 0$, $\delta_f(C_n) = 4\lfloor n/2 \rfloor - 4$ if $|d(v_h, v_i) - d(v_k, v_i)| = 1$. In fact, $d(v_h, v_k) = 1$ for all i and for all n but $i = \lfloor n/2 \rfloor$ or $-\lfloor n/2 \rfloor$, we have $|d(v_h, v_i) - d(v_k, v_i)| = 1$; if $d(v_h, v_k) \neq 1$, then there are four and two vertices such that $|d(v_h, v_i) - d(v_k, v_i)| = 1$ and the other vertices $|d(v_h, v_i) - d(v_k, v_i)| \geqslant 3$ and 2 in all n even and odd case, respectively.

Next, for the f satisfying (b), we have

$$\delta_f(C_n) \geqslant \sum_{\substack{i \neq h,k \\ i \in C}} \{|d(v_h, v_i) - d(v_k, v_i)| + |d(v_k, v_i) - d(v_{-h}, v_i)|\}$$

$$+ \sum_{\substack{i \neq h,k \\ i \in D}} \{|d(v_h, v_i) - d(v_k, v_{-i})| + |d(v_k, v_i) - d(v_{-h}, v_{-i})|\}.$$

By observation, we are able to see that at least one of the summands is larger than 2. Therefore, we conclude that $\delta_f(C_n) > 4\lfloor n/2 \rfloor - 4$ in this case. In conclusion that we have the lower bound $\delta_f(C_n) \geqslant 4\lfloor n/2 \rfloor - 4$ and the near automorphisms of C_n are $f \circ g$ and $g \circ f$ where $f = (v_0 v_1)$ and g is an automorphism of C_n, (since C_n is a vertex-transitive graph, we prefer $(v_0 v_1)$ to any transposition of two adjacent vertices) and special for $n = 5$, $f = (v_0 v_1)$ or $(v_0 v_2)$ and for $n = 6$, $f = (v_0 v_1)$, $(v_0 v_2)$ or $(v_0 v_3)$. This concludes the proof of the theorem. □

Finally, we would like to point out that the study of near automorphisms of paths and cycles does have some similarity. With the following proposition, we provide a short proof of $\pi(P_n) = 2n - 4$.

Proposition 2.7. *$\pi(P_n) \geqslant \pi(C_n)$, the equality holds only when n is even.*

Proof. Let $P_n = \langle v_1, v_2, \ldots, v_n \rangle$ and $C_n = (v_1, v_2, \ldots, v_n)$. Now, it is easy to see that $d_{P_n}(v_i, v_j) = |j - i|$ and $d_{C_n}(v_i, v_j) = \min\{|j - i|, n - |j - i|\}$. In order to prove the proposition, we will first show that for each permutation f of $V(P_n) = V(C_n)$ and for each pair of distinct vertices $\{x, y\}$, $|d_{P_n}(x, y) - d_{P_n}(f(x), f(y))| \geqslant |d_{C_n}(x, y) - d_{C_n}(f(x), f(y))|$. Clearly, if both $d_{P_n}(x, y)$ and $d_{P_n}(f(x), f(y))$ are not larger than $\lfloor n/2 \rfloor$, so are $d_{C_n}(x, y)$ and $d_{C_n}(f(x), f(y))$, the proof follows. On the other hand if both $d_{P_n}(x, y)$ and $d_{P_n}(f(x), f(y))$ are larger than $\lfloor n/2 \rfloor$, then $|d_{C_n}(x, y) - d_{C_n}(f(x), f(y))| = |n - d_{P_n}(x, y) - n + d_{P_n}(f(x), f(y))| = |d_{P_n}(x, y) - d_{P_n}(f(x), f(y))|$. Therefore, it is left to consider the case that one of them is larger than $\lfloor n/2 \rfloor$ and the other one is not larger than $\lfloor n/2 \rfloor$ or equivalently $d_{P_n}(x, y) > \lfloor n/2 \rfloor$ and $d_{P_n}(f(x), f(y)) \leqslant \lfloor n/2 \rfloor$ (by symmetry). Now, we have two subcases to consider.

(i) $d_{P_n}(x, y) + d_{P_n}(f(x), f(y)) \geqslant n$.

$$|d_{C_n}(x, y) - d_{C_n}(f(x), f(y))| = |n - d_{P_n}(x, y) - d_{P_n}(f(x), f(y))| = d_{P_n}(x, y) + d_{P_n}(f(x), f(y)) - n \leqslant d_{P_n}(x, y) - d_{P_n}(f(x), f(y)) = |d_{P_n}(x, y) - d_{P_n}(f(x), f(y))|.$$

(ii) $d_{P_n}(x, y) + d_{P_n}(f(x), f(y)) < n$.

$$|d_{C_n}(x, y) - d_{C_n}(f(x), f(y))| = |n - d_{P_n}(x, y) - d_{P_n}(f(x), f(y))| = n - d_{P_n}(x, y) - d_{P_n}(f(x), f(y)) \leqslant d_{P_n}(x, y) - d_{P_n}(f(x), f(y)) = |d_{P_n}(x, y) - d_{P_n}(f(x), f(y))|.$$

Note that the equalities in (i) and (ii) hold when $n = 2d_{P_n}(x, y)$ and $n = 2d_{P_n}(f(x), f(y))$, respectively. Therefore, n must be even. Thus, for each non-automorphism f, we have $\delta_f(P_n) \geqslant \delta_f(C_n)$. Hence, we left the case that when f is an automorphism of C_n but not an automorphism of P_n, $\delta_f(P_n) \geqslant 2n - 4$.

Clearly, $g(v_i) = v_{n-i+1}$ and $h(v_i) = v_{i+j} \pmod n$ are mirror reflection and rotation of C_n here, they can create all the automorphisms of C_n. Obviously, if $\{f(v_1), f(v_n)\} = \{v_1, v_n\}$ for some automorphism f of C_n, then f is also an automorphism of P_n. Otherwise, if $\{f(v_1), f(v_n)\} = \{v_j, v_{j+1}\}$ for $1 \leqslant j < n$, then $\{f(v_j), f(v_{j+1})\} = \{v_1, v_n\}$ or $\{f(v_{n-j}), f(v_{n-j+1})\} = \{v_1, v_n\}$, and $\delta_f(P_n) \geqslant \delta_f(v_1, v_n) + \delta_f(v_j, v_{j+1}) + \delta_f(v_{n-j}, v_{n-j+1}) = (n - 2) + (n - 2) = 2n - 4$. Thus this concludes the proof. □

Corollary 2.8. (Aitken [1]) $\pi(P_n) = 2n - 4$.

Proof. By Theorem 2.6 and Proposition 2.7, we can see that $\pi(P_n) \geqslant 2n - 4$. Then, Theorem 1.1 can be obtained by the fact that $\pi(P_n) \leqslant 2n - 4$ (Lemma 2.1). □

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