行政院國家科學委員會專題研究計畫成果報告

能量泛函極小映射之收斂性
Convergence of Energy Functional Minimizing Maps

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† 八十六年度及以前的一般國科會專題計畫(不含產學合作研究計畫)亦可選擇適用，惟較特殊的計畫如國科會規劃案等，請先洽得國科會各學術處同意。

一、中文摘要

在本文中，我們證明了由一一般緊緻里曼流型到一緊緻齊性流型半凸泛函 Euler-Langrange 演化方程組(含 p-調和映射)弱解的存在性。以及有界索伯列夫範序列弱解的緊緻性。

關鍵詞：緊緻里曼流型，緊緻齊性流型，半凸泛函， Euler-Langrange 演化方程組，p-調和映射，弱解

Abstract

In this paper, we will construct a weak solution for the heat flow associated with certain quasiconvex functionals from a Euclidean domain into a homogeneous space with a left invariant metric. In particular, p-harmonic heat flow for any $p > 1$.

Keywords: compact Riemannian manifold,
compact homogeneous space,
Euler-Lagrange evolution equation,
weak solution

1. Introduction

F. Bethuel, J. M. Coron, J. M. Ghidaglia and A. Soyeur, in [BCGS], constructed weak solutions for the heat flow associated with relaxed energies for harmonic maps between B^3 and S^2. Y. Chen and M. Struwe, in [C1] and [CS], studied the existence of a weak solution for the harmonic heat flow into any smooth compact manifold and proved a partial regularity result (see also [CH1]). J. Keller, J. Rubinstein and P. Sternberg ([KRS]) also proved the existence of a weak solution of harmonic heat flow. For p-harmonic heat flow with $p>2$, the existence of weak solutions into spheres was proved by Y. Chen, M. C. Hong and N. Hungerbuhler ([CHH] see also [H1]) and by X. Cheng ([CH2]). Recently, M. Li([LM2])proved the existence of a weak solution for p -harmonic heat flows into homogeneous spaces with $1 < p \leq 2$.

The basic tool in [LM2] is the use of Hardy space on R^{m+1} and the key ingredient there is finding a correct function having the weak derivative such that this function lies in the local Hardy space. In this paper, we will use an elementary method, without using Hardy space technique, to prove the existence of a weak solution for heat flow associated with certain class of quasiconvex functionals into homogeneous spaces, and this includes the p-harmonic heat flow $\forall p > 1$ as a special case. The proof we give here is very close to the proof of the existence of a harmonic flow between B^3 and S^2, in [BCGS]; not only the construction of approximation solutions (cf. [HK]), but also the wedge product method (cf. [C1], [BCGS], etc.). The only difference is that, instead of considering the normal vectors, what we will use is the Killing tangent vector fields constructed by Helein (cf. [H1]). The main idea of this method is based on the observation that we can take limit from (3.3). Only after we have observed this fact, did it become possible to say something by rewriting the equation (3.4).

Let m, K be the positive integers, denoted by $M^{m\times K}$ the space of all real $m\times K$ matrices, and suppose $\Omega\subseteq R^K$ is a bounded open subset. Let (N,g) be an n-dimensional smooth homogeneous space with a left invariant metric g. We assume that (N,g) is isometrically embedded in R^K. Let $F:M^{m\times K}\to R$ be given and consider the functional

(1.1) $\mathrm{F}(v)\equiv\int_\Omega F(\nabla v)$

for $v:\Omega\to R^K$. We say that F is *quasiconvex*, if

(1.2) $\int_U F(A)\le\int_U F(A+D\varphi)$

for all smooth, bounded, open domains $U\subset R^m$, all matrices $A\in M^{m\times K}$, and all $\varphi\in C_0^1(U;R^K)$. The following theorem is due to Acerbi and Fusco.

Theorem: ([AF]) *Assume* $F:M^{m\times K}\to R$ *is continuous and*
(1.3) $0\le F(A)\le C(1+|A|^p)$, $A\in M^{m\times K}$
for some constant C *and* $1\le p<\infty$. *Then* F *is weakly sequentially lower semi-continuous on the Sobolev space* $W^{1,p}(\Omega,R^K)$ *if and only if* F *is quasiconvex.*

When $1<p<\infty$, this theorem and an additional coercivity assumption of the form

(1.4) $F(A)>C|A|^p$, $C>0$, $A\in M^{m\times K}$

implies the existence of at least one minimizer of F for given Dirichlet boundary conditions (cf. Morrey [MCB], [E1]).

Though out this paper, we will assume that F is quasiconvex and

(H1) $C_1|A|^p\le F(A)\le C_2(1+|A|^p)$,
$|DF(A)|\le C(1+|A|^p)$

$\forall A$ and $p>1$ and some constant C, C_1, C_2.

(H2) $\frac{\partial^2 F}{\partial a_i^\alpha\partial a_j^\beta}(A)\xi^\alpha\xi^\beta\zeta_i\zeta_j$
$\ge\gamma_0|A|^{p_0-2}|\xi|^2|\zeta|^2$,
$\forall A\ne 0, \forall\zeta\in R^m$, and $\forall\xi\in R^K$

for some constant $\gamma_0>0$ and $p_0>1$.

Since we shall study the maps into a target manifold N, we need a hypothesis that relates F to N:

(H3) $DF(A)=f(A)$ $\forall A\ne 0$

where $f:M^{m\times K}\setminus\{0\}\to M^{m\times m}$ is a map such that its (i,j) component $f_{i,j}=0$ for all $i\ne j$.

The hypothesis (H3) means that for $v:\Omega\to N$, $DF(\nabla v)$ lies in the tangent space $\mathrm{Tan}_v N$ and keeps certain ellipticity for the equation of critical points of $\mathrm{F}(*)$. Notice that we don't require that f is measurable, so it is easy to find examples which are much more general than $|A|^p$.

Under the hypothesis (H3), the critical point $u\in W^{1,p}(\Omega,N)$ of $\mathrm{F}(*)$ is the weak solution of

(1.5) $\mathrm{div}(DF(\nabla u))=\mathrm{A}_u(\nabla u,DF(\nabla u))$,

where A is the second fundamental form of N, and div$DF(*)$ means $\nabla_i(\frac{\partial F}{\partial a_i^\alpha}e^\alpha)$ with $\{e^\alpha\}$ denoting the standard basis of R^K.

We now consider the heat flow associated with F(*), i.e., for $u:\Omega\times R_+\to N$, study the problem

(1.6) $2\partial_t u-\mathrm{div}(DF(\nabla u))$
$= \mathrm{A}_u(\nabla u, DF(\nabla u)$

(1.7) $u(x,0)=u_0(x),\quad x\in\Omega,$

(1.8) $u(x,t)=u_0(x)\quad t>0, x\in\partial\Omega,$

where $u\in W^{1,p}(\Omega,N)$ is given.

Definition 1.1 *A map* $u:\Omega\times R_+\to N$ *is said to be the weak solution of (1.6)-(1.8), if u is defined a.e. on* $\Omega\times R_+$, $u\in L^\infty(0,\infty;W^{1,p}(\Omega))$, $\partial_t u\in L^2(\Omega\times R_+,R^K)$, *such that u satisfies (1.6) in the weak sense and (1.7), (1.8) in the trace sense.*

The main results we obtained are:

Theorem 1.2 *Assume* F *is quasiconvex satisfying* H1, H2. *Then there exists a weak solution to (1.6)-(1.8).*

Immediate consequences of this theorem:

Theorem 1.3 *Assume* F *is quasiconvex satisfying* H1 *and* H2. *Assume* $G:M^{m\times K}\to R$ *is quasiconvex and for* $1<\bar{p}<p$,

(1.9) $C_1|A|^{\bar{p}}\le G(A)\le C_2(1+|A|^{\bar{p}}),$

(1.10) $|G(A)|\le C_2(1+|A|^{\bar{p}-1})$

for all $A\in M^{m\times K}$. *Then there exists a weak solution to the heat flow associated with the energy functional*

(1.11) $\bar{\mathrm{I}}(v)\equiv\int_\Omega F(\nabla v)+G(\nabla v)$.

We will prove Theorem 1.2 by constructing a sequence of approximation solutions converging to a map which turns out to be the weak solution of (1.6)-(1.8). The proof of Theorem 1.3 is the same as Theorem 1.2, the only care we need to take is to consider some new terms in the equation.

The theorem below follows immediately from the proof of Theorem 1.2.

Theorem 1.4 *Assume* $\{u_l\}_{l=1}^\infty$ *is a sequence of weak solutions of (1.6)-(1.8) such that* $\partial_t u_l$ *is bounded in* $L^2(\Omega\times R_+,R^K)$, *and for any* $T>0$, $\nabla_i u_l$ *is bounded in* $L^p(\Omega\times(0,T),R^K)$, $\forall i$, *then there is a subsequence which converges to a weak solution of (1.6)-(1.8).*

2. Construction

In order to construct a weak solution to (1.6)-(1.8), we proceed as K. Horihata, N. Kikuchi in [HK] and F. Bethuel, J. M. Coron, J. M. Ghidaglia, A. Soyeur in [BCGS]. For $h\in(0,1)$, we define the sequence $\{u_k\}$ as follows. We assume that u_{k-1}, $k\ge1$, is known and define u_k to be the minimizer of the functional

(2.1) $\int_\Omega F(\nabla v)+\int_\Omega\frac{|v-u_{k-1}|^2}{h}$

under the constraint $v\in H^1(\Omega,N), v=u_0$ on $\partial\Omega$. Since F is quasiconvex, u_k exists and is the weak solution of the equation below:

(2.2) $\frac{2}{h}D\Pi_{u_k}(u_k-u_{k-1})-\mathrm{div}DF(u_k)$
$=\mathrm{A}_{u_k}(\nabla u_k, DF(\nabla u_k))$

where div $DF(*)$ means $\nabla_i(\frac{\partial F}{\partial a_i^\alpha}(*)e^\alpha)$ with $\{e^\alpha\}_{\alpha=1}^K$ the standard basis of R^K, Π is the nearest point projection from some uniform tubular neighborhood of N onto N, and $D\Pi_y$ is the orthogonal projection of R^K onto $\mathrm{Tan}_y N$ for any $y \in N$.

Define $\bar{u}_h : \Omega\times[0,\infty) \to N$ and $u_h : \Omega\times[0,\infty) \to N$ by setting for $(k-1)h < t \le kh$:

(2.3) $\quad \bar{u}_h(x,t) = u_k(x)$

(2.4) $$u_h(x,t) = \frac{t-(k-1)h}{h}u_k(x) + \frac{kh-t}{h}u_{k-1}(x)$$

With these notations, we can write the equation (2.2) as

(2.5) $$D\Pi_{\bar{u}_h}(\partial_t u_h) - \mathrm{div}(DF(\nabla\bar{u}_h)) = \mathrm{A}_{\bar{u}_h}(\nabla\bar{u}_h, DF(\partial_t\bar{u}_h)).$$

We have the following energy inequality:

(2.6) $$\mathrm{F}(u_k) + \sum_{l=1}^{k}\int_\Omega \frac{|u_l - u_{l-1}|^2}{h} \le \bar{\mathrm{I}}(u_0)$$

(2.7) $$\mathrm{F}(\bar{u}_h(hk)) + \int_0^{hk}\int_\Omega |\partial_t u_h|^2 \le \bar{\mathrm{I}}(u_0)$$

which implies

(2.8) $$\int_0^\infty\int_\Omega |\partial_t u_h|^2 \le \bar{\mathrm{I}}(u_0)$$

(2.9) $$\mathrm{F}(\bar{u}_h(t)) \le \bar{\mathrm{I}}(u_0), \quad \forall t \ge 0.$$

Therefore, up to the extraction of a subsequence, we may assume that for any $T>0$, $u_h \to u$ weakly in $W^{1,p}(\Omega\times(0,T), R^K)$, $\partial_t u_h \to \partial_t u$ weakly in $L^2(\Omega\times(0,T), R^K)$, $u_h \to u$ strongly in $L^p(\Omega\times(0,T), R^K)$, and $\nabla\bar{u}_h \to \nabla\varphi$ weakly in $L^p(\Omega\times(0,T), R^K)$. Since

(2.10) $$\int_0^T\int_\Omega |u_h - \bar{u}_h|^2 \le h^2\,\bar{\mathrm{I}}(u_0),$$

we see that $\bar{u}_h \to u$ strongly in $L^q(\Omega\times(0,T), R^K)$ with $q = \min(p,2)$, therefore $\nabla\varphi = \nabla u$ and $u(x,t) \in N$ a.e.. We will show that u is the weak solution of (1.6)-(1.9).

3. Compactness

We now prove Theorem 1.2. In [H1], Helein proved the existence of certain vector fields on homogeneous space and use them to rewrite the weakly harmonic map equation to induce the regularity. In [TW], T. Toro and C. Y. Wang used these vector fields and the Hardy space to prove the compactness of weakly p-harmonic maps. Helein's work tells us that there exist d smooth tangent vector fields $Y_1,\cdots,Y_d$ and d smooth Killing tangent vector fields $X_1,\cdots,X_d$ on N such that for any tangent vector field V in TN, we have

(3.1) $$V = \sum_{l=1}^{d} < X_l, V > Y_l$$

In particular for $DF(\nabla\bar{u}_h)$, we have

(3.2) $$DF(\nabla\bar{u}_h) = \sum_{l=1}^{d} < DF(\bar{u}_h), X_l(\bar{u}_h) > Y_l$$

We can rewrite the equation (2.5) as follows. For any $\varsigma \in C_0^\infty(\Omega\times(0,T), R)$, we have

$$2\int_0^T\int_\Omega D\Pi_{\bar{u}_h}(\partial_t u_h)\varsigma X_l(\bar{u}_h)dxdt$$
$$= -\int_0^T\int_\Omega DF(\nabla\bar{u}_h)\nabla\varsigma X_l dxdt$$
$$= \int_0^T\int_\Omega \mathrm{div}(< DF(\nabla\bar{u}_h), X_l >)\varsigma dxdt$$

Therefore, for each l,

(3.3) $$\mathrm{div}(< DF(\nabla\bar{u}_h), X_l(\bar{u}_h) >) = 2 < D\Pi_{\bar{u}_h}(\partial_t u_h), X_l(\bar{u}_h) > .$$

Let $h \to 0$, we have

(3.4) $$\mathrm{div} < DF(u), X_l(u) >= 2 < \partial_t u, X_l(u) >, \quad \forall l.$$

Thus,

(3.5) $$< 2\partial_t u - \mathrm{div}DF(\nabla u), X_l(u) >= 0, \quad \forall l.$$

This implies

(3.6) $2\partial_t u - (\mathrm{div} DF(\nabla u))^{\mathrm{T}}$

$$= \sum_{l=1}^{d} < 2\partial_t u - (\mathrm{div} DF(\nabla u))^{\mathrm{T}}, X_l > Y_l$$

$$= 0$$

Therefore, $\partial_t u - \mathrm{div} DF(\nabla u)$ is orthogonal to the tangent space, and u is the weak solution of (1.6)-(1.8) by the standard argument. Of course, we need to check (3.5). This can be done by using φY_l as the test function in (3.4), and it follows

(3.7) $\mathrm{div} DF(\nabla u) =$

$$2\partial_t u + < DF(\nabla u), X_l(u) > DY_l(u)$$

which says that $\mathrm{div} DF(\nabla u)$ has meaning and so (3.5) follows from (3.4).

Proposition 3.1 *For the map* $u : \Omega \times R_+ \to N$ *with* $u \in L^\infty(0, \infty; W^{1,p}(\Omega, N))$, $\partial_t u \in L^2(\Omega \times R_+, R^K)$, *the equation (1.6), (3.4), (3.5), (3.6) and (3.7) are equivalent.*

To pass from (3.3) to (3.4), we need the fact that $\nabla \bar{u}_h \to \nabla u$ a.e. on $\Omega \times R_+$. This is actually proved in [HLM] for $F(A) = |A|^p$. For the general quasiconvex functional F, it's not clear whether we have this fact. However, in our case, from the hypothesis H2 and H3, it is still true that $\nabla \bar{u}_h \to \nabla u$ by a slight modification of the proof about a compactness assertion in [HLM] (see also [E3], [CHN]).

Remark: From (3.3), we may also rewrite the equation as

(3.8) $\mathrm{div}(DF(\nabla \bar{u}_h)) =$

$$< DF(\nabla \bar{u}_h), X(\bar{u}_h) > DY(\bar{u}_h) +$$

$$2 < D\Pi_{\bar{u}h}(\partial_t u_h), X(\bar{u}_h) > Y(\bar{u}_h)$$

and try to pass to the limit, which should be (3.7). However, since $\partial_t \bar{u}_h$ does not exist, we can not prove the first term on the right hand side of (3.8) lies in the local Hardy space. The technique used in [LM2] is to prove that $< DF(\nabla \bar{u}_h, X(\bar{u}_h) > DY(\bar{u}_h)$ lies in $\mathrm{H}^1_{\mathrm{loc}}(R^{m+1})$.

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