行政院國家科學委員會專題研究計畫成果報告

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執行期限: 90年8月1日 至 91年7月31日

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計畫參與人員: 吳建賢

執行單位: 交通大學電機與控制工程系

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Second-Order Approximation for RLC Trees

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Abstract

我們以兩個極點、一個零點的二階轉移函數來近似 RLC 樹狀電路原轉移函數的前三個s次方項係數, 每個節點的s次方項係數, 可用 KVL 所表示的矩陣代入簡單的公式計算出來; 我們亦提出明確的公式來計算步階響應的延遲時間、上升時間、最大超越量$\cdots$等; 由模擬的結果得知, 我們提出的方法比僅用兩個極點來近似, 更能精確的估計系統的參數。

We propose two-pole one-zero second-order approximations for transfer function in RLC trees. The approximation matches the first three moments of the original transfer function. Fundamental loop matrix formulation of circuit equations allows efficient and simultaneous computations of moments, and thus approximations, at all the nodes. Explicit formulas for step response parameters such as delay time, rise time, overshoot, etc., are given. Simulations show improved accuracy over existing second-order approximations.

I INTRODUCTION

RLC trees are useful in modelling interconnect lines in VLSI circuits [2]. Step response parameters, such as delay time, at capacitor nodes in the tree are important for routing and wire sizing optimization. Low-order approximation for the corresponding transfer functions are required for estimating the parameters without solving the complete RLC tree equations.

Ismail, et.al. [2] proposed a second-order approximation that matches the first two moments of the original transfer function. The second-order transfer function, with unit DC gain, is completely characterized by the damping ratio ζ and undamped natural frequency ω_n. Estimates of various step response parameters such as delay time, rise time, overshoot, etc., are proposed. In an effort to improve the accuracy of the second-order approximation, we propose a more general two-pole and one-zero second-order approximation. The three parameters of transfer function are determined by matching the first three moments of the original transfer function. Simulation results show that the additional degree of freedom in the second-order transfer function indeed improves the accuracy of the approximation, in term of frequency response and step response, and thus also improves the accuracy of estimates of step response parameters.

The fundamental loop matrix formulation of circuit equations is ideal for RLC trees [1]. The matrix formulation is simple. It involves only diagonal matrices and matrices with zeros and ones. The computation of moment matrices of the transfer matrix from source to all the capacitor voltage is simple and very efficient. Since the moments for each capacitor node are computed simultaneously, the proposed method constructs approximate for every source-to-node transfer function.

The paper is organized as follows. In Section II, we compute the transfer matrix of the RLC tree using fundamental loop matrix. In Section III, we give a recursive formula for computing the moment matrices of the transfer matrix. The formula for damping ratio, undamped natural frequency, and zero location of the second-order approximation are given in Section IV. Explicit formulas for delay time, rise time, overshoot etc, are given in Section V. Simulation examples and comparisons are given in Section VI. Finally, Section VII is a brief conclusion.

II TRANSFER MATRIX OF RLC TREE

For an RLC tree, the tree graph is uniquely defined and it consists of the voltage source branch and the R and L branches. The capacitor branches are the links that defines the fundamental loops [1]. KVL equations of these fundamental loops and KCL equations at the capacitor nodes completely specify the interconnection of the tree.

To write the circuit equations, let's consider the simple RLC tree shown in Figure 1, where the input voltage source is v_s, the tree branch voltages are v_{t_1}, v_{t_2}, and v_{t_3}, and the link voltages are v_{l_1}, v_{l_2}, and v_{l_3}, respectively; the branch currents i_{t_1}, i_{t_2}, i_{t_3}, and the link currents i_{l_1}, i_{l_2}, and i_{l_3} are defined accordingly in associated reference directions [1]. Note that we treat each series RL connection as a single branch. The KVL equations for the three fundamental loops are

$$\begin{aligned} v_{t_1} + v_{l_1} &= v_s && \text{(loop 1)} \\ v_{t_1} + v_{t_2} + v_{l_2} &= v_s && \text{(loop 2)} \\ v_{t_1} + v_{t_3} + v_{l_3} &= v_s && \text{(loop 3)} \end{aligned}$$

In matrix form, the equations become

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} v_t \\ v_l \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} v_s \qquad (1)$$

where $v_t = [v_{t_1} \;\; v_{t_2} \;\; v_{t_3}]^T$, $v_l = [v_{l_1} \;\; v_{l_2} \;\; v_{l_3}]^T$, and $B \in \mathbf{R}^{3\times 6}$ is the fundamental loop matrix associated with the tree. The matrix B is partitioned as

$$B = \left[\begin{array}{ccc} F & \vdots & I \end{array} \right] \qquad (2)$$

where $F, I \in \mathbf{R}^{3\times 3}$. The ijth entry of F are either 1 or 0 depending on whether the jth tree branch is in loop i or not. Thus (1) becomes

$$Fv_t + v_l = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} v_s \qquad (3)$$

The KCL equations at capacitor nodes are

$$\begin{aligned} i_{t_1} &= i_{l_1} + i_{l_2} + i_{l_3} \\ i_{t_2} &= i_{l_2} \\ i_{t_3} &= i_{l_3} \end{aligned}$$

In matrix form, the equations become

$$i_t = F^T i_l \qquad (4)$$

where $i_t = [i_{t_1} \;\; i_{t_2} \;\; i_{t_3}]^T$ and $i_l = [i_{l_1} \;\; i_{l_2} \;\; i_{l_3}]^T$. The branch equations for the tree branches and the link branches are respectively

$$v_{t_i} = R_i i_{t_i} + L_i \frac{di_{t_i}}{dt} \qquad i = 1, 2, 3 \quad (5)$$

and

$$i_{l_i} = C_i \frac{dv_{l_i}}{dt} \qquad i = 1, 2, 3 \qquad (6)$$

The equation (3), (4), (5), and (6) thus completely describe the RLC tree in Figure 1.

For a general RLC tree with n sections, the equations are similar. The vectors i_t, i_l, v_t, and v_l now all have n components; the fundamental loop matrix $B \in \mathbf{R}^{n\times 2n}$ is partitioned similarly as

$$B = \left[\begin{array}{ccc} F & \vdots & I \end{array} \right]$$

where the ikth element, f_{ik}, of $F \in \mathbf{R}^{n\times n}$ is

$$f_{ik} = \begin{cases} 1 & \text{if the } k\text{th tree branch is in the } i\text{th loop} \\ 0 & \text{if the } k\text{th tree branch is not in the } i\text{th loop} \end{cases}$$

The KVL equations and KCL equations in matrix form are respectively

$$Fv_t + v_l = Ev_s \qquad (7)$$

and

$$F^T i_l = i_t \qquad (8)$$

where $E = [1 \; \cdots \; 1]^T \in \mathbf{R}^{n\times 1}$. The branch equations in matrix form are

$$v_t = Ri_t + L\frac{di_t}{dt} \qquad (9)$$

and

$$i_l = C\frac{dv_l}{dt} \qquad (10)$$

where $R = \mathrm{diag}(R_1, R_2, \cdots, R_n)$, $L = \mathrm{diag}(L_1, L_2, \cdots, L_n)$, and $C = \mathrm{diag}(C_1, C_2, \cdots, C_n)$ are diagonal matrices with element values on the diagonal entries.

To write the equations relating v_l, the capacitor voltage, to v_s, the input voltage source, we substitute (8) and (9) into (7) to get

$$F(RF^T i_l + LF^T \frac{di_l}{dt}) + v_l = Ev_s \qquad (11)$$

and substitute (10) into (11) to get

$$FRF^T C\frac{dv_l}{dt} + FLF^T C\frac{d^2 v_l}{dt^2} + v_l = Ev_s \qquad (12)$$

Taking Laplace transform of (12) with zero initial conditions to get

$$(FLF^T Cs^2 + FRF^T Cs + I)V_l(s) = EV_s(s)$$

where $V_l(s) = \mathcal{L}(v_l(t))$ and $V_s(s) = \mathcal{L}(v_s(t))$. Thus the transfer matrix from v_s to the capacitor node voltages v_l is

$$H(s) = \frac{V_l(s)}{V_s(s)} = (FLF^T Cs^2 + FRF^T Cs + I)^{-1}E$$

Note that $H(s)$ is a rational matrix of dimension $n \times 1$.

III MOMENT COMPUTATION

The kth moment m_k of the transfer matrix $H(s)$ is defined as the coefficient of the term s^k in the power series expansion, at $s = 0$, of $H(s)$. Since

$$H(s) = H(0) + H'(0)s + \frac{1}{2!}H''(0)s^2 + \frac{1}{3!}H^{(3)}(0)s^3 + \cdots$$

the moment of $H(s)$ are

$$m_k = \frac{1}{k!}H^{(k)}(0) \qquad k = 0, 1, 2, \cdots$$

To compute the moments m_k, let's writes

$$H(s) = A(s)^{-1}E$$

where $A(s) = FLF^TCs^2 + FRF^TCs + I$. Since $A(0) = I$, we have $m_0 = H(0) = E$. By computation,

$$\begin{aligned}
\frac{dA^{-1}}{ds}|_{s=0} &= -A^{-1}\frac{dA}{ds}A^{-1}|_{s=0} = -FRF^TC \\
\frac{d^2A^{-1}}{d^2s}|_{s=0} &= -[-2A^{-1}\frac{dA}{ds}A^{-1}\frac{dA}{ds}A^{-1} + \\
&\quad A^{-1}\frac{d^2A}{ds^2}A^{-1}]|_{s=0} \\
&= 2FRF^TCFRF^TC - 2FLF^TC \\
\frac{d^3A^{-1}}{d^3s}|_{s=0} &= -[6A^{-1}\frac{dA}{ds}A^{-1}\frac{dA}{ds}A^{-1}\frac{dA}{ds}A^{-1} \\
&\quad -3A^{-1}\frac{d^2A}{ds^2}A^{-1}\frac{dA}{ds}A^{-1} \\
&\quad -3A^{-1}\frac{dA}{ds}A^{-1}\frac{d^2A}{ds^2}A^{-1}]|_{s=0} \\
&= -6FRF^TCFRF^TCFRF^TC + \\
&\quad 6FLF^TCFRF^TC + 6FRF^TCFLF^TC
\end{aligned}$$

Note that A(s) is a matrix of polynomials of degree 2, therefore $d^3A(s)/ds^3 = 0$. From the above result, the first three moment matrices are:

$$m_1 = -FRF^TCm_0 \quad (13)$$
$$m_2 = -FRF^TCm_1 - FLF^TCm_0 \quad (14)$$
$$m_3 = -FRF^TCm_2 - FLF^TCm_1 \quad (15)$$

Note that the second and third moments depend on the previous two moments. It can be shown that the recursive formula holds true for other high-order moments as well, that is, for $k \geq 2$,

$$m_k = -FRF^TCm_{k-1} - FLF^TCm_{k-2} \quad (16)$$

The formula (16) computes the moments of the transfer matrix $H(s)$ and thus the moments of the transfer functions from the source to all capacitor node are simultaneously computed. It is clear that similar recursive formula for each source-to-capacitor node transfer function can be obtained from (16). To see this, let's consider the simple 3-section RLC tree in Figure 1. Let m_j^i be the jth moment of the transfer function to node i, where $i, j = 1, 2, 3$. The first moment is

$$\begin{aligned}
m_1 &= -FRF^TCm_0 \\
&= -\begin{bmatrix} R_1C_1 + R_1C_2 + R_1C_3 \\ R_1C_1 + (R_1+R_2)C_2 + R_1C_3 \\ R_1C_1 + R_1C_2 + (R_1+R_3)C_3 \end{bmatrix} \\
&= \begin{bmatrix} m_1^1 \\ m_1^2 \\ m_1^3 \end{bmatrix} \quad (17)
\end{aligned}$$

where $F \in \mathbf{R}^{3\times 3}$ is defined in (2). The second moment is

$$\begin{aligned}
m_2 &= -FRF^TCm_1 - FLF^TCm_0 \\
&= -\begin{bmatrix} R_1C_1m_1^1 + R_1C_2m_1^2 + R_1C_3m_1^3 \\ R_1C_1m_1^1 + (R_1+R_2)C_2m_1^2 + R_1C_3m_1^3 \\ R_1C_1m_1^1 + R_1C_2m_1^2 + (R_1+R_3)C_3m_1^3 \end{bmatrix} \\
&\quad -\begin{bmatrix} L_1C_1 + L_1C_2 + L_1C_3 \\ L_1C_1 + (L_1+L_2)C_2 + L_1C_3 \\ L_1C_1 + L_1C_2 + (L_1+L_3)C_3 \end{bmatrix}
\end{aligned}$$

The third moment is

$$\begin{aligned}
m_3 &= -FRF^TCm_2 - FLF^TCm_1 \\
&= -\begin{bmatrix} R_1C_1m_2^1 + R_1C_2m_2^2 + R_1C_3m_2^3 \\ R_1C_1m_2^1 + (R_1+R_2)C_2m_2^2 + R_1C_3m_2^3 \\ R_1C_1m_2^1 + R_1C_2m_2^2 + (R_1+R_3)C_3m_2^3 \end{bmatrix} \\
&\quad -\begin{bmatrix} L_1C_1m_2^1 + L_1C_2m_2^2 + L_1C_3m_2^3 \\ L_1C_1m_2^1 + (L_1+L_2)C_2m_2^2 + L_1C_3m_2^3 \\ L_1C_1m_2^1 + L_1C_2m_2^2 + (L_1+L_3)C_3m_2^3 \end{bmatrix}
\end{aligned}$$

From above the recursive formula, the moments of the transfer function to each node can also be computed recursively:

$$\begin{aligned}
m_1^i &= -\sum_{k=1}^{3} C_kR_{ik} \\
m_2^i &= -\sum_{k=1}^{3} C_kR_{ik}m_1^k - \sum_{k=1}^{3} C_kL_{ik} \\
m_3^i &= -\sum_{k=1}^{3} C_kR_{ik}m_2^k - \sum_{k=1}^{3} C_kL_{ik}m_1^k
\end{aligned}$$

where R_{ik} is the sum of common resistance from the input to node i and k and L_{ik} is the sum of common inductance from the input to node i and k.

It can be shown that the same formula holds for general tree with n section and for moments of any order. That is

$$m_j^i = -\sum_{k=1}^{n} C_kR_{ik}m_{j-1}^k - \sum_{k=1}^{n} C_kL_{ik}m_{j-2}^k$$

for $i = 1, 2, \cdots, n$, and $j \geq 2$.

IV SECOND-ORDER APPROXIMATION

We now consider matching the first three moments to obtain a second-order approximation. The three parameters of the second-order approximation we should determine are the damping ratio ζ, undamped natural frequency ω_n, and zero location $-z$. We consider scalar transfer function in this section, since an approximation of a transfer matrix is obtained component by component. Suppose the first three moments of an RLC tree transfer function m_1, m_2, and m_3 are given.

The transfer function of the two-pole one-zero second-order approximation, with unit DC-gain, has the form

$$\begin{aligned}
h(s) &= \frac{\omega_n^2(s+z)}{z(s^2 + 2\zeta\omega_n s + \omega_n^2)} \\
&= \frac{1 + \frac{1}{z}s}{1 + \frac{2\zeta}{\omega_n}s + \frac{1}{\omega_n^2}s^2} \quad (18)
\end{aligned}$$

The power series expansion of $h(s)$ can be obtained by long division:

$$h(s) = 1 + \frac{\omega_n - 2\zeta z}{z\omega_n} s + \frac{4\zeta^2 z - 2\zeta\omega_n - z}{z\omega_n^2} s^2 + \frac{-\omega_n + 4\zeta z + 4\zeta^2\omega_n - 8\zeta^3 z}{z\omega_n^3} s^3 + \cdots$$

The moment matching equation are

$$m_1 = \frac{\omega_n - 2\zeta z}{z\omega_n} \quad (19)$$

$$m_2 = \frac{4\zeta^2 z - 2\zeta\omega_n - z}{z\omega_n^2} \quad (20)$$

$$m_3 = \frac{-\omega_n + 4\zeta z + 4\zeta^2\omega_n - 8\zeta^3 z}{z\omega_n^3} \quad (21)$$

Equations (19), (20), and (21) uniquely determine ω_n, ζ, and z. The solutions are

$$\omega_n = \sqrt{\frac{m_1^2 - m_2}{m_2^2 - m_1 m_3}} \quad (22)$$

$$\zeta = -\frac{1}{2m_1\omega_n}(m_2\omega_n^2 + 1) \quad (23)$$

$$z = \frac{\omega_n}{m_1\omega_n + 2\zeta} \quad (24)$$

Hence given the moments m_1, m_2, and m_3 the formulas (22), (23), and (24) determine the undamped natural frequency ω_n , the damping ratio ζ, and zero location $-z$.

V Step Response Parameters

The unit-step response of the second-order transfer function $h(s)$, in (18), is

$$s(t) = 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}[(\zeta - r)\sin\omega_d t + \sqrt{1-\zeta^2}\cos\omega_d t] \quad (25)$$

where $\sigma = \zeta\omega_n$, $r = \omega_n/z$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$. The delay time t_d is defined as the time it takes the signal to rise to 50% of its final value. The rise time t_r is defined as the time it takes the signal to rise from 10% to 90% of its final value. To simplify expressions, let $t' = \omega_n t$, and $g(t') = f(t)$, hence (25) becomes,

$$g(t') = 1 - \frac{e^{-\zeta t'}}{\sqrt{1-\zeta^2}}[(\zeta - r)\sin(\sqrt{1-\zeta^2}t') + \sqrt{1-\zeta^2}\cos(\sqrt{1-\zeta^2}t')] \quad (26)$$

We note that (26) has two variables, ζ and r, only and that exact explicit formulas for delay time and rise time do not exist.

To find an approximate explicit formula for delay time, we determined the normalized delay time t_d' for different values of ζ and r via simulations. Least squares curve fitting is then used. First for each r we fit the normalized delay time t_d' as a function ζ by $1/(a\zeta + b)$, where a and b depend on the value r. We then fit the parameter a and b by second-order polynomials. The result is

$$t_d'(r, \zeta) = \frac{1}{p_d(r)\zeta + q_d(r)} \quad (27)$$

The same curve fitting method is used in estimating the rise time. The result is

$$t_r'(r, \zeta) = \frac{1}{p_r(r)\zeta + q_r(r)} \quad (28)$$

where $p_d(r) = -0.0051r^2 - 0.5989r - 0.3652$, $q_d(r) = 0.5355r^2 + 0.9136r + 0.9542$, $p_r(r) = -0.3886r^2 - 0.1123r - 0.6959$, and $q_r(r) = 0.6064r^2 + 0.0762r + 0.9707$. Consequently, the delay time, t_d and rise time t_r are

$$t_d = \frac{t_d'}{\omega_n} \quad (29)$$

$$t_r = \frac{t_r'}{\omega_n} \quad (30)$$

The peak time can be found by differentiating $s(t)$ in (25) and then set $s'(t) = 0$. The peak time that the maximum overshoot occurs is

$$t_p = \frac{\pi - \theta}{\omega_d} \quad (31)$$

where $\theta = \tan^{-1}(r\sqrt{1-\zeta^2}/1 - r\zeta)$, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$. The maximum overshoot, M_o obtained by substituting (31) into (25) is

$$M_o = \sqrt{1 - r\zeta + r^2}e^{-\frac{\zeta(\pi-\theta)}{\sqrt{1-\zeta^2}}} \quad (32)$$

VI An Example

To examine the effectiveness of the proposed second-order approximation, we consider the RLC tree shown in Figure 2. The RLC tree has 6 sections and is considered in [2].

Step response of each capacitor node for the original transfer function, the proposed approximation, and the approximation reported in [2] are computed using Matlab. The results are shown in Figure 3. In each plot, the solid line is the exact response, the dot line is the response of the proposed approximation, and the dot-dashed line is the response of the approximation reported in [2]. The plots show improved approximation over that reported in [2]. We also note that for this example the proposed approximation gives step responses very close to the original.

The delay time t_d and rise time t_r for step response of each node are shown in Table 1 and Table 2, respectively. In general, the proposed method and the explicit formulas (29) and (30) give better estimate over that proposed in [2]. Over all, the formula (29) gives estimates of delay time with error less than 20%; the formula (30) gives estimates of rise time with error less than 30%. Figure 4 shows the frequency magnitude response of the approximation error transfer function at each node. Again, the proposed approximations show improved accuracy over [2].

Table 1. delay time, in *ns*

node	exact	proposed	formula (29)	[2]
node 1	0.1235	0.1470	0.1473	0.1751
node 2	0.1953	0.2111	0.2212	0.2143
node 3	0.2021	0.2065	0.2066	0.2124
node 4	0.2608	0.2647	0.2658	0.2553
node 5	0.2720	0.2705	0.2717	0.2613
node 6	0.2575	0.2612	0.2624	0.2526

Table 2. rise time, in *ns*

node	exact	proposed	formula (30)
node 1	0.2561	0.2343	0.2296
node 2	0.2134	0.2608	0.2545
node 3	0.2575	0.2668	0.2601
node 4	0.1943	0.2558	0.2521
node 5	0.2385	0.2720	0.2590
node 6	0.2295	0.2633	0.2592

VII CONCLUSION

We propose a method to obtain second-order approximations for transfer functions in RLC trees. The two-pole one-zero approximation is shown to give improved accuracy over the existing second-order approximations in terms of step response, frequency response, estimated delay time and rise time. The results can be used to quickly estimate signal delay and other parameters. In view of the accuracy obtained, the second-order model can also be used in dynamic simulation to replace the original tree.

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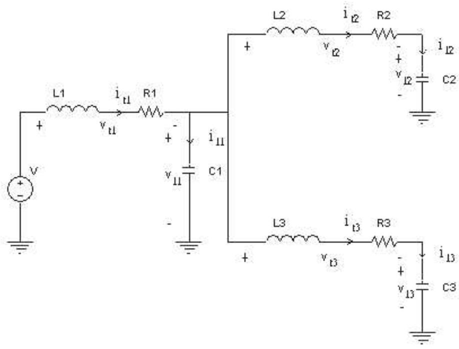


Figure 1: A simple RLC tree

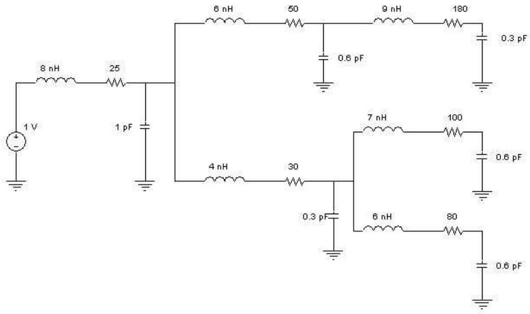
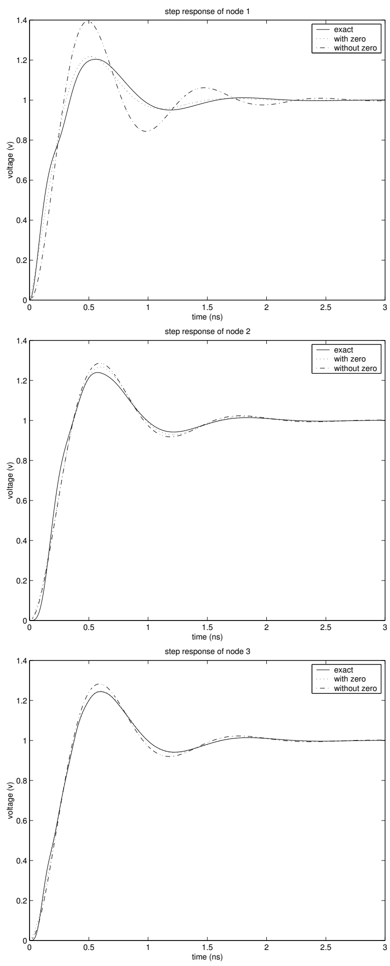


Figure 2: An example of an RLC tree



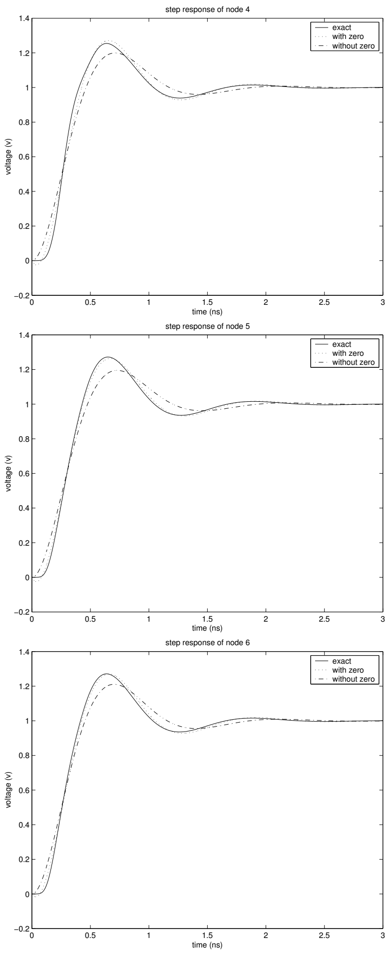
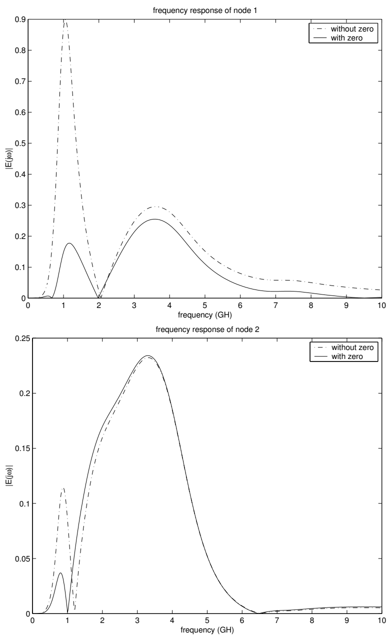


Figure 3: The step response of each node



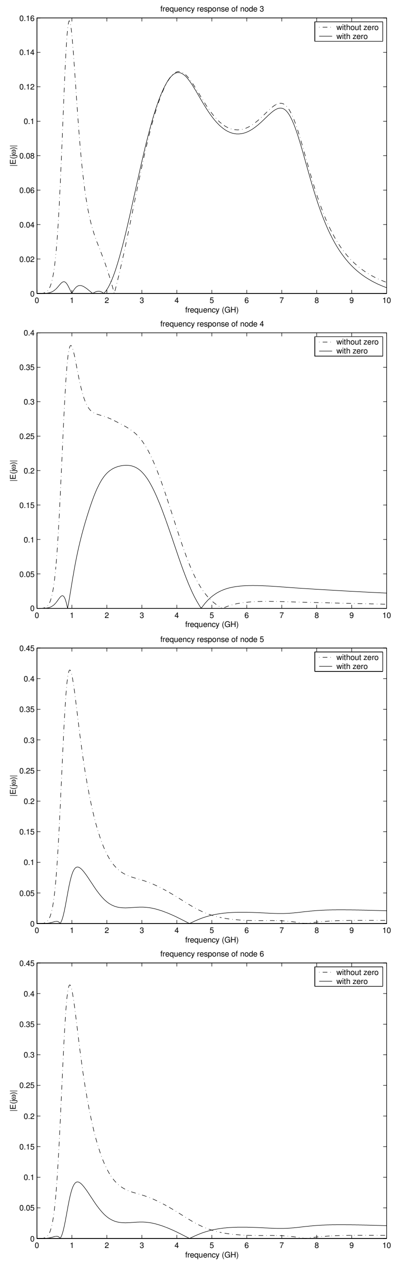


Figure 4: The error of the frequency response of each node