行政院國家科學委員會專題研究計畫成果報告

M/G/s 排隊系統的極限機率解析解

計畫編號：NSC 90-2118-M-009-007

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一、中文摘要

我們找到 *M/G/s* 排隊系統的極限機率，方法是以遞回方式先找 *M/G/1* 的解，繼而分兩部分找 *M/G/s* 的解．一部份是隊伍未形程，一部份是隊伍形程，以這兩部分的交集為基礎，再找全體解。

關鍵詞：極限機率，*M/G/s* 排隊系統

Abstract

We present in a recursive form the limiting probabilities in steady state for the number of customers in the *M/G/s* system. Our method includes first finding the limiting probabilities for the number of customers in the *M/G/*1 system and its variations, next exploring the *M/G/s* system by separately conditioning the number of customers on being less than or equal to *s* and on being greater than or equal to *s*, and last relating these two conditioned results. Moreover, we obtain an asymptotic behavior of the (time) limiting probabilities as the number of customers in the system goes large.

Keywords: LIMITING PROBABILITIES, *M/G/s* QUEUES.

1.The *M/G/*1 queues

Consider an *M/G/*1 queueing system. The customers arrive in accordance with a Poisson process with rate λ and the service times are independent and have common distribution function *G*. Assume that *G* is continuous with hazard rate function

$$\Lambda(x)=\frac{g(x)}{\overline{G}(x)}$$ where $g(x)$ is the density of *G* and the moment generating function of *G* exists. Denote $X(t)$ the number of customers at time *t* in the system. For $k \ge 0$ we let $p_k=\lim_{t\to\infty}P(X(t)=k)$ and

$$r_k=P(\textit{an arriving customer in steady state finds the system in state } k)$$

Theorem 1

The limiting probabilities of the number of customers of an *M/G/*1 queue can be written as the following recursive form, for $j\ge 2$,

(1)

$$\pi_{j+1}=\frac{\pi_j}{\int_0^\infty e^{-\lambda t}dG(t)}\left(1-\int_0^\infty \lambda t e^{-\lambda t}dG(t)\right)-\frac{1}{\int_0^\infty e^{-\lambda t}dG(t)}\left[\pi_{j-1}\int_0^\infty\frac{(\lambda t)^2e^{-\lambda t}}{2!}dG(t)+\cdots\right.$$

$$\left.+\pi_2\int_0^\infty\frac{(\lambda t)^{j-1}e^{-\lambda t}}{(j-1)!}dG(t)+(\pi_0+\pi_1)\int_0^\infty\frac{(\lambda t)^j e^{-\lambda t}}{j!}dG(t)\right]$$

with

$$\pi_0=1-\lambda EY\ ,$$

$$\pi_1=\frac{\pi_0}{\int_0^\infty e^{-\lambda t}dG(t)}\left[1-\int_0^\infty e^{-\lambda t}dG(t)\right],\text{ and}$$

$$\pi_2=\frac{\pi_0+\pi_1}{\int_0^\infty e^{-\lambda t}dG(t)}\left(1-\int_0^\infty e^{-\lambda t}dG(t)-\int_0^\infty \lambda t e^{-\lambda t}dG(t)\right)$$

.

Theorem 2

For an *M/G/*1 queue, the sequence of limiting probabilities $\pi_i,\ i\ge 0$, satisfy

$0 \le \liminf_{j\to\infty} \frac{\pi_{j+1}}{\pi_j} \le \frac{1}{c} \le \limsup_{j\to\infty} \frac{\pi_{j+1}}{\pi_j} \le 1$ where c is the unique solution to the equation $x = \int_0^\infty e^{\lambda t(x-1)} dG(t)$ other than $x=1$.

3.The *M/G/s* Queues.

Theorem 3

For any nonnegative integer *m*, the limiting probabilities of the number of customers of a delayed *m*-dependent *M/F,G/*1 queue can be written as the following recursive form,

for $j \ge 2$,

(8)
$$\pi_{j+1}^{(1)} = \frac{\pi_j^{(1)}}{\int_0^\infty e^{-\lambda t} dG(t)} \left(1 - \int_0^\infty \lambda t e^{-\lambda t} dG(t)\right) - \frac{1}{\int_0^\infty e^{-\lambda t} dG(t)}$$
$$\left[\pi_{j-1}^{(1)} \int_0^\infty \frac{(\lambda t)^2 e^{-\lambda t}}{2!} dG(t) + \cdots + \pi_2^{(1)} \int_0^\infty \frac{(\lambda t)^{j-1} e^{-\lambda t}}{(j-1)!} dG(t) + \pi_1^{(1)} \int_0^\infty \frac{(\lambda t)^j e^{-\lambda t}}{j!} dG(t) + \pi_0^{(1)} \int_0^\infty \frac{(\lambda t)^j e^{-\lambda t}}{j!} dF(t)\right]$$

with

$$\pi_0^{(1)} = \frac{1-\lambda EY}{1+\lambda(EZ - EY)},$$

$$\pi_1^{(1)} = \frac{\pi_0^{(1)}}{\int_0^\infty e^{-\lambda t} dG(t)} \left[1 - \int_0^\infty e^{-\lambda t} dF(t)\right], \text{ and}$$

$$\pi_2^{(1)} = \frac{1}{\int_0^\infty e^{-\lambda t} dG(t)}$$
$$\left(\pi_0^{(1)} \int_0^\infty 1 - e^{-\lambda t} - \lambda t e^{-\lambda t} dF(t) + \pi_1^{(1)} \int_0^\infty 1 - e^{-\lambda t} - \lambda t e^{-\lambda t} dG(t)\right)$$

where $Y \sim G$ and $Z \sim F$. Moreover, the result of Theorem 2 also holds here disregard the distribution function *F*.

Theorem 4

For any nonnegative integers *m* and *n*, the limiting probabilities of the number of customers waiting in the queue of an *m*-dependent *M/G/*1 system conditioned on queueing state are the same as the limiting probabilities of the number of customers in a delayed *n*-dependent *M/F,G/*1 system where *F* satisfies the conditions

$$\frac{\int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{k+1}}{(k+1)!} dG(t)}{1 - \int_0^\infty e^{-\lambda t} dG(t)} = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^k}{k!} dF(t), \text{ for}$$

all $k \ge 0$.

Theorem 5

In steady state, the unordered ages at each departing time of the remaining (*s*-1) customers of a conditioned *M/G/s* system on queueing state are distributed as (*s*-1) independent random variables with common density the equilibrium density of *G*.

Theorem 6

The limiting probabilities of an *M/G/s* queue can be written as

$$\pi_0^{(s)} = \frac{\hat{\pi}_0^{(1)}}{\hat{\pi}_0^{(1)} \left(\sum_{i=0}^{s-1} (\lambda EY)^i / i!\right) + (\lambda EY)^s / s!},$$

$$\pi_j^{(s)} = \frac{(\lambda EY)^j}{j!} \pi_0^{(s)}, \ 0 \le j \le s \text{ and}$$

$$\pi_j^{(s)} = \frac{\hat{\pi}_{j-s}^{(1)} (\lambda EY)^s / s!}{\hat{\pi}_0^{(1)} \left(\sum_{i=0}^{s-1} (\lambda EY)^i / i!\right) + (\lambda EY)^s / s!},$$

$j \ge s$,

where $Y \sim G$ and $\{\hat{\pi}_j^{(1)}, j \ge 0\}$ satisfy

for $j \ge 3$,

$$\hat{\pi}_j^{(1)} = \frac{\hat{\pi}_{j-1}^{(1)}}{\int_0^\infty e^{-\lambda t} dQ(t)} \left(1 - \int_0^\infty \lambda t e^{-\lambda t} dQ(t)\right) - \frac{1}{\int_0^\infty e^{-\lambda t} dQ(t)}$$
$$\left[\hat{\pi}_{j-2}^{(1)} \int_0^\infty \frac{(\lambda t)^2 e^{-\lambda t}}{2!} dQ(t) + \cdots\right.$$

$$+\hat{\pi}_1^{(1)}\int_0^\infty \frac{(\lambda t)^{j-1}e^{-\lambda t}}{(j-1)!}dQ(t)$$

$$+\hat{\pi}_0^{(1)}\int_0^\infty \frac{(\lambda t)^{j}e^{-\lambda t}}{j!}dQ(t)\Bigg/\left(1-\int_0^\infty e^{-\lambda t}dQ(t)\right)$$

with $\hat{\pi}_2^{(1)} = \dfrac{\hat{\pi}_1^{(1)}\left(1-\int_0^\infty \lambda t e^{-\lambda t}dQ(t)\right)}{\int_0^\infty e^{-\lambda t}dQ(t)}$,

$$-\frac{\hat{\pi}_0^{(1)}\int_0^\infty \frac{(\lambda t)^2 e^{-\lambda t}}{2!}dQ(t)}{\int_0^\infty e^{-\lambda t}dQ(t)\left(1-\int_0^\infty e^{-\lambda t}dQ(t)\right)},$$

$$\hat{\pi}_1^{(1)} = \hat{\pi}_0^{(1)}\frac{\left(1-\int_0^\infty e^{-\lambda t}+\lambda t e^{-\lambda t}dQ(t)\right)}{\int_0^\infty e^{-\lambda t}dQ(t)\left(1-\int_0^\infty e^{-\lambda t}dQ(t)\right)}$$ and

$$\hat{\pi}_0^{(1)} = \frac{1-\lambda EU}{\lambda EU}\left[\frac{1}{\int_0^\infty e^{-\lambda t}dQ(t)}-1\right]$$

where $U \sim Q$ and

$$\overline{Q}(t) = \overline{G}(t)\left(\int_0^\infty \frac{\overline{G}(t+x)}{EY}dx\right)^{s-1}$$

$$= \overline{G}(t)\left(\int_t^\infty \frac{\overline{G}(x)}{EY}dx\right)^{s-1}.$$ Moreover,

$\{\pi_j^{(s)}, j \ge 0\}$ satisfy

$$0 \le \liminf_{j\to\infty}\frac{\pi_{j+1}^{(s)}}{\pi_j^{(s)}} \le \frac{1}{c} \le \limsup_{j\to\infty}\frac{\pi_{j+1}^{(s)}}{\pi_j^{(s)}} \le 1$$ where

c is the unique solution to the equation

$x = \int_0^\infty e^{\lambda t(x-1)}dQ(t)$ other than $x = 1$.

Corollary 1

A necessary and sufficient condition for an *M/G/s* queue to be ergodic is

$$\hat{\pi}_0^{(1)} = \frac{1-\lambda EU}{\lambda EU}\left(\frac{1}{\int_0^\infty e^{-\lambda t}dG(t)}-1\right) > 0$$ or,

equivalently, $\lambda EU < 1$ where $U \sim Q$.

Corollary 2

For an *M/G/s* queue in steady state,

$$P(queueing\ \ state) = \sum_{i\ge s}\pi_i^{(s)} = \frac{\pi_s^{(s)}}{\hat{\pi}_0^{(1)}}$$

$$= \frac{(\lambda EY)^s / s!}{\hat{\pi}_0^{(1)}\left(\sum_{i=0}^{s-1}(\lambda EY)^i / i!\right)+(\lambda EY)^s / s!}.$$

Reference

[1] Kleinrock, L. (1975) Queueing Systems, Vol. I: Theory, John Wiley and Sons.

[2] Ross, M. S. (1996) Stochastic Processes, 2nd Ed. John Wiley and Sons.

附件：封面格式

行政院國家科學委員會補助專題研究計畫成果報告

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