Necessary and sufficient conditions for the stability of a sleeping top described by three forms of dynamic equations

Zheng-Ming Ge
Department of Mechanical Engineering, National Chiao Tung University, Hsinchu, Taiwan, Republic of China
(Received 31 October 2007; revised manuscript received 28 January 2008; published 15 April 2008)

Necessary and sufficient conditions for the stability of a sleeping top described by dynamic equations of six state variables, Euler equations, and Poisson equations, by a two-degree-of-freedom system, Krylov equations, and by a one-degree-of-freedom system, nutation angle equation, is obtained by the Lyapunov direct method, Ge-Liu second instability theorem, an instability theorem, and a Ge-Yao-Chen partial region stability theorem without using the first approximation theory altogether.

DOI: 10.1103/PhysRevE.77.046606 PACS number(s): 45.20.−d, 45.40.Cc

I. INTRODUCTION

The stability of a sleeping top is a classical problem which appears in the standard courses of classical mechanics [1,2]. Routh [3], Klein [4], and Grammel [5] studied this problem from various heuristic points of view. In 1946, Chetaev [6,9] first strictly studied the problem described by a two-degree-of-freedom system, Krylov equations [7]. By the Lyapunov direct method, he obtained the sufficient condition of conditional direction stability :

$$C^2\omega^2 > 4Amga, \tag{1}$$

where C is the axial moment of inertia of the top about a fixed point, A is the equatorial moment of inertia of the top about the fixed point, ω is the angular velocity of the top about the symmetric axis of the vertical sleeping top, m is the mass of the top, a is the distance between the center of gravity of the top and the fixed point, and g is the gravity acceleration. In 1954, Chetaev [8,9] studied the same problem by Euler equations and Poisson equations for six state variables and obtained the same sufficient condition of unconditional stability by the Lyapunov direct method. In 1979 using the same equations, Ge [10,11] obtained the necessary and sufficient condition of unconditional stability of a sleeping top

$$C^2\omega^2 \geq 4Amga \tag{2}$$

by the Lyapunov direct method and first approximation theory, and corrected the error of Loitsyanskii and Lurie [12], Rumjantsev [13], and Magnus [14]. They declared that the necessary and sufficient condition for stability of a sleeping top is $C^2\omega^2 > 4Amga$.

In this paper, the necessary and sufficient condition for unconditional stability and conditional direction stability of the sleeping top is obtained by using Euler equations and Poisson equations, and by using Krylov equations by the Lyapunov direct method, Ge-Liu second instability theorem [15], Ge theorem for determining the definiteness of functions [16], and an instability theorem. The necessary and sufficient condition of conditional nutation angle stability is obtain by using the nutation angle equation by the Lyapunov direct method and Ge-Yao-Chen (GYC) partial region stability theorem [17,18]. In this paper, the first approximation theory has not been used altogether.

This paper is organized as follows. In Sec. II, a necessary and sufficient condition for unconditional stability of a sleeping top by using Euler equations and Poisson equations is obtained by the Lyapunov direct method and Ge-Liu second instability theorem. In Sec. III, the same condition of conditional direction stability is obtained by using Krylov equations by the Lyapunov direct method and an instability theorem. In Sec. IV, the same condition of conditional nutation angle stability is obtained by using a nutation angle equation by the Lyapunov direct method and GYC partial region stability theorem. In Sec. V, conclusions are drawn.

II. STABILITY OF A SLEEPING TOP DESCRIBED BY EULER EQUATIONS AND POISSON EQUATIONS

A. Euler equations and Poisson equations

In Fig 1, O is the fixed point of a symmetric top. $Ox_1y_1z_1$ is an inertial frame with vertical axis z_1. $Oxyz$ is a body frame fixed with the symmetric top and coincides with the principal axes of inertia of the top. A, B, and C are the principal moments of inertia of the top about the Ox, Oy, and Oz axes, respectively. The conditions for a Lagrange top are

$$A = B, \;\; x = o, \;\; y = 0, \;\; z = a > 0, \tag{3}$$

where x, y, and z are the coordinates of the center of gravity of the Lagrange top in the $Oxyz$ frame. Rigid body motion about a fixed point with condition (3) is called a Lagrange case. Let p, q, and r be the projections of the angular velocity vector of the Lagrange top on three principal axes Ox, Oy, and Oz, respectively, $\gamma_1, \gamma_2, \gamma_3$ be the direction cosines between Ox, Oy, Oz and the vertical axis Oz_1, respectively. The

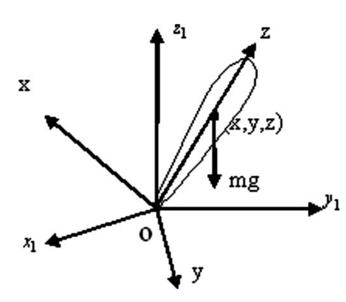

FIG. 1. Rigid body motion in the Lagrange case.

dynamic equations for a Lagrange top are the combination of Euler equations and Poisson equations:

$$A\frac{dp}{dt}+(C-A)qr=mga\gamma_2,$$

$$A\frac{dq}{dt}+(A-C)rp=-mga\gamma_1,$$

$$C\frac{dr}{dt}=0,$$

$$\frac{d\gamma_1}{dt}=r\gamma_2-q\gamma_3,$$

$$\frac{d\gamma_2}{dt}=p\gamma_3-r\gamma_1,$$

$$\frac{d\gamma_3}{dt}=q\gamma_1-p\gamma_2. \tag{4}$$

We shall study the stability of a solution, a vertical permanent rotation:

$$p=0,\ \ q=0,\ \ r=\omega,\ \ \gamma_1=0,$$

$$\gamma_2=0,\ \ \gamma_3=1 \tag{5}$$

of system (4). A Lagrange top with condition (5) is called a sleeping top. Let

$$p=\xi,\ \ q=\eta,\ \ r=\omega+\varsigma,$$

$$\gamma_1=\alpha,\ \ \gamma_2=\beta,\ \ \gamma_3=1+\delta, \tag{6}$$

where ξ, η, ς, α, β, and δ are the disturbances of six state variables in Eq (4).

B. Sufficient condition of unconditional stability

In the Lagrange case, the first integrals of Eq. (4) are

$$A(p^2+q^2)+Cr^2+2mga\gamma_3=h,$$

$$A(p\gamma_1+q\gamma_2)+Cr\gamma_3=k,$$

$$\gamma_1^2+\gamma_2^2+\gamma_3^2=1,$$

$$r=\omega, \tag{7}$$

where h, k, and ω are constants determined by initial conditions of Eq. (4). For the differential equations of disturbances ξ, η, ς, α, β, and δ, the corresponding first integrals are

$$V_1=A(\xi^2+\eta^2)+C(\varsigma^2+2\omega\xi)+2mag\delta,$$

$$V_2=A(\xi\alpha+\eta\beta)+C(\delta\varsigma+\omega\delta+\varsigma),$$

$$V_3=\alpha^2+\beta^2+\delta^2+2\delta(=0),$$

$$V_4=\varsigma. \tag{8}$$

The positive definite Lyapunov function given by Chetaev when $C^2\omega^2>4Amga$ is

$$V_{\rm I}=V_1+2\lambda V_2-(mga+C\omega\lambda)V_3+\mu V_4^2,$$

$$-2(C\omega+C\lambda)V_4=A\xi^2+2\lambda A\xi\alpha,$$

$$-(mga+C\omega\lambda)\alpha^2+A\eta^2+2\lambda A\eta B,$$

$$-(mga+C\omega\lambda)\beta^2+(C+\mu)\varsigma^2+2\lambda C\delta\varsigma,$$

$$-(mga+C\omega\lambda)\delta^2, \tag{9}$$

where $\mu=C(C-A)/A$, $\lambda=-C\omega/2A$.

We have $\dot{V}_{\rm I}=0$. By Lyapunov stability theorem, the null solution of $\xi,\eta,\varsigma,\alpha,\beta,$ and δ is stable, i.e., the solution (5) is stable. Equation (1) is the sufficient condition of stability for a sleeping top, which is given by Chetaev [8,9]. Since six disturbances correspond to the whole six states of dynamic equations, Euler equations, and Possion equations, we call this stability unconditional stability. In this case, both the magnitude and the direction of the angular velocity vector are stable.

When $C^2\omega^2\geq 4Amga$, Ge [10,11] gave another positive definite Lyapunov function

$$V=V_{\rm I}+V_{\rm II}, \tag{10}$$

where

$$V_{\rm II}=C\omega\left(V_3-\frac{4}{\omega}V_4\right)^2=C\omega\left(\alpha^2+\beta^2+\delta^2+2\delta-\frac{4}{\omega}\varsigma\right)^2 \tag{11}$$

is a positive semidefinite function of ξ, η, ς, α, β, and δ. $\dot{V}=0$ also. The Lyapunov stability theorem is satisfied, the sufficient condition for stability of a sleeping top is now

$$C^2\omega^2\geq 4Amga. \tag{12}$$

When $C^2\omega^2<4Amga$, by first approximation theory, the sleeping top is unstable. In this paper, instead of using first approximation theory, the Ge-Liu second instability theorem is used to prove that when $C^2\omega^2<4Amga$, the sleeping top is unstable.

C. Ge-Liu second instability theorem

In 1999, Ge and Liu [15] gave two instability theorems. The second of them is as follows.

Consider a nonautonomous vector differential equation

$$\dot{x}=f(t,x(t)) \qquad \forall\, t\geq 0, \tag{13}$$

where $x\in R^n$ and $f:R_+\times R^n\to R^n$ is continuous. Let $x=0$ be an equilibrium point for the system described by Eq. (13). Then $f(t,0)=0,\forall t\geq 0$. We can prove the following [15].

Theorem. If there exists a C^n function $V:R_+\times R^n\to R$, a ball $B_r=\{x\in R^n,\|x\|<r\}$, an open set $\Omega\subset B_r$, such that

(i) $0<V(t,x)\leq L<\infty$, for some L, $\forall t\geq t_0$, $\forall x\in\Omega$.

(ii) $0 \in \partial\Omega$ (the boundary of Ω).

(iii) $\dot{V}(t,x) > 0, \dot{V}(t,x)$ is uniformly continuous in t, $\forall t \geq t_0$, $\forall x \in \Omega$.

(iv) (a) There exists an even $n \geq 2$, such that for some nonempty set $G \subset \partial\Omega \cap B_r$,

$$V^i(t,x) = 0 \quad \text{for} \quad 1 \leq i \leq n-1,$$

$$\exists \gamma > 0, \quad V^{(n)}(t,x) \geq \gamma, \quad \forall t \geq t_0, \quad \forall x \in G.$$

(b) $V(t,x)=0$, $\forall t \geq t_0$, $\forall x \in \partial\Omega \cap B_r - G$,

Here $V^{(*)}(t,x)$ denotes the $(*)$th time derivative of V with respect to time. Then the equilibrium point 0 of the system (13) is unstable.

D. Sufficient condition for instability and necessary and sufficient condition for stability

Since $C^2\omega^2 < 4Amga$, choose

$$\lambda = -\frac{C\omega}{4A} \tag{14}$$

and let

$$\begin{matrix} b \\ c \end{matrix} = \frac{C\omega \pm \sqrt{C^2\omega^2 + 4(4Amga - C^2\omega^2)}}{4A},$$

$$\begin{matrix} d \\ e \end{matrix} = \frac{C\omega \pm \sqrt{C^2\omega^2 + 4(4Amga - C^2\omega^2)}}{4C}. \tag{15}$$

Since $4Amga > C^2\omega^2, b, d$ are positive, and c, e, are negative. Now

$$V_{\rm I} = A(\xi - b\alpha)(\xi - c\alpha) + A(\eta - b\beta)(\eta - c\beta) + \frac{C^2}{A}(\varsigma - d\delta)(\varsigma - e\delta)$$

is an indefinite function.

The positive definite Lyapunov function V is chosen as

$$V = V_{\rm I}^2 + \alpha^2 + \beta^2 + \delta^2, \tag{16}$$

$$\dot{V} = \frac{d}{dt}(\alpha^2 + \beta^2 + \delta^2), \tag{17}$$

since $V_{\rm I}$ is a first integral. By the third equation of Eq. (8), the sixth equation of Eq. (4), and Eq. (6)

$$\dot{V} = -2\frac{d\delta}{dt} = -2(\eta\alpha - \xi\beta) = 2(\xi\beta - \eta\alpha) \tag{18}$$

is indefinite. There exists Ω in which $\xi\beta > \eta\alpha$, $V > 0$, and $\dot{V} > 0$, where $0 \in \partial\Omega$. Since V does not contain t explicitly, conditions (i)–(iii) in the above theorem are satisfied. We shall prove that (iv, a) is also satisfied. $\ddot{V}$ can be found as

$$\ddot{V} = 2(\xi\dot{\beta} + \beta\dot{\xi} - \eta\dot{\alpha} - \alpha\dot{\eta}) = 2\left(\xi^2 - \frac{C}{A}\omega\xi\alpha - \frac{C}{A}\omega\beta\eta + \frac{mga}{A}\beta^2 + \eta^2 + \frac{mga}{A}\alpha^2\right) + 2\left(\xi^2\delta - \xi\alpha\beta - \frac{C}{A}\beta\eta\varsigma + \eta^2\delta - \frac{(C-A)}{A}\alpha\xi\varsigma\right). \tag{19}$$

On the boundary of Ω, $\partial\Omega$, $\dot{V}=0$, i.e.,

$$\xi\beta - \eta\alpha = 0. \tag{20}$$

We can prove that $\ddot{V} > 0$ on $\partial\Omega$. There are many cases satisfying Eq. (20).

(a) $\eta = f\xi, \beta = f\alpha$, where f can take an arbitrary positive value except zero. From Eq. (19),

$$\ddot{V} = 2(1+f^2)\left(\xi^2 - \frac{C}{A}\omega\xi\alpha + \frac{mga}{A}\alpha^2\right) + 2(1+f^2)\left(\xi^2\delta - \frac{C}{A}\xi\alpha\varsigma\right). \tag{21}$$

Since $4Amga > C^2\omega^2$, the second order terms of $\ddot{V}$ are a positive definite function of ξ, α, while the third order terms of $\ddot{V}$ have no influence on the definiteness of $\ddot{V}$. Therefore $\ddot{V} > 0$. When $\eta = f\beta$, $\xi = f\alpha$, or $\xi = f\beta$, $\eta = f\alpha$, $\beta = \alpha$ it can be proved similarly that $\ddot{V} > 0$.

(b) $\beta = \eta = 0$. Now

$$\ddot{V} = 2\left(\xi^2 - \frac{C\omega}{A}\xi\alpha + \frac{mga}{A}\alpha^2\right) + 2\left(\xi^2\delta - \frac{C}{A}\alpha\xi\delta\right) \tag{22}$$

is a positive definite function of ξ, α, i.e., $\ddot{V} > 0$.

When $\beta = \alpha = 0$; $\xi = \eta = 0$ or $\xi = \alpha = 0$, it can also be easily obtained that $\ddot{V} > 0$.

(c) $\beta = \alpha = \eta = 0$. Now

$$\ddot{V} = 2\xi^2 + 2\xi^2\delta \tag{23}$$

is a positive definite function of $\xi, \ddot{V} > 0$. When $\alpha = \beta = \xi = 0$, $\beta = \xi = \eta = 0$, or $\alpha = \xi = \eta = 0$ it can also be easily obtained that $\ddot{V} > 0$. By the above results, (iv,a) of the theorem is proved.

Since V is positive definite, $\partial\Omega \cap B_r - G = 0$, (iv,b) need not be proved. V satisfies the Ge-Liu second instability theorem, the sufficient condition of instability for the sleeping top is

$$C^2\omega^2 < 4Amga. \tag{24}$$

Together with the above result of Sec. II B, we conclude that the necessary and sufficient condition for unconditional stability of the sleeping top is

$$C^2\omega^2 \geq 4Amga. \tag{25}$$

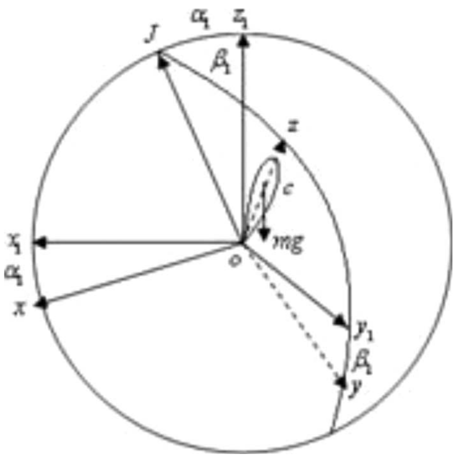

FIG. 2. Sleeping top described by Krylov equations.

III. CONDITIONAL DIRECTION STABILITY OF A SLEEPING TOP DESCRIBED BY KRYLOV EQUATIONS

A. Krylov equations

In Fig. 2, $Ox_1y_1z_1$ is the inertial frame, where Oz_1 is the vertical axis. Oz is the dynamic symmetrical axis of the sleeping top with center of gravity c, where $Oc=a$. The direction of Oz is determined by two angles: α_1, the angle between the projection of Oz on the Ox_1z_1 vertical plane, OJ, and vertical axis Oz_1; β_1 is the angle between OJ and Oz_1. The moving frame $Oxyz$ does not participate in the spin motion of the sleeping top. The motion of Oz is described by $\dot{\alpha}_1$ and $\dot{\beta}_1$. From Fig. 2, $(x,x_1)=\alpha_1$, $(y,y_1)=\beta_1$. ϕ is the spin angle of the sleeping top about Oz. The angular velocity of the sleeping top Ω is

$$\Omega = \omega_1 + \omega_2 + \omega_3, \tag{26}$$

where $\omega_1=\dot{\alpha}_1$, $\omega_2=\dot{\beta}_1$, and $\omega_3=\dot{\phi}$. The projections of Ω on the principal axes Oz,Ox,Oy are, respectively,

$$p = \dot{\phi} + \dot{\alpha}_1 \sin \beta,$$

$$q = -\dot{\beta}_1, r = \dot{\alpha}_1 \cos \beta_1. \tag{27}$$

The kinetic and potential energies of the top are

$$T = [C(\dot{\phi} + \dot{\alpha}_1 \sin \beta_1)^2 + A(\dot{\beta}_1^2 + \dot{\alpha}_1^2 \cos^2 \beta_1)]/2, \tag{28}$$

$$\Pi = mga \cos \alpha_1 \cos \beta_1, \tag{29}$$

where $\gamma=(z,z_1)$. By spherical trigonometry, $\cos \gamma = \cos \alpha_1 \cos \beta_1$. A first integral corresponding to cyclic coordinate ϕ is

$$G_Z = Ap = A(\dot{\phi} + \dot{\alpha}_1 \sin \beta_1) = \text{const}, \tag{30}$$

where G_z is the projection of the angular momentum $\mathbf{G}$ of the top of the Oz axis. Only under condition (30), two-degree-of-freedom Lagrange equations for α_1,β_1, Krylov equations can be obtained:

$$A\ddot{\alpha}_1 \cos \beta_1 - 2A\dot{\alpha}_1\dot{\beta}_1 \sin \beta_1 + C\omega\dot{\beta}_1 = mga \sin \alpha_1,$$

$$A\ddot{\beta}_1 + A\dot{\alpha}_1^2 \sin \beta_1 \cos \beta_1 - C\omega\dot{\alpha}_1 \cos \beta_1 = mga \sin \beta_1 \cos \alpha_1. \tag{31}$$

They correspond to four first order differential equations of four state variables α_1, β_1, $\dot{\alpha}_1$, and $\dot{\beta}_1$, which have zero solution $\alpha_1=\beta_1=\dot{\alpha}_1=\dot{\beta}_1=0$. This solution corresponds to undisturbed sleeping top motion. Therefore Eq. (31) is the differential equation of disturbances α_1, β_1, $\dot{\alpha}_1$, and $\dot{\beta}_1$.

B. Sufficient condition of conditional direction stability

There exist two other first integrals

$$T + \Pi = \frac{1}{2}[C(\dot{\phi} + \dot{\alpha}_1 \sin \beta_1)^2 + A(\dot{\beta}_1^2 + \dot{\alpha}_1^2 \cos^2 \beta_1)] + mga \cos \alpha_1 \cos \beta_1 = \text{const}, \tag{32}$$

$$G_{z_1} = C\omega \cos \alpha_1 \cos \beta_1 + A(\dot{\beta} \sin \alpha_1 - \dot{\alpha} \cos \alpha_1 \cos \beta_1 \sin \beta_1), \tag{33}$$

where G_{z_1} is the projection of $\mathbf{G}$ on the Oz_1 axis. Form the other two first integrals by Eqs. (30), (32), and (33):

$$W_1 = T + \Pi - \frac{G_z^2}{C} = \frac{1}{2}C(\dot{\alpha}_1^2 \cos^2\beta_1 + \dot{\beta}_1^2) + mga(\cos \alpha_1 \cos \beta_1 - 1) = \text{const}, \tag{34}$$

$$W_2 = G_{z_1} = A(\dot{\beta}_1 \sin \alpha_1 - \dot{\alpha}_1 \cos \alpha_1 \cos \beta_1 \sin \beta_1) + C\omega(\cos \alpha_1 \cos \beta_1 - 1) = \text{const}. \tag{35}$$

They become zero when the $\alpha_1=\beta_1=\dot{\alpha}_1=\dot{\beta}_1=0$. Lyapunov function is chosen as

$$V = W_1 - \lambda W_2, \tag{36}$$

where λ is a constant to be determined to make V positive definite. Express V in series:

$$V = \frac{1}{2}[A\dot{\alpha}_1^2 + 2A\lambda\dot{\alpha}_1\beta_1 + (C\omega\lambda - mga)\beta_1^2] + \frac{1}{2}[A\dot{\beta}_1^2 - 2A\lambda\dot{\beta}_1\alpha_1 + (C\omega\lambda - mga)\alpha_1^2] + \text{H.O.T.} \tag{37}$$

The degrees of higher order terms (H.O.T.) are no less than four. When λ is chosen as

$$\lambda = C\omega/2A \tag{38}$$

V is a positive definite function of α_1, β_1, $\dot{\alpha}_1$, $\dot{\beta}_1$, and $\dot{V}=0$. Lyapunov stability theorem is satisfied. Therefore when $C^2\omega^2>4Amga$, the sleeping top is conditionally stable. Since four disturbances α_1, β_1, $\dot{\alpha}_1$, and $\dot{\beta}_1$ are not all arbitrary, condition (30)must be satisfied, so we call this stability conditional stability. In this case only the direction of the angular velocity vector of the sleeping top is proven to be stable. When

$$C^2\omega^2 = 4Amga \tag{39}$$

V becomes

$$V = V_2 + V_4 + \text{H.O.T.} = \frac{1}{2}A(\dot{\alpha}_1 + C\omega\beta_1/2A)^2 + \frac{1}{2}A\left(\dot{\beta}_1 - \frac{C\omega}{2A}\alpha_1\right)^2 + V_4 + \text{H.O.T.}, \tag{40}$$

where V_2 is a second degree positive semidefinite function and V_4 is a fourth degree function. When

$$\dot{\alpha}_1 = -\lambda_2\beta_1, \quad \dot{\beta}_1 = \lambda_2\alpha_1 \tag{41}$$

$V_2=0$. Substituting Eq. (41) in V_4, after a complicated calculation we obtain

$$V_4 = \frac{mga}{8}(\alpha_1^2 + \beta_1^2)^2. \tag{42}$$

Now

$$V = mga(\alpha_1^2 + \beta_1^2)^2/8 + \text{H.O.T.}, \tag{43}$$

where H.O.T. are terms of α_1, β_1 of degree no less than six. V_4 in Eq. (42) is positive definite for α_1, β_1. We can prove that V in Eq. (40) is positive definite for $\alpha_1, \beta_1, \dot{\alpha}_1, \dot{\beta}_1$ [16]. Lyapunov stability theorem is satisfied. Therefore when $C^2\omega^2=4Amga$, the sleeping top is stable. It is concluded that the sufficient condition of conditional direction stability is $C^2\omega^2 \geq 4Amga$.

When $C^2\omega^2 < 4Amga$, by Lyapunov first approximation theory, the sleeping top is unstable. In this paper, instead of using first approximation theory, a different instability theorem is used to prove that when $C^2\omega^2 < 4Amga$, the sleeping top is unstable.

C. Instability theorem

Consider an autonomous vector differential equation

$$\dot{x} = f(x(t)) \quad \forall\, t \geq 0, \tag{44}$$

where $x \in R^n$, and $f: R^n \to R^n$ is continuous. Let $x=0$ be an equilibrium point for the system described by Eq. (44). Then $f(0)=0$, $\forall t \geq 0$.

Theorem. If there exists a C^n positive definite function $V: R^n \to R$, a ball $B_r = \{x \in R^n \| x\| \leq r\}$, and

(i) There exists an open set $\Omega \subset B_r$ in which $\dot{V}(x) = |(O|x|^2)| > 0$.

(ii) $O \subset \partial\Omega$ (the boundary Ω).

(iii) For $\partial\Omega \cap B_r$, $\dot{V}(x) = O(|x|^4)$, $\ddot{V}(x) = |O(|x|^2)|$, then the equilibrium O of Eq. (24) is unstable.

Proof. For any trajectory initiated in Ω, we assume that it can escape Ω by moving across $\partial\Omega$. When a trajectory approaches and touches $\partial\Omega$, $\dot{V}$ diminishes from $|O|x|^2|$ to $\dot{V} = O(|x|^4)$ by (i) and (iii), i.e., $\ddot{V}$ is negative; but by (iii), $\ddot{V} = |O(|x|^2)| > 0$. This shows that it is not true that $x(t)$ leaves Ω through $\partial\Omega$.

Next we prove that $x(t)$ must leave B_r through the sphere $\|x\| = r$. The initial point x_0 is in the interior of Ω and $V(x_0) = a > 0$. The trajectory $x(t)$ starting from $x(0) = x_0$ must leave Ω. To prove this fact, we notice that as long as $x(t)$ is inside Ω, $V(x) \geq a$, since $\dot{V}(x) > 0$ in Ω. Let $\gamma = \min\{\dot{V}(x) | x \in \Omega \cup \partial\Omega \quad \text{and} \quad V(x) \geq a\}$ which exists since the continuous function $\dot{V}(x)$ has a minimum over the compact set $\{x \in \Omega \cup \partial\Omega \quad \text{and} \quad V(x) \geq a\}$. Then $\gamma > 0$ and

$$V[x(t)] = V(x_0) + \int_0^t \dot{V}[x(s)]ds \geq a + \int_0^t \gamma ds = a + \gamma. \tag{45}$$

This inequality shows that $x(t)$ cannot stay forever in Ω because V is bounded on Ω. Hence $x(t)$ must leave Ω through the sphere $\|x\| = r$. The origin is unstable.

D. Sufficient condition for instability and the necessary and sufficient condition for conditional direction stability

When $C^2\omega^2 < 4Amga$, choose $\lambda = -C\omega/4A$ and use b, c, d, and e in Eq. (15), where b, d are positive, while c, e are negative. Now

$$V_\text{I} = A(\xi - b\alpha)(\xi - c\alpha) + A(\eta - b\beta)(\eta - c\beta) + \frac{C^2}{A}(\varsigma - d\delta)(\varsigma - e\delta).$$

The positive definite Lyapunov function is chosen as

$$V = \dot{\alpha}_1^2 + \dot{\beta}_1^2 + \alpha_1^2 + \beta_1^2. \tag{46}$$

Through Eq. (31), indefinite

$$\dot{V} = 2\frac{(mga + A)}{A}(\alpha_1\dot{\alpha}_1 + \beta_1\dot{\beta}_1) + O(|\alpha^4|) \tag{47}$$

is obtained. There exists Ω in which $\dot{V} = |O(|\alpha_1|^2)| > 0$. We have

$$\ddot{V} = 2\frac{(mga + A)}{A}\left[\frac{mga(\alpha_1^2 + \beta_1^2)}{A} + \frac{C\omega}{A}(\beta_1\dot{\alpha}_1 - \alpha_1\dot{\beta}_1) + \dot{\alpha}_1^2 + \dot{\beta}_1^2\right]. \tag{48}$$

In Ω, when $\alpha_1 + \xi\beta_1 > 0$, $\dot{\alpha}_1 - \dot{\beta}_1/\xi > 0$, where ξ takes any positive value except 0, then $\dot{V} = |O(|\alpha_1|^2)| > 0$. On $\partial\Omega$, $\alpha_1 + \xi\beta_1 = 0$, $\dot{\alpha}_1 - \dot{\beta}_1/\xi = 0$, $\dot{V} = O(|\alpha_1|^4)$, and $\ddot{V}$ becomes

$$\ddot{V} = 2\frac{(mga + A)}{A}\left[\frac{mga(\xi^2 + 1)}{A}\beta_1^2 + \frac{C\omega}{A}\left(\xi + \frac{1}{\xi}\right)\beta_1\dot{\beta}_1 + \left(\frac{1}{\xi^2} + 1\right)\dot{\beta}_1^2\right]. \tag{49}$$

By Sylvester theorem, since $4Amga > C^2\omega^2$, $\ddot{V}$ is positive definite, i.e., $\ddot{V} > 0$ for any ξ. Similarly, when $\alpha_1 - \xi\beta_1 > 0$, $\dot{\alpha}_1 + \dot{\beta}_1/\xi > 0$; $\alpha_1 + \xi\dot{\beta}_1 > 0$, $\dot{\alpha}_1 - \beta_1/\xi > 0$; $\alpha_1 - \xi\dot{\beta}_1 > 0$, $\dot{\alpha}_1 + \beta_1/\xi > 0$, we can also prove that $\ddot{V} > 0$. The above theorem is satisfied. When $C^2\omega^2 < 4amg$, the motion is unstable. It is concluded that the necessary and sufficient condition for conditional direction stability of a sleeping top is also $C^2\omega^2 \geq 4Amga$.

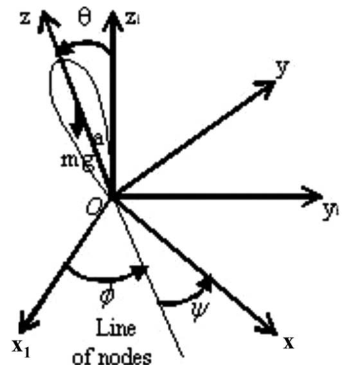

FIG. 3. Three generalized coordinates ϕ, ψ, and θ.

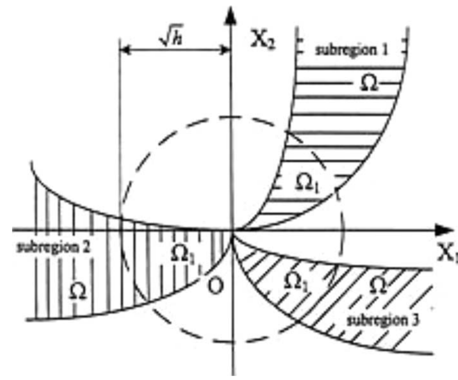

FIG. 4. Partial regions Ω and Ω_1.

IV. CONDITIONAL NUTATION ANGLE STABILITY FOR A SLEEPING TOP DESCRIBED BY NUTATION ANGLE EQUATION

A. Nutation angle equation

In Fig. 3, the symmetric top motion can be described by the Lagrange equation of three generalized coordinates, precession angle ϕ, spin angle ψ, and nutation angle θ. Since ϕ,ψ are cyclic coordinates, there are two corresponding first integrals:

$$(I_1 \sin^2 \theta + I_3 \cos^2 \theta)\dot{\phi} + I_3\dot{\psi}\cos\theta = \text{const},$$

$$I_3(\dot{\psi} + \dot{\phi}\cos\theta) = \text{const}, \tag{50}$$

where I_1 is the equatorial principal moment of inertia, and I_3 is the axial principal moment of inertia. By Eq. (50), $\phi,\dot{\phi},\psi,\dot{\psi}$ are absent in the only dynamic equation, nutation angle equation. For the sleeping top case, the energy equation is [1]

$$\dot{u}^2 = (1-u^2)[\beta(1+u-a^2)], \tag{51}$$

where $u=\cos\theta$, $\beta=2mga/A$, $a=C\omega/A$. Taking the time derivative of Eq. (51), the nutation angle equation is obtained:

$$\ddot{u} = -(a^2+\beta)u + \frac{3}{2}\beta u^2 + a^2 - \frac{\beta}{2}. \tag{52}$$

For the sleeping top, $u=\cos\theta=\cos 0=1$. Let $u=1+u'$, where u' is disturbance, the standard form equations of disturbances become

$$\dot{u}' = v',$$

$$\dot{v}' = (2\beta - a^2)u' + \frac{3}{2}\beta u'^2. \tag{53}$$

B. GYC partial region stability theorem

Ge, Yao, and Chen [17,18] gave a stability theorem on the partial region of the neighborhood (whole space for global stability) of the origin.

Consider an autonomous differential equation

$$\dot{x} = f(x), \tag{54}$$

where $x\in R^n$, and $f:R^n\to R^n$ is continuous and satisfies the Lipschitz condition. Let $x=0$ be an equilibrium point for the system described by Eq. (56), and $f(0)=0$.

We are only interested in stability of this zero solution on the partial region Ω (including the boundary) of the neighborhood of the origin which in general may consist of several subregions as shown in Fig. 4. It is stipulated that the state point cannot go out of Ω.

Definition. The equilibrium point $x=0$ of Eq. (54) is stable on Ω if for each $\varepsilon>0$ there is $\delta=\delta(\varepsilon)>0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall\, t \geq 0, \tag{55}$$

where $x\neq 0$ is any point in Ω.

Let us consider a continuously differentiable function $V(x)$ given on $\Omega_1=\Omega\cap H$ where H is the region $\|x\|\leq h>0$. If $V(x)>0$ in Ω_1 and $V(0)=0$ expect at origin, $V(x)$ is positive definite. If $V(x)\geq$ in Ω_1 and $V(0)=0$, $V(x)$ is positive semidefinite.

Theorem. If $V(x)$ is positive definite, $\dot{V}(x)$ through Eq. (54) is negative semidefinite, $x=0$ is stable in Ω.

The proof of this theorem is similar to that of the Lyapunov stability theorem [19].

C. Necessary and sufficient condition for conditional nutation angle stability

There are three cases.

(a) $\alpha^2-2\beta>0$.

The positive definite Lyapunov function is chosen as

$$V = \frac{v'^2}{2} + \frac{(\alpha^2-2\beta)u'^2}{2} - \frac{\beta u'^3}{2}. \tag{56}$$

The time derivative of $\dot{V}$ through any solution of Eq. (53) is

$$\dot{V} = v'\left[(2\beta-\alpha^2)u' + \frac{3\beta u'^2}{2}\right] - \left[(2\beta-\alpha^2)u' + \frac{3\beta u'^2}{2}\right]v'$$
$$= 0.$$

By Lyapunov stability theorem, the motion is stable.

(b) $\alpha^2-2\beta=0$.

Equation (53) becomes

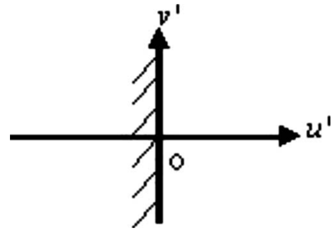

FIG. 5. Partial region.

$$\dot{u}' = v',$$

$$\dot{v}' = \frac{3\beta u'^2}{2}. \tag{57}$$

Since $u=\cos 0=1$ is the maximum value of $\cos\theta$, $u'=u-1$ is always negative. The partial region is the left half plane of the $u'v'$ plane as shown in Fig. 5.

The partial region positive definite Lyapunov function is chosen as

$$V = \frac{v'^2}{2} - \frac{\beta u'^3}{2}. \tag{58}$$

The time derivative of V through any solution of Eq. (57) is

$$\dot{V} = \frac{3v'\beta u'^2}{2} - \frac{3\beta u'^2 v'}{2} = 0. \tag{59}$$

By GYC partial region stability theorem, the motion is stable.

(c) $\alpha^2 - 2\beta < 0$.

The indefinite Lyapunov function is chosen as

$$V = u'v'. \tag{60}$$

The time derivation of V through any solution of Eq. (53) is

$$\dot{V} = v'\dot{u}' + \dot{v}'u' = v'^2 + u'\left[(2\beta - \alpha^2)u' + \frac{3\beta u'^2}{2}\right] = v'^2 + (2\beta - \alpha^2)u'^2 + \frac{3\beta u'^3}{2}, \tag{61}$$

which is positive definite. By Lyapunov first instability theorem, the motion is unstable.

From above results, we obtain that the necessary and sufficient condition for conditional nutation angle stability is also

$$C^2\omega^2 \geq 4Amga. \tag{62}$$

Since two conditions in Eq. (50) must be satisfied, the stability is called the conditional nutation angle stability.

V. CONCLUSIONS

The necessary and sufficient condition for the stability of a sleeping top described by three forms of dynamic equations is obtained. For dynamic equations of six stable variables, Euler equations, and Poisson equations, unconditional stability is obtained by the Lyapunov direct method and the Ge-Liu second instability theorem. For dynamic equations of a two-degree-of-freedom system, Krylov equations, conditional direction stability is obtained by the Lyapunov direct method and a different instability theorem. For dynamic equations of a one-degree-of-freedom system, a nutation angle equation, conditional nutation angle stability is obtained by the Lyapunov direct method and GYC partial region stability theorem. The necessary and sufficient condition for a sleeping top obtained from the above three cases is the same:

$$C^2\omega^2 \geq 4Amga. \tag{63}$$

By using the direct method, unconditional instability, conditional direction stability for $C^2\omega^2 \geq 4Amga$, conditional direction instability, and three cases for conditional nutation angle stability and instability, the results were obtained in this paper.

The classical problem of classical mechanics has been studied for more than 100 years and is solved in this paper by the direct method only without the use of first approximation theory.

ACKNOWLEDGMENTS

This research was supported by the National Science Council, Republic of China under Grant No. NSC 96-2221-E-009-145-MY3.

[1] H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics*, 3rd ed. (Addison Wesley, San Francisco, 2002), pp. 210–220.
[2] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed. (Springer-Verlag, New York, 1989), pp. 154–155.
[3] E. J. Routh, *The Advance Part of a Treatise on the Dynamics of a System of Rigid Bodies* (Dover, London, 1955).
[4] F. Klein, Bull. Am. Math. Soc. **3**, 129 (1897); **3**, 292 (1897).
[5] R. Grammel, Math. Z. **6**, 124 (1920).
[6] N. G. Chetaev, Prikl. Mat. Mekh. **10** (l), (1946), in Russian.
[7] A. N. Krylov, *Rotating Motion of an Oval Shell in Flight (Complete Works)* (Gostehizdat, Leningrad, 1929) (in Russian).
[8] N. G. Chetaev, Prikl. Mat. Mekh. **18**(1), 123 (1954) in Russian.
[9] N. G. Chetayev, *The Stability of Motion* (Pergamon Press, New York, 1961).
[10] Z.-M. Ge, J. Shanghai Jiaotong Univ. **13**(1), 53 (1979).
[11] Z.-M. Ge and Y.-J. Wu, J. Appl. Mech. **51**, 430 (1984).
[12] L. G. Loitsjansky and A. I. Lure, *Course of Theoretical Mechanics* (Gostehizdat, Moscow, 1948), pp. 572–573.

[13] V. V. Rumjantsev, Prikl. Mat. Mekh. **19**, 339 (1957).
[14] K. Magnus, Prikl. Mat. Mekh. **22**, 173 (1958)
[15] Z.-M. Ge and W.-P. Liu, Jpn. J. Appl. Phys. **38**, 3793 (1999).
[16] Z.-M. Ge, Appl. Math. Mech. **2**, 689 (1981).
[17] Z.-M. Ge, C.-W. Yao, and H. K. Chen, J. Chin. Soc. Mech. Eng. **15**, 140 (1994).
[18] Z.-M. Ge, J.-K. Yu, and H. K. Chen, Jpn. J. Appl. Phys. **37**, 2762 (1998).
[19] H. K. Khalil, *Nonlinear Systems*, 3rd ed. (Prentice Hall, Englewood Cliffs, NJ, 2002), pp. 144–145.