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Sphere-separable partitions of multi-parameter elements

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Abstract

We show the optimality of sphere-separable partitions for problems where n vectors in d-dimensional space are to be partitioned into p categories to minimize a cost function which is dependent in the sum of the vectors in each category; the sum of the squares of their Euclidean norms; and the number of elements in each category. We further show that the number of these partitions is polynomial in n. These results broaden the class of partition problems for which an optimal solution is guaranteed within a prescribed set whose size is polynomially bounded in n. Applications of the results are demonstrated through examples.

Keywords: Partitions; Combinatorial optimization; Separation; Polynomial bounds

1. Introduction

We consider partition problems where a finite set $N = \{1, \ldots, n\}$ is to be partitioned into p sets so as to minimize some cost function. An (*ordered*) *p-partition* of N is a vector $\pi = (\pi_1, \ldots, \pi_p)$ where $\pi_1, \ldots, \pi_p$ are disjoint sets whose union is N; we refer to $\pi_1, \ldots, \pi_p$ as the *parts* of π and to the vector $\langle\pi\rangle = (|\pi_1|, \ldots, |\pi_p|)$ (where $|\pi_j|$ is the cardinality of the part π_j) as the *shape* of π. Problems where partitions' shapes are restricted to given sets are referred to as *constrained-shape partition problems*, and instances of such problems in which partitions' shapes are prescribed are referred to as *single-shape partition problems*. Finally, problems where partitions' shapes are unrestricted are referred to as *open-shape partition problems*.

It is assumed throughout that each element i in N is associated with a vector $A^i \in R^d$ whose coordinates represent attributes of i, and the cost associated with a partition depends on the sum of the attributes of the elements that are assigned to each part. Specifically, for each p-partition $\pi = (\pi_1, \ldots, \pi_p)$ we let

$$A^\pi = \left[\sum_{i\in\pi_1} A^i, \ldots, \sum_{i\in\pi_p} A^i\right] \in R^{d\times p}, \tag{1}$$

the (i, j) element of A^π represents the total value of attribute i of the elements assigned to the part π_j. In each of the partition problems we consider, the cost $F(\pi)$ associated with a partition π is a function of A^π (and possibly other

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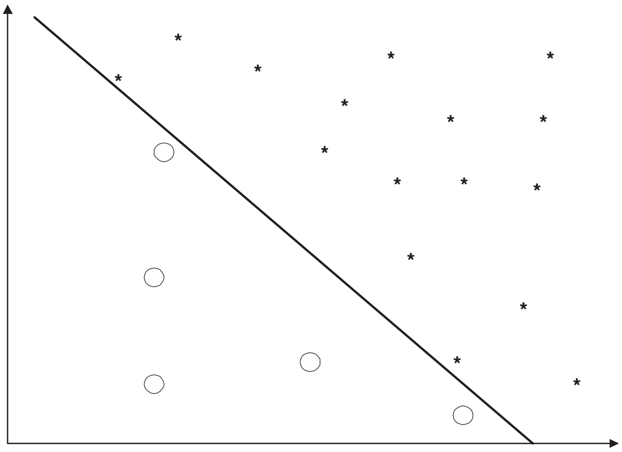

Fig. 1. A two-dimensional hyperplane-separable 2-partition.

characteristics of π). For simplicity, we assume throughout that the A^i's are distinct; see [10] for approaches that relax this assumption.

The number of p-partitions of N is p^n, in particular, it is exponential in n. It is therefore useful to identify classes of partition problems for which it is possible to guarantee the existence of optimal partitions having some (pre-identified) structure that is satisfied by a polynomial number (in n) of partitions; here, "structure" is used synonymously to "property". For a systematic study of various partition-properties see [8,12] and references therein.

One useful property of partitions is hyperplane-separability. Specifically, a p-partition π is *hyperplane-separable* if for each pair of indices $j, k = 1, \ldots, p$ with $j < k$, there exists a vector $b \in R^{1\times d}$ and a scalar γ such that for (the closed half-space) $C = \{x \in R^d : b^{\mathrm{T}}x \leqslant \gamma\}$, either

$$\{A^i : i \in \pi_j\} \subseteq C \quad \text{and} \quad \{A^i : i \in \pi_k\} \cap C = \emptyset \tag{2}$$

or

$$\{A^i : i \in \pi_k\} \subseteq C \quad \text{and} \quad \{A^i : i \in \pi_j\} \cap C = \emptyset; \tag{3}$$

See Fig. 1. Partition problems that are guaranteed to have hyperplane-separable optimal partitions were identified in [3] and a polynomial bound (in n) on the number of hyperplane-separable partition was derived in [10] (see Theorem 1 and the following remarks). A comprehensive set of examples of partition problems that satisfy the assumptions of [3] is provided in [12] and references therein.

A p-partition π is *convex-separable* if for each pair of indices $j, k = 1, \ldots, p$ with $j < k$, there exists a closed convex set C such that either (2) or (3) are satisfied; see Fig. 2. An equivalent requirement asserts that, respectively, either $[\mathrm{conv}\{A^i : i \in \pi_j\}] \cap \{A^i : i \in \pi_k\} = \emptyset$ or $[\mathrm{conv}\{A^i : i \in \pi_k\}] \cap \{A^i : i \in \pi_j\} = \emptyset$; this formulation is used in [4,5] where the adjective *nested* is used. A p-partition is *connected* if the relation over pairs (j, k) defined by $\mathrm{conv}\{A^i : i \in \pi_j\} \cap \{A^i : i \in \pi_k\} = \emptyset$ is acyclic. Fig. 3 (cf., the example that proves Theorem 4.1 in [8]) demonstrates that convex-separability does not imply connectedness (a statement in [4], possibly a typo, incorrectly asserts that convex-separability and connectedness are equivalent).

Convex-separability of optimal partitions was explored in [4,5]. In particular, it was established in Theorem 3.4 of [5] that every optimal partition of a single-shape partition problems with cost function

$$F^1(\pi) = \sum_{j=1}^{p} w_j \sum_{i,t\in\pi_j} \|A^i - A^t\|^2 \quad \text{for every partition } \pi, \tag{4}$$

with $w_1, \ldots, w_p$ as arbitrary positive numbers and with $\|\cdot\|$ as the l_2 norm in R^d is convex separable (the special case with $w_i = 1$ for each $i = 1, \ldots, p$ was established in Theorem 1.2 of [4]). Also, Theorem 3.1 of [5] shows that

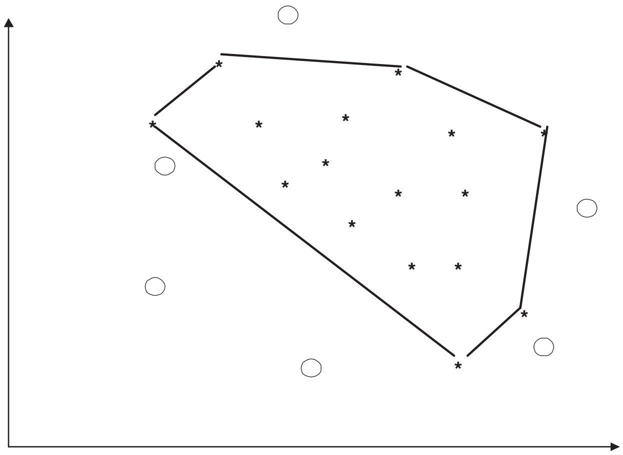

Fig. 2. A two-dimensional convex-separable 2-partition.

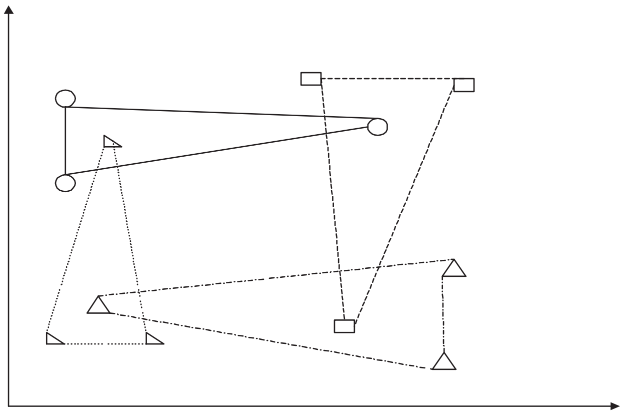

Fig. 3. A two-dimensional convex-separable 4-partition which is not connected.

optimal partitions for single-shape partition problem with cost function

$$F^2(\pi) = \sum_{j=1}^{p} w_j \sum_{i\in\pi_j} \|A^i - \frac{1}{|\pi_j|}\sum_{t\in\pi_j} A^t\|^2 \quad \text{for every partition } \pi \tag{5}$$

are convex-separable. The introduction of multivariate convex-separability along with the last result resolved a question, posed in [6] and left open for four decades, about extending the special instance of the above result (with $w_i = 1$ for each $i = 1, \ldots, p$) from $d = 1$ to $d > 1$. Partition problems with cost functions given by (5) have applications for the multivariate stratification problem.

When $d = 1$, the number of convex-separable p-partitions is known to equal $(1/(n-p+1))\binom{n-1}{p-1}\binom{n}{p}$ (see [9]); in particular, the number is polynomial in n. But, when $d > 1$, the set of convex-separable partitions may coincide with the set of all p^n partitions. For example, this is the case when all the A^i's are on the surface of a common ball (cf., [8, Theorem 3.6]); as none of the vectors is in the convex hull of any subset of the others (see Fig. 4), all partitions satisfy a more demanding property than convex separability asserting that for all corresponding j, k and C both (2) and (3) are satisfied (this property, is called *nonpenetrating*). As it is possible for convex-separability to impose no restrictions, results that establish the existence of convex-separable optimal partitions have limited (computational) usefulness.

The purpose of the current paper is to consider a property of partitions, referred to as "sphere-separability" – it is a relaxation of hyperplane-separability that admits at most a polynomial number of partitions while the class is

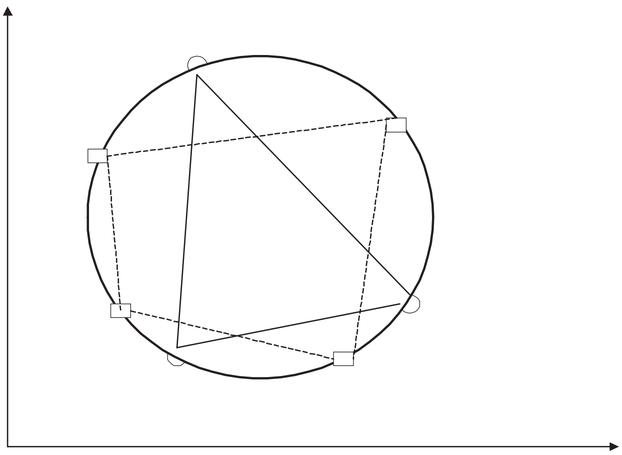

Fig. 4. A two-dimensional convex-separable partition with A^i's on a unit ball.

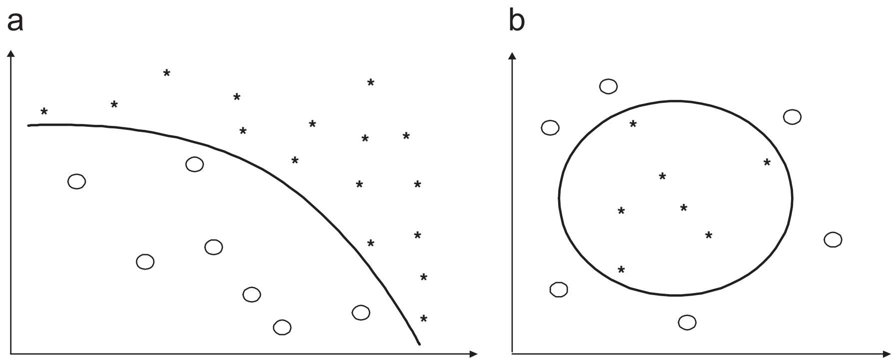

Fig. 5. Examples of two-dimensional sphere-separable partitions.

guaranteed to contain optimal partitions for optimization problems for which there need not exist hyperplane-separable optimal partitions. Specifically, a p-partition π is *sphere-separable* if for each pair of indices $j, k = 1, \ldots, p$ with $j < k$ there exists a vector $b \in R^d$ and a scalar γ such that either (2) or (3) are satisfied by the set C given by either $\{x \in R^d : b^T x \leqslant \gamma\}$ or $\{x \in R^d : \|x - b\| \leqslant \gamma\}$; see Fig. 5.

For $d = 1$, sphere-separability clearly coincides with convex-separability. For $d > 1$, sphere-separability implies convex-separability, but, Fig. 4 demonstrates a convex-separable partition which is not sphere-separable (each sphere that contains all the circles must contain at least two squares and each sphere that contains all the squares must contain at least two circles—a formal verification of this fact is not included and is left to the reader). Thus, sphere-separability is strictly more restrictive than convex-separability. In fact, in the forthcoming Theorem 2 we provide a polynomial bound (in n) on the number of sphere-separable partitions (whereas the best general bound on the number of convex-separable partitions is p^n). Theorem 2 also identifies a class of partition problems for which sphere-separable optimal partitions are guaranteed to exist, and a subclass for which *all* optimal partitions are sphere-separable. Cost functions over partitions for which this result applies include the functions F^1 and F^2 given, respectively, in (4) and in (5) where each w_j is allowed to be a function of the shape of the underlying partition π (e.g., $w_j(|\pi_j|) = 1/|\pi_j|$). Another function to which the results applies is the function

$$F^3(\pi) = \sum_{j=1}^{p} w_j \sum_{i \in \pi_j} \|A^i - u^j\|^2 \quad \text{for every partition } \pi, \tag{6}$$

with $u^1, \ldots, u^p$ as prescribed d-vectors and with $w_1, \ldots, w_p$ as in (4) and (5).

We review and update results about hyperplane-separability in Section 2 and present our main results about sphere-separability in Section 3. Concluding remarks are provided in Section 4.

2. Hyperplane-separability

In this section we record extensions of known results about hyperplane-separable partitions.

The first lemma records a useful hierarchy for the presence of properties in optimal solutions for single-shape and constrained-shape partition problems (surprisingly, the simple result seems to be missing from the literature—see the paragraph preceding Theorem 1 and the example that follows that theorem).

Lemma 1. *Consider a cost function F over p-partitions and a property Q of p-partitions such that for each single-shape partition problem with cost function F, Q is satisfied by some (every) optimal partition. Then, for each constrained-shape partition problem with cost function F, Q is satisfied by some (every) optimal partition.*

Proof. We first establish the lemma with "some" rather than "every". Consider a partition problem where F is to be minimized over the set of partitions whose shape must lie in a prescribed set Γ of integer p-vectors whose coordinate sum is n. Let π be an optimal partition for this problem. Consider the single-shape partition problem where F is to be minimized over p-partitions with shape $\langle\pi\rangle$; π is obviously optimal for this problem. By assumption, Q is satisfied by some optimal partition for this single-shape partition problem, say π'. As both π and π' are optimal for the same partition problem, $F(\pi)=F(\pi')$. Now, as $\langle\pi'\rangle=\langle\pi\rangle \in \Gamma$ and $F(\pi)=F(\pi')$, π' is feasible and optimal for the underlying constrained-shape partition problem; so π' satisfies Q and is optimal for that problem. The case with"every" replacing "some" follows from the above arguments and the observation that π must satisfy Q as it is optimal for the corresponding single-shape partition problem. □

Remark. Lemma 1 can be modified by considering any particular set Γ of nonnegative, integer p-vectors whose coordinates sum to n. Specifically, the arguments that prove Lemma 1 show that if for every $(n_1, \ldots, n_p) \in \Gamma$, property Q of partitions is satisfied by some/every optimal partition corresponding to each single-shape problem with the cost function F defined over partitions whose shape is $(n_1, \ldots, n_p)$, then Q must also be satisfied by some/every optimal partition of the constrained-shape problem with cost function F over partitions whose shape must be in Γ. The modification allows one to restrict the domain of F.

Theorem 1 records a polynomial bound on the number of hyperplane-separable partitions (obtained in [10]), and a generalization of a sufficient condition for the optimality of hyperplane-separable partitions (obtained in [3] for partition problems with shape-constraints restricted to lower and upper bounds and with objective functions that are shape-independent). Let $Z_\oplus$ be the set of nonnegative integers.

Theorem 1. *Assume that the A^i's are distinct and d and p are fixed.*

(i) *The number of hyperplane-separable partitions is bounded by* $O[n^{d\binom{p}{2}}]$.
(ii) *If the cost function F over the partitions is given by $F(\pi)=g(\langle\pi\rangle, A^\pi)$ for each partition π where g is a real-value function on $Z_\oplus^p \times R^{d\times p}$ which is quasi-concave in its $d \times p$ real variables for each fixed value of its integer variables, then each constrained-shape partition problem has a hyperplane-separable optimal partition; further, if the A^i's are nonzero and the quasi-concavity holds strictly, then every optimal partition is hyperplane-separable.*

Proof. Part (i) was established in [10]. As for part (ii), Lemma 1 implies that it suffices to consider single-shape problems. Consider a partition problem with cost function F where partitions are restricted to have shape $(n_1, \ldots, n_p)$. One can view $n_1, \ldots, n_p$ as parameters of the cost function, that is, write $F(\pi) = h_{n_1,\ldots,n_p}(A^\pi)$, where $h_{n_1,\ldots,n_p}$ is the real-valued function on $R^{d\times p}$ with $h_{n_1,\ldots,n_p}(x) = g[(n_1, \ldots, n_p), x]$ for each $x \in R^{d\times p}$. The assumption in part (ii) asserts that $h_{n_1,\ldots,n_p}$ is quasi-concave and therefore Theorem 4 of [3] implies the existence of an optimal partition which is hyperplane-separable. Under the assumptions of the second part of (ii), $h_{n_1,\ldots,n_p}$ is strictly quasi-concave and the A^i's are nonzero; the conclusion that every optimal partition is hyperplane-separable then follows from Theorem 7 of [3] using similar result to those used above. □

Remark.

(1) Beyond the bound in part (i) of Theorem 1, [10] provides a strongly polynomial algorithm (in n with d and p fixed) for enumerating the set of all A^π's associated with hyperplane-separable partitions, but with the restriction that it is independent of the integer variables.

(2) Ref. [13] identifies a smaller set of partitions than the hyperplane-separable ones which is guaranteed to contain an optimal partition for open-shape problems with cost function F as in Theorem 1.

(3) The bound of part (i) of Theorem 1 was improved in [2] to $O(n^{d-1})$ when $p=2$ and to $O(n^p)$ when $d=2$. Further, [1] confirmed that $O[n^{d\binom{p}{2}}]$ is the best upper bound when $p \geqslant 3$ and $d \geqslant 3$.

(4) Part (ii) of Theorem 1 includes applications that were *not* covered by [3], e.g., constrained-shape partition problems with objective function that depends on the shape. See also the application of Theorem 2 to the objective function F^2.

(5) The assumption that the A^i's are distinct is clearly necessary for part (ii) of Theorem 1 (e.g., no separability can be expected when all A^i's coincide). Ref. [10] establishes the conclusions of Theorem 1 when duplicate A^i's are allowed under a weaker notion of hyperplane-separability asserting that $|\{A^i : i \in \pi_j\} \cap \{A^i : i \in \pi_k\}| \leqslant 1$ for each distinct $j, k \in \{1, \dots p\}$ with the separability inequalities "$b^T A^i < b^T A^t$" holding whenever $A^i \neq A^t$.

(6) The conclusions of part (ii) of Theorem 1 apply when quasi-concavity is relaxed to edge-quasi-concavity; see [11]. This extension (with $d=1$) applies to the maximization of system-reliability in an assembly problem, see [11,12]. But, as the sum of quasi-concave functions need not be quasi-concave, the most useful applications of part (ii) of Theorem 1 have g concave in the corresponding variables.

(7) The remark following Lemma 1 allows one to extend the conclusion of part (ii) of Theorem 1 to cost functions which are defined only over restricted shapes, for example, to shapes with no empty parts (which allows for g to include expressions like $1/|\pi_j|$).

(8) An alternative approach to proving the conclusions of part (ii) of Theorem 1 is to consider for each $i = 1, \dots, n$ the vector A^i by $\bar{A}^i = \binom{A^i}{1}$ and for each partition π the matrix $\bar{A}^\pi \in R^{(d+1)\times p}$ defined by (1) with the $\bar{A}^i$'s replacing the A^i's. $F(\pi) = g(\langle\pi\rangle, A^\pi)$ can then be expressed as $f(\bar{A}^\pi)$ for a corresponding function f, and results of [3] can be applied directly. But, this approach requires the stronger assumption that asserts that g is (strictly) quasi-concave in all of its $(d+1)p$ variables.

3. Sphere-separability

This section provides modifications of the conclusions of Theorem 1 that apply to sphere-separability.

We will consider partition problems where the cost associated with a partition π depends on A^π and on $\Sigma_{i\in\pi_1} \|A^i\|^2, \dots, \Sigma_{i\in\pi_p} \|A^i\|^2$. To capture such functions, let $\widehat{A^i} = \binom{A^i}{\|A^i\|^2} \in R^{d+1}$ for $i = 1, \dots, n$, and let $\widehat{A^\pi}$ be defined by (1) for each partition π with the $\widehat{A^i}$'s substituting for the A^i's. We next use the $\widehat{A^i}$'s to establish a reduction of sphere-separability to hyperplane-separability.

Lemma 2. *A partition π is sphere-separable if and only if it is hyperplane-separable under the problem where the vectors associated with the elements are $\widehat{A}^1, \dots, \widehat{A}^n$, respectively.*

Proof. We first observe that for every $x, y, u \in R^d$,

$$\|x+u\| < \|y+u\| \quad \text{if and only if} \quad \|x\|^2 + 2u^T x < \|y\|^2 + 2u^T y. \tag{7}$$

Assume that π is a hyperplane-separable partition for the problem where the items are associated, respectively, with $\widehat{A}^1, \dots, \widehat{A}^n$, and let $j, k \in \{1, \dots, p\}$ be distinct indices. It follows that there exists a vector $b \in R^d$ and scalar β such that

$$(b^T, \beta)\begin{pmatrix} A^i \\ \|A^i\|^2 \end{pmatrix} < (b^T, \beta)\begin{pmatrix} A^t \\ \|A^t\|^2 \end{pmatrix} \quad \text{for each } i \in \pi_j \quad \text{and} \quad t \in \pi_k. \tag{8}$$

Now, if $\beta=0$, then (8) implies the original definition of hyperplane-separability with respect to the vector b. Alternatively, if $\beta > 0$, (using (7) with $x = R^i$ and $u = b/2\beta$) (8) can be rewritten as

$$\left\| A^i + \frac{b}{2\beta} \right\|^2 < \left\| A^t + \frac{b}{2\beta} \right\|^2 \quad \text{for each } i \in \pi_j \quad \text{and} \quad t \in \pi_k. \tag{9}$$

With γ as any positive scalar satisfying

$$\left\| A^i + \frac{b}{2\beta} \right\|^2 < \gamma^2 < \left\| A^t + \frac{b}{2\beta} \right\|^2 \quad \text{for each } i \in \pi_j \quad \text{and} \quad t \in \pi_k. \tag{10}$$

We have that (2) is satisfied with C as the sphere

$$C = \left\{ x \in R^d : \left\| x - \left(-\frac{b}{2\beta} \right) \right\| \leqslant \gamma \right\}. \tag{11}$$

Finally, if $\beta < 0$, the inverse of the inequalities of (9) hold. With γ as any positive scalar for which the inverse inequalities of (10) hold, (3) is satisfied with C as the sphere given in (11).

Next assume that π is a sphere-separable partition and let $j, k \in \{1, \ldots, p\}$ be distinct indices. Now, if (2) or (3) holds with $C = \{x \in R^d : b^{\mathrm{T}} x \leqslant \gamma\}$, then either the inequalities of (8) or their reversed version hold with $\beta = 0$. Alternatively, if (2) holds with $C = \{x \in R^d : \|x - b\| \leqslant \gamma\}$, then (10) holds with $\beta = -\frac{1}{2}$ and with the first inequality holding weakly, from which we deduce (9) and (8). Symmetric argument applies when (3) holds with $C = \{x \in R^d : \|x - b\| \leqslant \gamma\}$. □

The reduction of sphere-separability to hyperplane-separability allows one to use Theorem 1 to explore sphere-separability. We state two results (the first of which corrects an error in [8]).

Theorem 2. *Assume that the A^i's are distinct and d and p are fixed.*

(i) *The number of sphere-separable partitions is bounded by* $\mathrm{O}[n^{(d+1)\binom{p}{2}}]$.
(ii) *If the cost function F over partitions is given by $F(\pi) = g(\langle\pi\rangle, \widehat{A}^{\pi})$ for each partition π where g is a real-value function on $Z_{\oplus}^p \times R^{(d+1)\times p}$ which is quasi-concave in its $(d+1) \times p$ real variables for each fixed value of its integer variables, then each single-shape, open-shape and constrained-shape partition problem has a sphere-separable optimal partition; further, if the quasi-concavity holds strictly, then every optimal partition is sphere-separable.*

Proof. Part (i) is immediate from Lemma 1 and part (i) of Theorem 1. Next, for part (ii), Lemma 1 shows that it suffices to consider single-shape partition problems. As in the proof of Theorem 1, one can view the prescribed shape as parameters of the cost function and write $F(\pi) = h_{n_1,\ldots,n_p}(\widehat{A}^{\pi})$ for each partition π where $(n_1, \ldots, n_p)$ is the prescribed shape and $h_{n_1,\ldots,n_p}$ is the real-valued function on $R^{(d+1)p}$ with $h_{n_1,\ldots,n_p}(x) = g[(n_1, \ldots, n_p), x]$ for each $x \in R^{(d+1)\times p}$. Our assumptions imply that $h_{n_1,\ldots,n_p}$ is quasi-concave, and therefore part (ii) of Theorem 1 implies the existence of an optimal partition which is hyperplane-separable for the problem where the $\widehat{A}^i$'s are the vectors associated with the elements. By Lemma 2, any such partition is sphere-separable. The conclusions under strict quasi-concavity and nonzero A^i's follow from similar arguments. □

The remarks that follow Theorem 1 apply, with corresponding adjustments, to Theorem 2.

We next apply part (ii) of Theorem 2 to the functions F^1, F^2 and F^3 defined, respectively, by (4), (5) and (6).

Corollary 1. *Assume that the A^i's are distinct and the function over partitions is either F^1 or F^2 or F^3 as defined, respectively, by* (4), (5) *and* (6). *Then each constrained-shape partition problem has a sphere-separable optimal partition.*

Proof. The functions F^1, F^2 and F^3 satisfy, respectively, the quasi-convexity assumptions of part (ii) of Theorem 2 with respect to the functions g^1, g^2 and g^3 listed in Table 1, where $Z = (Z_1, \ldots Z_p)$ are their p integer variables and $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X^1 \ldots X^p \\ Y^1 \ldots Y^p \end{pmatrix}$ are their $(d+1) \times p$ real variables. The conclusions of the corollary are now immediate from part (ii) of Theorem 2. □

Table 1
Examples of cost functions leading to sphere-separable partitions

k	Equivalent form of $F^k(\pi)$	$g^k\left[Z, \binom{X}{Y}\right] = g^k\left[Z_1, \ldots, Z_p, \binom{X^1}{Y^1}, \ldots, \binom{X^p}{Y^p}\right]$
1	$\sum_{j=1}^{p} 2w_j\vert\pi_j\vert[\sum_{i\in\pi_j}\Vert A^i\Vert^2 - \Vert\sum_{i\in\pi_j} A^i\Vert^2]$	$\sum_{j=1}^{p} 2w_j(Y^j - \Vert X^j\Vert^2)$
2	$\sum_{j=1}^{p} w_j[\sum_{i\in\pi_j}\Vert A^i\Vert^2 - \frac{1}{\vert\pi_j\vert}\Vert\sum_{i\in\pi_j} A^i\Vert^2]$	$\sum_{j=1}^{p} w_j(Y^j - \frac{1}{Z_j}\Vert X^j\Vert^2)$
3	$\sum_{j=1}^{p} w_j[\sum_{i\in\pi_j}\Vert A^i\Vert^2 - (\sum_{i\in\pi_j} A^i)^{\mathrm{T}}u^j + \vert\pi_j\vert\Vert u^j\Vert^2]$	$\sum_{j=1}^{p} w_j[Y^j - 2(X^j)^{\mathrm{T}}u^j + Z_j\Vert u^j\Vert^2]$

The conclusions of Corollary 1 sharpen results of [4,5] that were reviewed in the Introduction in two ways. First, the earlier results established only convex-separability rather than the more restrictive sphere-separability. Second, they considered only special instances of the partition-optimization problems considered in Corollary 1. For the significance of partition optimization problems with cost functions given by F^1 and F^2 see the references cited in the Introduction.

4. Concluding remarks

This paper studies sphere-separability of partitions and position this property in the context of other types of separability that have been investigated in recent years. In particular, we identify conditions under which the (polynomial) class of sphere-separable partitions includes optimal solutions. Such partitions might be useful in situations where one needs to classify objects into distinct categories, where either the absolute or the relative number of objects in each category is set a priori (hence, single-shape problems) and it is important that the objects in each class will be as close as possible to the “center of gravity” of the class. The simplicity offered by the sphere structure may enable the development of useful approximation algorithms, cf. [7].

References

[1] N. Alon, S. Onn, Separable partitions, Discrete Appl. Math. 91 (1999) 39–51.
[2] S. Aviram, N. Lev-Tov, S. Onn, U.G. Rothblum, Vertex characterization of partition polytopes of planar point sets, Discrete. Appl. Math. 125 (2002) 1–15.
[3] E.R. Barnes, A.J. Hoffman, U.G. Rothblum, Optimal partitions having disjoint convex and conic hulls, Math. Programming 54 (1992) 69–86.
[4] E. Boros, P.L. Hammer, On clustering problems with connected optima in Euclidean spaces, Discrete Math. 75 (1989) 81–88.
[5] E. Boros, F.K. Hwang, Optimality of nested partitions and its applications to cluster analysis, SIAM J. Optam. 6 (1995) 1153–1162.
[6] W.D. Fisher, On grouping for maximum homogeneity, J. Amer. Statist. Assoc. 53 (1958) 789–798.
[7] M. Hassin, S. Rubinstein, Approximation algorithms for the metric maximum clustering problem with given cluster sizes, Oper. Res. Lett. 31 (2003) 179–184.
[8] F.K. Hwang, J.S. Lee, Y.C. Liu, U.G. Rothblum, Sortability of vector partitioning, Discrete. Math. 263 (2003) 129–142.
[9] F.K. Hwang, C.L. Mallows, Enumerating consecutive and nested partitions, J. Combin. Theory Ser. A 70 (1995) 323–333.
[10] F.K. Hwang, S. Onn, U.G. Rothblum, A polynomial time algorithm for shaped partition problems, SIAM J. Optam. 10 (1999) 70–81.
[11] F.K. Hwang, U.G. Rothblum, Directional-quasi-convexity, Schur-convexity and the optimality of consecutive partitions, Math. Oper. Res. 21 (1996) 540–554.
[12] F.K. Hwang, U.G. Rothblum, Partitions: Optimality and Clustering, 2007, forthcoming.
[13] S. Onn, L.J. Schulman, The vector partition problem for convex objective functions, Math. Oper. Res. 26 (2001) 583–590.