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Short Paper

Hamiltonian Connectivity, Pancyclicity and 3^*-Connectivity of Matching Composition Networks*

SHIN-SHIN KAO, JUI-CHIA WU AND YUAN-KANG SHIH[1]
Department of Applied Mathematics
Chung Yuan Christian University
Chungli, 320 Taiwan
E-mail: skao@math.cycu.edu.tw
[1]*Department of Computer Science*
National Chiao Tung University
Hsinchu, 300 Taiwan

In this paper, we discuss many properties of graphs of *Matching Composition Networks* (*MCN*) [16]. A graph in *MCN* is obtained from the disjoint union of two graphs G_0 and G_1 by adding a perfect matching between $V(G_0)$ and $V(G_1)$. We prove that any graph in *MCN* preserves the hamiltonian connectivity or hamiltonian laceability, and pancyclicity of G_0 and G_1 under simple conditions. In addition, if there exist three internally vertex-disjoint paths between any pair of distinct vertices in G_i for $i \in \{0, 1\}$, then so it is the case in any graph in *MCN*. Since *MCN* includes many well-known interconnection networks as special cases, such as the Hypercube Q_n, the Crossed cube CQ_n, the Twisted cube TQ_n, the Möbius cube MQ_n, and the Hypbercube-like graphs HL_n, our results apply to all of the above-mentioned networks.

Keywords: hypercube-like graphs, perfect matching, hamiltonian-connected, pancyclic, 3^*-connected

1. INTRODUCTION

The *hypercube* Q_n is one of the most popular network topologies due to its attractive properties such as regularity, edge and vertex symmetry, and strong connectivity. However, the hypercube does not have the smallest diameter for its resources. By twisting some pairs of edges in a hypercube, many hypercube variants are proposed to reduce the diameter [1, 7-9]. Cull and Larson [6] surveyed many hypercube variants and concluded that they are all hamiltonian. Vaidya *et al.* [24] further introduced the class of hypercube-like graphs HL_n defined as follows: $HL_1 = K_2$, where K_2 is the two-vertex complete graph. A graph in HL_n is obtained from the disjoint union of two graphs G_0 and G_1 in HL_{n-1} by adding a perfect matching between $V(G_0)$ and $V(G_1)$. Obviously, the family HL_n contains most of hypercube variants and preserves the recursive structure of the hypercube.

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Instead of studying hypercube-like graphs in HL_n, which require the number of vertices being 2^n, Lai *et al.* [16] proposed a class of *Matching Composition Networks*, denoted by *MCN*. Let G_0 and G_1 be two graphs with $|V(G_0)| = |V(G_1)| = m$, $MCN^0 = \{G_0, G_1\}$, and MCN^1 be the set of all graphs obtained by adding a perfect matching between $V(G_0)$ and $V(G_1)$. Obviously, $|V(G)| = 2m$ for $G \in MCN^1$. Similarly, MCN^k denotes the set of all graphs obtained by adding a perfect matching between any two graphs in MCN^{k-1} for $k \geq 2$ and $|V(G)| = 2^k \cdot m$ for $G \in MCN^k$. We use the symbol *MCN* for the collection of graphs in MCN^k for all $k \geq 0$. It is obvious that *MCN* includes many well-known interconnection networks as special cases, such as the Hypercube Q_n, the Crossed cube CQ_n, the Twisted cube TQ_n, the Möbius cube MQ_n, and the Hypbercube-like graphs HL_n by recursively applying the construction [16]. The Folded Petersen Cube Networks $FPQ_{n,k}$ [21], where $FPQ_{n,k} = FPQ_{n-1,k} \times K_2$ and $|V(FPQ_{n,k})| = 2^n \cdot 10^k$ for $n \geq 0$ and $k \geq 0$, is also a family of graphs belonging to *MCN*. In this article, we prove that a graph $G \in MCN$ composed by G_0 and G_1 preserves the hamiltonian connectivity and pancyclicity of G_0 and G_1 under simple conditions. In addition, if there exist three internally vertex-disjoint paths between any pair of distinct vertices in G_i for $i \in \{0, 1\}$, then so it is the case in G. The results give a unified way of proving many properties of the hypercube variants. In fact, the study of these characters is shared by all networks with a recursive structure as of the hypercube-like graphs.

2. PRELIMINARIES

For the graph definitions and notations we follow [2]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set* of G. Two vertices u and v are *adjacent* if $(u, v) \in E$. A *path* P between two vertices v_0 and v_k is represented by $P = \langle v_0, v_1, v_2, \ldots, v_k\rangle$ where each pair of consecutive vertices are connected by an edge. We shall say that *P joins* v_0 to v_k. The *length* of a path P is the number of edges in P. We also write the path $\langle v_0, v_1, v_2, \ldots, v_k\rangle$ as $\langle v_0, P_1, v_i, v_{i+1}, \ldots, v_j, P_2, v_t, \ldots, v_k\rangle$, where P_1 is the path $\langle v_0, v_1, \ldots, v_{i-1}, v_i\rangle$ and P_2 is the path $\langle v_j, v_{j+1}, \ldots, v_{t-1}, v_t\rangle$. Hence, it is possible to write a path $\langle v_0, v_1, P, v_1, v_2, \ldots, v_k\rangle$ if the length of P is zero. If a path $Q = \langle v_0, v_1, v_2, \ldots, v_k\rangle$, then Q^{-1} denotes the path $\langle v_k, v_{k-1}, \ldots, v_1, v_0\rangle$. A *hamiltonian path* between u and v, where u and v are two distinct vertices of G, is a path joining u to v that visits every vertex of G exactly once. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A *hamiltonian graph* is a graph with a hamiltonian cycle.

A graph $G = (V, E)$ is *connected* if there is a path between any two distinct vertices in G. A graph $G = (V, E)$ is *hamiltonian connected* if there is a hamiltonian path between any two distinct vertices in G. A graph $G = (B \cup W, E)$ is *bipartite* if $V(G) = B \cup W$, $B \cap W = \emptyset$, and $E(G)$ is a subset of $\{(u, v) \mid u \in B, v \in W\}$. A bipartite graph $G = (B \cup W, E)$ is *balanced* if $|B| = |W|$. Let G be a balanced bipartite graph. Since any hamiltonian path in G consists of the same number of vertices of the two partite sets, there exists no hamiltonian path between two vertices belonging to the same partite set of G. Thus G is not hamiltonian connected. We say that a bipartite graph G is *hamiltonian laceable* if there is a hamiltonian path between any pair of vertices $\{x, y \mid x \in B, y \in W\}$.

A graph is *pancyclic* if it contains a cycle of every length from 3 to $|V(G)|$ inclusive. A graph is *r-pancyclic* if it contains a cycle of every length from r to $|V(G)|$ inclusive. The concept of pancyclic graphs is proposed by Bondy [3]. It is known that there is no odd cycle in any bipartite graph. Hence, any bipartite graph is not pancyclic. For this reason, the concept of bipancyclicity is proposed [20]. A bipartite graph is *bipancyclic* if it contains a cycle of every even length from 4 to $|V(G)|$ inclusive. It is proved that the hypercube is bipancyclic [17, 23].

An l-container $C(u, v)$ in a graph G is a set of l internally vertex-disjoint paths between two distinct vertices u and v. An l-container $C(u, v)$ in a graph G is an l-container such that every vertex of G is on some path in $C(u, v)$. It follows from Menger's Theorem [19] that in a k-connected graph, there exist k internally vertex-disjoint paths between any pair of distinct vertices. A graph G is 3^*-*connected* if there exists a 3^*-container between any two distinct vertices of G. A graph G is *bi*-3^*-*connected* if G is a bipartite graph and there exists a 3^*-container between any two distinct vertices from the opposite partite sets of G. It is shown in [14, 15] that if $G = (B \cup W, E)$ is a bi-3^*-connected graph, then $|B| = |W|$.

In this article, we always let $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ be two graphs such that $|V_0| = |V_1| = t$, where $t \geq 2$ is an integer. Let M be an arbitrary perfect matching between $V(G_0)$ and $V(G_1)$. That is, M is a set of t edges with one endpoint in G_0 and the other endpoint in G_1. Define $G = (G_0, G_1; M)$ such that $V(G) = V_0 \cup V_1$ and $E(G) = E_0 \cup E_1 \cup M$. We use the notation $(x, \bar{x})$ for any edge of M. Thus for $i = 0$ or 1, if x is a vertex of G_i, then $\bar{x}$ denotes the corresponding vertex of x in G_{1-i} under M. For $i \in \{0, 1\}$, if G_i is a bipartite graph, then $G_i = (B_i \cup W_i, E_i)$ where B_i and W_i denote the bipartite sets of G_i. G is obtained by connecting G_0 and G_1 with $M = \{(x, y) \mid x \in V_0, y \in V_1\}$ and $|M| = t$. Thus G is bipartite if G contains no odd cycle, and G is nonbipartite if otherwise. If G is a bipartite graph, we let $G = (B \cup W, E)$ where B and W denote the bipartite sets of G. Here we only consider the cases where $|B_i| = |W_i|$ for $i \in \{0, 1\}$ and $|B| = |W|$.

Definition 1 Let $G = (B \cup W, E)$. We say that G satisifies the two-path property if for given $\{u, u'\} \subset B$ and $\{v, v'\} \subset W$, there exist two vertex disjoint paths P and Q such that P joins u to v, Q joins u' to v', and $P \cup Q$ covers all vertices of G.

3. HAMILTONIAN CONNECTIVITY AND HAMILTONIAN LACEABILITY

Chang *et al.* [4] proved that the hypercube Q_n satisfies the two-path property for $n \geq 2$. Park and Chwa [22] proved that the hypercube-like graph HL_n satisfies the two-path property for $n \geq 2$, and that every HL_n is hamiltonian laceable or hamiltonian connected depending on whether it is bipartite or not. By the similar deduction as in [22], we can prove the following lemma and theorems.

Lemma 1 Let G_i be a bipartite hamiltonian laceable graph satisfying the two-path property for $i \in \{0, 1\}$. Let $G = (G_0, G_1; M)$ be a bipartite graph. Then G satisfies the two-path property.

Theorem 1 Let G_0 and G_1 be two bipartite hamiltonian laceable graphs. Then $G = (G_0,$

G_1; M) is hamiltonian laceable if G is a bipartite graph, or is hamiltonian connected if G is a nonbipartite graph.

Theorem 2 Let G_0 and G_1 be two nonbipartite hamiltonian connected graphs. Then $G = (G_0, G_1; M)$ is hamiltonian connected.

Theorem 3 Let G_0 be a bipartite hamiltonian laceable graph satisfying the two-path property and G_1 be a nonbipartite hamiltonian connected graph. Then $G = (G_0, G_1; M)$ is hamiltonian connected.

By brute force, it is easy to check that $FPQ_{1,1}$ is hamiltonian connected. Thus we know $FPQ_{n,1}$ is hamiltonian connected for all $n \geq 1$ using Theorem 2 recursively. Consequently, with Theorems 1-3, every graph $G = (G_0, G_1; M)$ in *MCN* preserves the hamiltonian laceability or hamiltonian connectivity of G_0 and G_1 depending on it is bipartite or not.

4. PANCYCLICITY AND BI-PANCYCLICITY

Theorem 4 Let G_0 and G_1 be two graphs that are bipartite, hamiltonian laceable and suppose that both G_0 and G_1 are bipancyclic with $|V(G_i)| = t \geq 4$ for $i = 0, 1$. Then $G = (G_0, G_1; M)$ is bipancyclic if G is a bipartite graph.

Proof: It is known that $V(G) = B \cup W$, $V_i = B_i \cup W_i$ for $i = 0, 1$ and $|V_i| = t$ for some even integer $t \geq 4$. Obviously, $|V(G)| = 2t$. Since G is bipartite, we need to construct a cycle of length l for every even integer l with $4 \leq l \leq 2t$. G_1 is bipancyclic, so there is a cycle of length l for every even integer l with $4 \leq l \leq t$. It suffices to construct cycles of length n for every even integer n with $t + 2 \leq n \leq 2t$. Let $\mathcal{C}$ be a hamiltonian cycle of G_0 and $\mathcal{C} = \langle u_1, u_2, \ldots, u_t, u_1\rangle$. Without loss of generality, suppose that $u_1 \in W_0$. Let $l' = n - t$. Obviously, l' is an even integer with $2 \leq l' \leq t$, thus $u_{l'} \in B_0$. Since G is bipartite, $\overline{u}_1, \overline{u}_{l'} \in V(G_1)$ and they belong to the opposite partite sets of G_1. Since G_1 is hamiltonian laceable, there is a hamiltonian path Q between $\overline{u}_1$ and $\overline{u}_{l'}$ in G_1. Thus $\langle u_1, \overline{u}_1, Q, \overline{u}_{l'}, u_{l'}, u_{l'-1}, u_{l'-2}, \ldots, u_1\rangle$ is cycle in G with length $(l' - 1) + 2 + (t - 1) = l' + t = n$. Thus we obtain the cycle with length n for every even integer n with $t + 2 \leq n \leq 2t$. ❑

Recursively applying the results in section 3 and Theorem 4, we show that every bipartite graph $G = (G_0, G_1; M)$ in *MCN* preserves the bipancyclicity of G_0 and G_1. In particular, the hypercube Q_n and the hypercube-like graph HL_n are bipancyclic for $n \geq 2$ [22, 23].

Lemma 2 For $i \in \{0, 1\}$, let G_i be a hamiltonian connected graph and $G_i - \{x\}$ is hamiltonian connected for any $x \in V(G_i)$. Suppose that $|V(G_i)| = t \geq 3$. Let $G = (G_0, G_1; M)$. Then $G - \{v\}$ is hamiltonian connected for any $v \in V(G)$.

Proof: Given three distinct vertices x, y and z of G, we want to show that there is a hamiltonian path between x and y in $G - \{z\}$. There are three cases.

Case 1: $\{x, y, z\} \subset V(G_0)$. Since $G_0 - \{z\}$ is hamiltonian connected, there exists a hamiltonian path P in $G_0 - \{z\}$ between x and y. Let $P = \langle x, x', P_1, y\rangle$ with $x \neq x'$. Since G_1 is hamiltonian connected, there exists a hamiltonian path Q in G_1 between $\overline{x}$ and $\overline{x}'$. Thus $\langle x, \overline{x}, Q, \overline{x}', x', P_1, y\rangle$ is a hamiltonian path between x and y in $G - \{z\}$.

Case 2: $\{x, y\} \subset V(G_0)$ and $z \in V(G_1)$. Since G_0 is hamiltonian connected, there exists a hamiltonian path P in G_0 between x and y. Let $P = \langle x = x_0, x_1, x_2, \ldots, x_{t-2}, x_{t-1} = y\rangle$. Since $t \geq 3$, there exists an integer k with $0 \leq k \leq t - 2$ such that $\{\overline{x}_k, \overline{x}_{k+1}\} \cap \{z\} = \emptyset$. Since $G_1 - \{z\}$ is hamiltonian connected, there exists a hamiltonian path Q in $G_1 - \{z\}$ between $\overline{x}_k$ and $\overline{x}_{k+1}$. Thus $\langle x, x_1, \ldots, x_k, \overline{x}_k, Q, \overline{x}_{k+1}, x_{k+1}, x_{k+2}, \ldots, y\rangle$ is a hamiltonian path between x and y in $G - \{z\}$.

Case 3: $\{x, z\} \subset V(G_0)$ and $y \in V(G_1)$. Since $t \geq 3$, there is a vertex u in $G_0 - \{x, z\}$ with $\overline{u} \neq y$. Since $G_0 - \{z\}$ is hamiltonian connected, there is a hamiltonian path P between x and u. Since G_1 is hamiltonian connected, there is a hamiltonian path Q between $\overline{u}$ and y. Thus $\langle x, P, u, \overline{u}, Q, y\rangle$ is a hamiltonian path between x and y in $G - \{z\}$. ❑

Theorem 5 Let G_0 and G_1 be two graphs that are hamiltonian connected and r-pancyclic for some integer $r \geq 3$. For $i \in \{0, 1\}$, $G_i - \{x\}$ is hamiltonian connected for any $x \in V(G_i)$. Suppose that $|V(G_i)| = t \geq 3$. Let $G = (G_0, G_1; M)$. Then G is r-pancyclic and $G - \{v\}$ is hamiltonian connected for any $v \in V(G)$.

Proof: That $G - \{v\}$ is hamiltonian connected for any $v \in V(G)$ is proved in Lemma 2. Since $|V(G)| = 2t$, to prove the pancyclicity, we need to construct a cycle of length n for every integer n with $r \leq n \leq 2t$. Since G_1 is r-pancyclic, there is a cycle of length n for every integer n with $r \leq n \leq t$. It suffices to construct cycles of length n for every integer n with $t + 1 \leq n \leq 2t$. Let $\mathcal{C}$ be a hamiltonian cycle of G_0 and $\mathcal{C} = \langle u_1, u_2, \ldots, u_t, u_1\rangle$. Let $l' = n - t$. Obviously, l' is an integer with $1 \leq l' \leq t$.

Case 1: $2 \leq l' \leq t$. Since G_1 is hamiltonian connected, there is a hamiltonian path Q between $\overline{u}_1$ and $\overline{u}_{l'}$ in G_1. Thus $\langle u_1, \overline{u}_1, Q, \overline{u}_{l'}, u_{l'}, u_{l'-1}, u_{l'-2}, \ldots, u_1\rangle$ is cycle in G with length $(l' - 1) + 2 + (t - 1) = l' + t$. Thus we obtain the cycle with length n for $t + 2 \leq n \leq 2t$.

Case 2: $l' = 1$. Let $v \in V(G_1) - \{\overline{u}_1, \overline{u}_2\}$. Since $G_1 - \{v\}$ is hamiltonian connected, there is a hamiltonian path Q between $\overline{u}_1$ and $\overline{u}_2$ in $G_1 - \{v\}$. Obviously, $|Q| = t - 2$. Thus $\langle u_1, \overline{u}_1, Q, \overline{u}_2, u_2, u_1\rangle$ is cycle in G with length $1 + 2 + (t - 2) = t + 1$. Thus we obtain the cycle with length n for $n = t + 1$. ❑

Recursively applying the results in section 3, Lemma 2 and Theorem 5, we show that a graph $G = (G_0, G_1; M)$ in MCN preserves the pancyclicity of G_0 and G_1. Suppose that $n \geq 4$ and H is a graph in the class of CQ_n, TQ_n, and MQ_n. With the above theorem, it is easy to prove that $H - \{v\}$ is hamiltonian connected for any $v \in V(H)$ and H is 4-pancyclic, *i.e.*, H contains cycles of any length at least four. The corresponding properties are studied in [5, 10-13, 25, 26]. Applying the results to hypercube-like graphs, we show the existence of cycles of any length at least four in HL_n, as was mentioned in [22].

Theorem 6 Let G_0 be a bipartite graph that is hamiltonian laceable and bipancyclic. Let G_1 be a hamiltonian connected and r-pancyclic graph for some integer $r \geq 3$. Suppose that $G_1 - \{x\}$ is hamiltonian connected for any $x \in V(G_1)$. Then $G = (G_0, G_1; M)$ is r-pancyclic.

Proof: Obviously, $|V_i| = t$ for $i = 0, 1$ and $|V(G)| = 2t$. We need to construct a cycle of length n for every integer n with $r \leq n \leq 2t$. Since G_1 is r-pancyclic, there is a cycle of length n for every integer n with $r \leq n \leq t$. It suffices to construct cycles of length n for $t + 1 \leq n \leq 2t$. Let $\mathcal{C}$ be a hamiltonian cycle of G_0 and $\mathcal{C} = \langle u_1, u_2, \ldots, u_t, u_1\rangle$. Without loss of generality, suppose that $u_1 \in W_0$. Let $l' = n - t$. Obviously, l' is an integer with $1 \leq l' \leq t$. The cycle construction follows the two cases as in Theorem 5. ❑

By Theorem 6, we know that a graph $G = (G_0, G_1; M)$ in MCN composed by letting $G_0 = Q_n$ a hypercube and $G_1 = HL_n$ some hypercube-like graph is 4-pancyclic, *i.e.* G contains cycles of length l for any integer l satisfying $4 \leq l \leq 2^{n+1}$.

5. 3^*-CONNECTIVITY AND BI-3^*-CONNECTIVITY

Theorem 7 Let G_0 and G_1 be two graphs that are bipartite, hamiltonian laceable and satisfy the two-path property. Then $G = (G_0, G_1; M)$ is bi-3^*-connected if G is a bipartite graph and is 3^*-connected if G is a nonbipartite graph.

Proof: It is known that $V(G) = B \cup W$, $V(G_0) = B_0 \cup W_0$, and $V(G_1) = B_1 \cup W_1$.

Case 1: Suppose that G is a bipartite graph. Given $u, v \in V(G)$ such that u and v belong to the opposite partite sets, we need to find a 3^*-container of G between u and v. Let it be $A = \{P_1, P_2, P_3\}$. We construct A as follows.

Case 1.1: $u, v \in G_0$. Without loss of generality, let $u \in W_0$ and $v \in B_0$. Obviously, $\bar{u} \in B_1$ and $\bar{v} \in W_1$. Since G_0 is hamiltonian laceable and hence it is hamiltonian, there is a 2^*-container $\{P_1, P_2\}$ of G_0 between u and v. Since G_1 is hamiltonian laceable, there is a hamiltonian path P of G_1 between $\bar{u}$ and $\bar{v}$. We set $P_3 = \langle u, \bar{u}, P, \bar{v}, v\rangle$.

Case 1.2: $u \in G_0$ and $v \in G_1$ with $\bar{u} = v$. Without loss of generality, let $u \in W_0$. Obviously, $v \in B_1$. We can choose two distinct vertices $x \in W_0 - \{u\}$ and $y \in B_0$. Obviously, $\bar{x} \in B_1$ and $\bar{y} \in W_1$. Since G_0 is hamiltonian laceable, there is a hamiltonian path Q of G_0 between x and y. Again, there is a hamiltonian path R of G_1 between $\bar{x}$ and $\bar{y}$. We write $Q = \langle x, Q_1, u, Q_2, y\rangle$ and $R = \langle \bar{x}, R_1, v, R_2, \bar{y}\rangle$. We set $P_1 = \langle u, Q_1^{-1}, x, \bar{x}, R_1, v\rangle$, $P_2 = \langle u, Q_2, y, \bar{y}, R_2^{-1}, v\rangle$, and $P_3 = \langle u, v\rangle$.

Case 1.3: $u \in G_0$ and $v \in G_1$ with $\bar{u} \neq v$. Without loss of generality, let $u \in W_0$ and $v \in B_1$. Obviously, $\bar{v} \in W_0$. We can choose a vertex $x \in B_0$. Since G_0 is hamiltonian laceable, there is a hamiltonian path Q of G_0 between x and $\bar{v}$. Again there is a hamiltonian path R of G_1 between $\bar{x}$ and $\bar{u}$. We write $Q = \langle x, Q_1, u, Q_2, \bar{v}\rangle$ and $R = \langle \bar{x}, R_1, v, R_2, \bar{u}\rangle$. We set $P_1 = \langle u, \bar{u}, R_2^{-1}, v\rangle$, $P_2 = \langle u, Q_1^{-1}, x, \bar{x}, R_1, v\rangle$, and $P_3 = \langle u, Q_2, \bar{v}, v\rangle$.

Case 2: Suppose that G is a nonbipartite graph. Let u and v be any two distinct vertices of G. Without loss of generality, we assume that $u \in W_0$ and $\overline{u} \in B_1$. We need to find a 3^*- container of G between u and v. Let $A = \{P_1, P_2, P_3\}$ be the 3^*-container of G between u and v. A is constructed as follows.

Case 2.1: $v \in V_0$ and $\overline{v} \in W_1$. Since G_0 is hamiltonian laceable and hence it is hamiltonian, there is a 2^*-container $\{P_1, P_2\}$ of G_0 between u and v. Since G_1 is hamiltonian laceable, there is a hamiltonian path P of G_1 between $\overline{u}$ and $\overline{v}$. We set $P_3 = \langle u, \overline{u}, P, \overline{v}, v\rangle$.

Case 2.2: $v \in W_0$ and $\overline{v} \in B_1$. We can choose a vertex $x \in B_0$ such that $\overline{x} \in W_1$ and choose a vertex $y \in W_0$ such that $\overline{y} \in W_1$. Since G_0 is hamiltonian laceable, there is a hamiltonian path Q of G_0 between x and y. Without loss of generality, we write $Q = \langle x, Q_1, p, Q_2, q, Q_3, y\rangle$ where $\{p, q\} = \{u, v\}$. By the two-path property, there are two paths T_1 and T_2 of G_1 such that (1) T_1 joins $\overline{x}$ to $\overline{q}$, (2) T_2 joins $\overline{p}$ to $\overline{y}$, (3) $T_1 \cap T_2 = \phi$, and (4) $T_1 \cup T_2$ covers all vertices of G_1. We set $P_1 = \langle p, Q_2, q\rangle$, $P_2 = \langle p, Q_1^{-1}, x, \overline{x}, T_1, \overline{q}, q\rangle$, $P_3 = \langle p, \overline{p}, T_2, \overline{y}, y, Q_3^{-1}, q\rangle$.

Case 2.3: $v \in B_0$ and $\overline{v} \in B_1$. We can choose a vertex $x \in B_0$ such that $\overline{x} \in W_1$ and choose a vertex $y \in W_0$ such that $\overline{y} \in W_1$. Since G_0 is hamiltonian laceable, there is a hamiltonian path Q of G_0 between x and y. Without loss of generality, we write $Q = \langle x, Q_1, p, Q_2, q, Q_3, y\rangle$ where $\{p, q\} = \{u, v\}$. By the two-path property, there are two paths T_1 and T_2 of G_1 such that (1) T_1 joins $\overline{x}$ to $\overline{q}$, (2) T_2 joins $\overline{p}$ to $\overline{y}$, (3) $T_1 \cap T_2 = \phi$, and (4) $T_1 \cup T_2$ covers all vertices of G_1. We set $P_1 = \langle p, Q_2, q\rangle$, $P_2 = \langle p, Q_1^{-1}, x, \overline{x}, T_1, \overline{q}, q\rangle$, $P_3 = \langle p, \overline{p}, T_2, \overline{y}, y, Q_3^{-1}, q\rangle$.

Case 2.4: $v \in V_1$ and $\overline{u} \neq v$. Obviously $u \in W_0$, $\overline{u} \in B_1$, and $\overline{v} \in V_0$. We can choose a vertex $x \in V_0$ such that (1) x and $\overline{v}$ belong to the opposite bipartition of G_0, and (2) $\overline{x} \in W_1$. Since G_0 is hamiltonian laceable, there is a hamiltonian path Q of G_0 between x and $\overline{v}$. Similarly, since G_1 is hamiltonian laceable, there is a hamiltonian path T of G_1 between $\overline{x}$ and $\overline{u}$. Write $Q = \langle x, Q_1, u, Q_2, \overline{v}\rangle$ and $T = \langle \overline{x}, T_1, v, T_2, \overline{u}\rangle$. We set $P_1 = \langle u, \overline{u}, T_2^{-1}, v\rangle$, $P_2 = \langle u, Q_2, \overline{v}, v\rangle$, and $P_3 = \langle u, Q_1^{-1}, x, \overline{x}, T_1, v\rangle$.

Case 2.5: $v \in V_1$ and $v = \overline{u}$. Obviously, $u \in W_0$ and $\overline{u} \in B_1$, we can choose a vertex $x \in W_0$ such that $\overline{x} \in W_1$ and choose a vertex $y \in B_0$ such that $\overline{y} \in B_1$. Since G_0 is hamiltonian laceable, there is a hamiltonian path Q of G_0 between x and y. Again, there is a hamiltonian path T of G_1 between $\overline{x}$ and $\overline{y}$. Without loss of generality, we write that $Q = \langle x, Q_1, u, Q_2, y\rangle$ and $T = \langle \overline{x}, T_1, v, T_2, \overline{y}\rangle$. We set $P_1 = \langle u, v\rangle$, $P_2 = \langle u, Q_1^{-1}, x, \overline{x}, T_1, v\rangle$, and $P_3 = \langle u, Q_2, y, \overline{y}, T_2^{-1}, v\rangle$. ❑

Theorem 8 Let G_0 and G_1 be two hamiltonian connected graphs. Then $G = (G_0, G_1; M)$ is 3^*-connected.

Proof: Let u and v be any two distinct vertices of G. We need to find a 3^*-container of G between u and v. Let $A = \{P_1, P_2, P_3\}$ be the 3^*-container of G between u and v. A is constructed as follows.

Case 1: $v \in G_0$. Since G_0 is hamiltonian connected and hence it is hamiltonian, there is a 2^*-container $\{P_1, P_2\}$ of G_0 between u and v. Since G_1 is hamiltonian connected, there is a hamiltonian path P of G_1 between $\overline{u}$ and $\overline{v}$. We set $P_3 = \langle u, \overline{u}, P, \overline{v}, v\rangle$.

Case 2: $u, v \in G_0$ and $v \in G_1$ with $\overline{u} = v$. We can choose two distinct vertices x and y in $G_0 - \{u\}$. Since G_0 is hamiltonian connected, there is a hamiltonian path Q of G_0 between x and y. Similarly, there is a hamiltonian path R of G_1 between $\overline{x}$ and $\overline{y}$. We write $Q = \langle x, Q_1, u, Q_2, y\rangle$ and $R = \langle \overline{x}, R_1, v, R_2, \overline{y}\rangle$. We set $P_1 = \langle u, Q_1^{-1}, x, \overline{x}, R_1, v\rangle$, $P_2 = \langle u, Q_2, y, \overline{y}, R_2^{-1}, v\rangle$, and $P_3 = \langle u, v\rangle$.

Case 3: $u \in G_0$ and $v \in G_1$ with $\overline{u} \neq v$. We can choose a vertex x in $G_0 - \{u, \overline{v}\}$. Since G_0 is hamiltonian connected, there is a hamiltonian path Q of G_0 between x and $\overline{v}$. Similarly there is a hamiltonian path R of G_1 between $\overline{x}$ and $\overline{u}$. We write $Q = \langle x, Q_1, u, Q_2, \overline{v}\rangle$ and $R = \langle \overline{x}, R_1, v, R_2, \overline{u}\rangle$. We set $P_1 = \langle u, \overline{u}, R_2^{-1}, v\rangle$, $P_2 = \langle u, Q_1^{-1}, x, \overline{x}, R_1, v\rangle$, and $P_3 = \langle u, Q_2, \overline{v}, v\rangle$. ❑

As was mentioned in section 3, $FPQ_{1,1}$ is hamiltonian connected. Using Theorem 8 recursively, we prove that $FPQ_{n,1}$ is 3^*-connected for all $n \geq 1$.

Theorem 9 Let G_0 be a bipartite hamiltonian laceable graph satisfying the two-path property and G_1 be a nonbipartite hamiltonian connected graph. Then $G = (G_0, G_1; M)$ is 3^*-connected.

Proof: It is known that $V(G_0) = B_0 \cup W_0$. Let u and v be any two distinct vertices of G. We construct a 3^*-container, $A = \{P_1, P_2, P_3\}$, of G between u and v as follows.

Case 1: $u, v \in G_0$. Since G_0 is hamiltonian laceable and hence it is hamiltonian, there is a 2^*-container $\{P_1, P_2\}$ of G_0 between u and v. Since G_1 is hamiltonian connected, there is a hamiltonian path P of G_1 between $\overline{u}$ and $\overline{v}$. We set $P_3 = \langle u, \overline{u}, P, \overline{v}, v\rangle$.

Case 2: $u, v \in G_1$. Without loss of generality, we assume that $\overline{u} \in W_0$.

Case 2.1: $\overline{v} \in W_0$. We can choose two distinct vertices $x, y \in B_0$. Since G_1 is hamiltonian connected, there is a hamiltonian path Q of G_1 between $\overline{x}$ and $\overline{y}$. Without loss of generality, we write $Q = \langle \overline{x}, Q_1, u, Q_2, v, Q_3, \overline{y}\rangle$. By the two-path property, there are two paths T_1 and T_2 of G_0 such that (1) T_1 joins $\overline{u}$ to y, (2) T_2 joins x to $\overline{v}$, (3) $T_1 \cap T_2 = \phi$, and (4) $T_1 \cup T_2$ covers all vertices of G_0. We set $P_1 = \langle u, Q_2, v\rangle$, $P_2 = \langle u, Q_1^{-1}, \overline{x}, x, T_2, \overline{v}, v\rangle$, and $P_3 = \langle u, \overline{u}, T_1, y, \overline{y}, Q_3^{-1}, v\rangle$.

Case 2.2: $\overline{v} \in B_0$. Since G_1 is hamiltonian connected and hence it is hamiltonian, there is a 2^*-container $\{P_1, P_2\}$ of G_1 between u and v. Since G_0 is hamiltonian laceable, there is a hamiltonian path P of G_0 between $\overline{u}$ and $\overline{v}$. We set $P_3 = \langle u, \overline{u}, P, \overline{v}, v\rangle$.

Case 3: $u \in G_0$ and $v \in G_1$ with $\overline{u} = v$. Without loss of generality, we assume that $u \in W_0$. We can choose a vertex x in $W_0 - \{u\}$ and a vertex $y \in B_0$. Since G_0 is hamiltonian laceable, there is a hamiltonian path Q of G_0 between x and y. Since G_1 is hamiltonian connected, there is a hamiltonian path T of G_1 between $\overline{x}$ and $\overline{y}$. Without loss of generality, we write $Q = \langle x, Q_1, u, Q_2, y\rangle$ and $T = \langle \bar{x}, T_1, v, T_2, \bar{y}\rangle$. We set $P_1 = \langle u, v\rangle$, $P_2 = \langle u, Q_1^{-1}, x, \bar{x}, T_1, v\rangle$, and $P_3 = \langle u, Q_2, y, \bar{y}, T_2^{-1}, v\rangle$.

Case 4: $u \in G_0$ and $v \in G_1$ with $\bar{u} \neq v$. Without loss of generality, we assume that $u \in W_0$. We can choose a vertex $x \in V_0$ belonging to the opposite partite set of $\bar{v}$. Since G_0 is hamiltonian laceable, there is a hamiltonian path Q of G_0 between x and $\bar{v}$. Since G_1 is hamiltonian connected, there is a hamiltonian path T of G_1 between $\bar{x}$ and $\bar{u}$. Without loss of generality, we write $Q = \langle x, Q_1, u, Q_2, \bar{v}\rangle$ and $T = \langle \bar{x}, T_1, v, T_2, \bar{u}\rangle$. We set $P_1 = \langle u, Q_2, \bar{v}, v\rangle$, $P_2 = \langle u, \bar{u}, T_2^{-1}, v\rangle$, and $P_3 = \langle u, Q_1^{-1}, \bar{x}, \bar{x}, T_1, v\rangle$. ❑

With section 3, Theorems 7-9, we show that every graph $G = (G_0, G_1; M)$ in MCN preserves the 3^*-connectivity of G_0 and G_1. In particular, every hypercube-like graph HL_n is 3^*-connected, which was studied by Lin [18].

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Shin-Shin Kao (高欣欣) received the B.S. degree in the Department of Mathematics from the National Tsing Hua University, Hsinchu, Taiwan, R.O.C. in 1990. She received her M.S. and Ph.D. degrees from the department of Mathematics in the University of California, Los Angeles, U.S.A. in 1992 and 1995, respectively. She has been working in the department of Applied Mathematics in Chung Yuan Christian University, Chungli, Taiwan, R.O.C. since 1995. Currently she is a full professor and the chairman of the department since August, 2006. Her research interests include interconnection networks, fault-tolerant problems, and graph theory applications.

Jui-Chia Wu (吳瑞家) received the B.S. degree and the M.S. degree in the Department of Applied Mathematics from Chung Yuan Christian University, Chungli, Taiwan, R.O.C. in 2002 and 2006, respectively. His research interests include interconnection networks, fault-tolerant problems and graph theory applications.

Yuan-Kang Shih (石園鋼) received the B.S. degree in the Department of Mathematics from Fu Jen Catholic University, Hsinchuang, Taipei County, Taiwan, R.O.C. in 2004. He received his M.S. degree from the Department of Applied Mathematics from Chung Yuan Christian University, Chungli, Taiwan, R.O.C. in 2006. He is now a student in the Ph.D. program in the College of Computer Science in the National Chiao Tung University, Hsinchu, Taiwan, R.O.C. His research interests include interconnection networks, fault-tolerant problems, and graph theory applications.