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Microscopic analysis of desired-speed car-following stability

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Abstract

A desired-speed car-following model is proposed in this paper. The driver's desired speed is considered so that the model can reflect the driver's characteristics and the difference between drivers. The model can explain why different drivers maintain different speeds or spacing under the same condition. The necessary and sufficient conditions for stability are discussed, and they indicate that if the driver's desired speed is close to equilibrium speed, traffic is stable. Otherwise, if the difference between the driver's desired speed and equilibrium speed is large, traffic may be unstable.

Keywords: Desired-Speed; Car-following; Equilibrium; Stability

1. Introduction

Car-following models describe both the space-time behavior of vehicles and their interactions individually on a single lane. After car-following for a long time, the speed or spacing of the vehicle might be kept at a particular value or changed again and again over time. The fundamental diagram (shown as Fig. 1.) of traffic flow indicates that traffic flow is unstable at low speed (i.e., under heavy traffic), and vice versa. Why traffic is unstable under heavy traffic?

Traffic stability can be analyzed from the viewpoint of macroscopic traffic flow. Zhang [1] found that various instability criteria can be reduced to a single criterion derived from first order waves traveling faster than slow second order waves in the higher order theories. Nagatani [2] pointed out when the density is larger than a critical value, the traffic becomes unstable. Yi et al. [3] derived a nonlinear traffic flow stability criterion using a wavefront expansion technique. Jiang and Wu [4] found that stability depends on the equilibrium speed density relationship, and it is also affected by the sensitivity parameters in the corresponding car-following model.

Some researchers analyzed traffic stability from the viewpoint of microscopic traffic. It is found that unstable traffic is likely to occur under higher reaction time and higher sensitivity response based on

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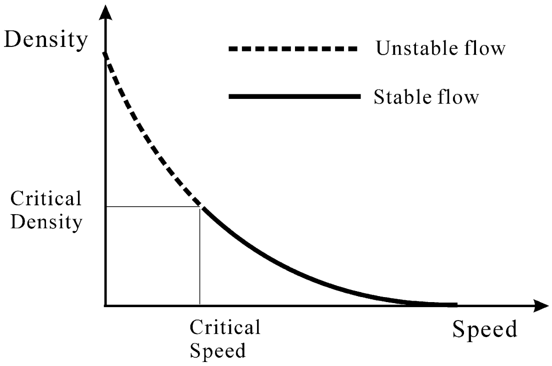

Fig. 1. Relationship among speed and density [9].

stimulus-response models [5–8]. This cannot explain why heavy traffic is unstable, unless drivers have different reaction time or different sensitivity response under different spacings.

The purpose of this study was to analyze traffic stability from the viewpoint of drivers' behavior, hence a car-following model was developed. A microscopic traffic flow model should describe the difference between drivers. Drivers may keep different velocities or different spacings under the same condition. The proposed model can describe the difference between drivers and explain why heavy traffic is unstable.

The remaining parts of the paper are organized as follows: a desired-speed car-following model is proposed in Section 2. The stability analysis is presented in Section 3. Numerical examples are presented in Section 4. Finally, this paper is concluded in Section 5.

2. A car-following model

2.1. Model assumptions

Car-following models describe the movements of vehicles individually. Different drivers have different behaviors. Drivers' behaviors are influenced by vehicle interaction, driver characteristics, and external environment. If there is no lead vehicle, a vehicle will run at a specific speed (its desired speed) and it is only influenced by driver characteristics and external environment. Different drivers have different desired speeds, hence the desired speed reflects the difference between individual drivers.

If there is a lead vehicle, and as the spacing decreases, the following vehicle may slow down its speed so that it cannot run at its desired speed. According to literature, the speed of a following vehicle is dependent on the speed of the lead vehicle, the speed of itself, and the spacing between vehicles. Hence, variables of the proposed model are desired speed, the speed of the lead vehicle, the speed of itself, and the spacing between vehicles.

The proposed model assumes that repulsion and thrust act on the following vehicle. The following vehicle then decides its appropriate velocity under the thrust and the repulsion. The concept of the proposed model is shown as Fig. 2. The model assumptions are listed below:

(1) *Thrust*: The desired speed is the thrust, and it makes the following vehicle move forward. If there is no lead vehicle, the vehicle will run at its desired speed.
(2) *Repulsion*: Because the lead vehicle makes the following vehicle unable to run at its desired speed, the lead vehicle is considered to be giving repulsion to the follower. The repulsion is related to the speed

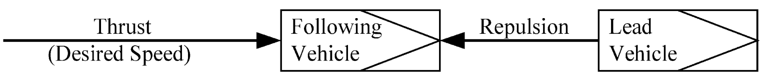

Fig. 2. An illustration of the car-following concept.

of lead vehicle, the speed of follower, and the spacing. If the speed of lead vehicle is higher or the spacing is greater, the repulsion should be less because a driver will maintain a higher velocity under this condition. On the other hand, a driver may slow down if his speed is too fast, and vice versa. Therefore, the repulsion increases as follower's speed decreases.

(3) *Velocity decision*: The following vehicle decides its appropriate velocity under the thrust and repulsion. The appropriate velocity equals thrust minus repulsion.
(4) *Aggressiveness*: The proposed model assumes that the higher desired speed is, the more aggressive the driver is. The driver with the higher desired speed would maintain a higher speed or a shorter spacing under the same condition.
(5) *Safety*: As some drivers' behaviors are not safe, the proposed model assumes that drivers do not take safe-distance into consideration. They only consider the standstill spacing.

2.2. Modeling

The proposed model is shown as Eqs. (1)–(5). Eq. (1) is based on the condition that both the lead vehicle and the following one are running. The following vehicle will choose an appropriate speed, and the appropriate speed equals thrust minus repulsion. It is often observed on a road that the same condition results in different speeds for different drivers, and the difference is reflected by Eq. (1).

Eq. (2) is based on the condition that the speed of the lead vehicle is zero. The following vehicle decelerates its speed so that it can stop before collision. Eq. (3) is based on that the lead vehicle is moving and the following one stops, the follower will not start to move immediately. The follower usually keeps stopping, and waits to move until the spacing is greater than a specific spacing (i.e. the start spacing). Then, it will move at next time step and the acceleration is its desired start acceleration. Finally, shown as Eq. (4), if the following vehicle stops and the spacing is shorter than the start spacing, the following vehicle keeps stopping at next time step.

$$\widetilde{V}_{n,t+1} = v_{n,d}\left(1 - \exp\left(-\lambda\frac{(V_{n-1,t})^{\alpha}}{(V_{n,t})^{\beta}}\left(\frac{H_{n,t}-S_n}{L}\right)^{\gamma}\right)\right), \quad \text{for } V_{n-1,t} \neq 0 \text{ and } V_{n,t} \neq 0, \tag{1}$$

$$\widetilde{V}_{n,t+1} = V_{n,t} - \frac{(V_{n,t})^2}{2(H_{n,t}-S_n)}T, \quad \text{for } V_{n-1,t} = 0 \text{ and } V_{n,t} \neq 0, \tag{2}$$

$$\widetilde{V}_{n,t+1} = a_{n,d}T, \quad \text{for } V_{n-1,t} \neq 0 \text{ and } V_{n,t} = 0 \text{ and } H_{n,t} \geqslant Z_n, \tag{3}$$

$$\widetilde{V}_{n,t+1} = 0, \quad \text{for } V_{n,t} = 0 \text{ and } H_{n,t} < Z_n, \tag{4}$$

where $\widetilde{V}_{n,t+1}$ is the speed of the following vehicle at time step $t+1$, $v_{n,d}$ is the desired speed of the following vehicle, $V_{n,t}$ is the speed of the following vehicle n at time step t, $V_{n-1,t}$ is the speed of the lead vehicle $n-1$ at time step t, $H_{n,t}$ is the spacing between vehicle $n-1$ and vehicle n at time step t, S_n is the safe standstill spacing of the following vehicle n, λ, α, β, γ, L is the nonnegative parameters, T is the length of a time interval, it equals reaction time of drivers, $a_{n,d}$ is the desired start acceleration of the following vehicle n, Z_n is the minimum start spacing.

Aside from the repulsion and thrust, the speed of the following vehicle also depends on the performance of the vehicle. The acceleration of a vehicle should be between the vehicle's maximum and minimum acceleration. Therefore, the proposed model should be modified as Eq. (5), where $a_{\max}$ is the maximum acceleration of the following vehicle, $a_{\min}$ is the minimum acceleration (i.e. maximum deceleration) of the following vehicle, and T is the length of a time interval.

$$\begin{aligned} V_{n,t+1} &= \widetilde{V}_{n,t+1}, && \text{for } a_{\min} \leqslant a_{n,t+1} \leqslant a_{\max}, \\ V_{n,t+1} &= V_{n,t} + a_{\max}T, && \text{for } a_{n,t+1} > a_{\max}, \\ V_{n,t+1} &= V_{n,t} + a_{\min}T, && \text{for } a_{n,t+1} < a_{\min}. \end{aligned} \tag{5}$$

3. Stability analysis

In this section, stability between two moving cars is discussed. The discussion is on the stability of a following vehicle when its lead vehicle is in equilibrium state and the following vehicle has no acceleration limit. When the lead vehicle is not in equilibrium state, the following vehicle will never achieve equilibrium state.

3.1. State of equilibrium

A system is either in equilibrium or not. A vehicle is in equilibrium state if its speed and spacing never change as time passes. Equilibrium state is discussed below.

A car-following process can be considered as a dynamical system. The process of a car following a vehicle that runs at equilibrium velocity is the dynamical system presented as Eqs. (6)–(8).

$$\mathbf{X}_{n,t+1} = \begin{bmatrix} V_{n,t+1} \\ H_{n,t+1} \end{bmatrix} = \mathbf{F}(\mathbf{X}_{n,t}) = \mathbf{F}\begin{bmatrix} V_{n,t} \\ H_{n,t} \end{bmatrix}, \tag{6}$$

$$V_{n,t+1} = f(V_{n,t}, H_{n,t}) = v_{n,d}\left(1 - \exp\left(-\lambda \frac{(V_{n-1,e})^{\alpha}}{(V_{n,t})^{\beta}}\left(\frac{H_{n,t} - S_n}{L}\right)^{\gamma}\right)\right), \tag{7}$$

$$H_{n,t+1} = g(V_{n,t}, H_{n,t}) = H_{n,t} + 0.5T(2V_{n-1,e} - V_{n,t} - V_{n,t+1}), \tag{8}$$

where

$$0 < V_{n,t} \leqslant v_{n,d}, \quad S_n < H_{n,t}.$$

$V_{n-1,e}$ is the equilibrium velocity of the lead vehicle. For any initial state $\mathbf{X}_{n,0}$, Eq. (6) uniquely determines the state trajectory, $\mathbf{X}_{n,t}$, $t \geqslant 0$. According to Eqs. (7) and (8), the equilibrium state occurs when $V_{n,t} = V_{n,e} = V_{n-1,e} = V_{n,t+1}$. Thus, the equilibrium spacing between vehicle $n-1$ and vehicle n is shown as Eq. (9).

$$H_{n,e} = L\sqrt[r]{\frac{\ln\left(1 - \frac{V_{n-1,e}}{v_{n,d}}\right)}{-\lambda(V_{n-1,e})^{\alpha-\beta}}} + S_n. \tag{9}$$

Hence, the equilibrium state of the proposed car-following model is:

$$\mathbf{X}_{n,e} = (V_{n,e}, H_{n,e})^{\mathrm{T}} = \left(V_{n-1,e}, L\sqrt[r]{\frac{\ln\left(1 - \frac{V_{n-1,e}}{v_{n,d}}\right)}{-\lambda(V_{n-1,e})^{\alpha-\beta}}} + S_n\right)^{\mathrm{T}},$$

and it is the unique equilibrium state.

Eq. (9) indicates that the equilibrium spacing is only dependent on the desired speed of the following vehicle and the equilibrium speed of the lead vehicle. It also reflects the model assumption that the driver with higher desired speed maintains a shorter spacing under the same condition. It is a common traffic phenomenon that different drivers may keep different spacing under the same equilibrium speed.

If the desired speed of the following vehicle is less than the equilibrium speed, Eq. (9) becomes meaningless. This is reasonable, because the following vehicle will maintain its speed as its desired speed and depart from car-following process.

3.2. Necessary and sufficient conditions for linearized stability

In this section, the linearized stability is discussed. If equilibrium state is asymptotically stable, all nearby solutions actually converge to the equilibrium state as time tends to infinity [10]. When equilibrium state is asymptotically stable, the car-following process will lead to equilibrium state and traffic is regarded as stable traffic. Necessary and sufficient conditions for linearized stability are provided.

The Jacobian matrix of the proposed dynamical system is shown as Eq. (10).

$$DF(X_{n,t}) = \begin{bmatrix} \frac{\partial f(V_{n,t},H_{n,t})}{\partial V_{n,t}} & \frac{\partial f(V_{n,t},H_{n,t})}{\partial H_{n,t}} \\ \frac{\partial g(V_{n,t},H_{n,t})}{\partial V_{n,t}} & \frac{\partial g(V_{n,t},H_{n,t})}{\partial H_{n,t}} \end{bmatrix}. \tag{10}$$

The eigenvalues of $D\mathbf{F}(\mathbf{X}_{n,e})$ are the roots of the following characteristic equation:

$$\Lambda^2 - b\Lambda + c = 0, \tag{11}$$

where

$$b = -\frac{\partial f}{\partial V_{n,t}}(V_{n,e}, H_{n,e}) - \frac{\partial g}{\partial H_{n,t}}(V_{n,e}, H_{n,e}),$$
$$c = \frac{\partial f}{\partial V_{n,t}}(V_{n,e}, H_{n,e})\frac{\partial g}{\partial H_{n,t}}(V_{n,e}, H_{n,e}) - \frac{\partial f}{\partial H_{n,t}}(V_{n,e}, H_{n,e})\frac{\partial g}{\partial V_{n,t}}(V_{n,e}, H_{n,e}).$$

Theorem 1 [11]. *If all absolute values of eigenvalues of $D\mathbf{F}(\mathbf{X}_{n,e})$ are less than* 1, *equilibrium state is asymptotically stable.*

Theorem 2 [12]. *If equilibrium state is stable, all absolute values of eigenvalues of $D\mathbf{F}(\mathbf{X}_{n,e})$ are less than or equal to* 1.

Theorem 3 [13]. *Eq. (11) has both roots inside the unique circle if, and only if,*

$$2 > 1 + c > |b|.$$

Theorem 4

(a) *Necessary condition for linearized stability*
If equilibrium state $\mathbf{X}_{n,e} = (V_{n,e},H_{n,e})^{\mathrm{T}}$ of the dynamical system presented as Eqs. (6)–(8) is asymptotically stable,

$$(1-D_n)^{\left(1-\frac{1}{D_n}\right)} \leqslant \exp\left(\frac{1}{\beta}\right) \cap T \leqslant \left(1 - \left(\beta\left(\frac{1}{D_n}\right)[\ln(1-D_n)](1-D_n)\right)^{-1}\right) \cdot \left(\frac{2\beta(H_{n,e}-S_n)}{\gamma V_{n,e}}\right),$$

where $D_n = \frac{V_{n,e}}{v_{n,d}}$.

(b) *Sufficient condition for linearized stability*
If

$$(1-D_n)^{\left(1-\frac{1}{D_n}\right)} < \exp\left(\frac{1}{\beta}\right) \cap T < \left(1 - \left(\beta\left(\frac{1}{D_n}\right)[\ln(1-D_n)](1-D_n)\right)^{-1}\right) \cdot \left(\frac{2\beta(H_{n,e}-S_n)}{\gamma V_{n,e}}\right),$$

equilibrium state $\mathbf{X}_{n,e} = (V_{n,e},H_{n,e})^{\mathrm{T}}$ of the dynamical system presented as Eqs. (6)–(8) is asymptotically stable.

Proof. (a) *Necessary condition*

According to Theorem 2 and 3, if the presented dynamical system is stable, it implies $|b| \leqslant 1 + c$ and $c \leqslant 1$.

$$|b| \leqslant 1 + c \Rightarrow \frac{\partial f}{\partial V_{n,t}}(V_{n,e}, H_{n,e}) \geqslant -1. \tag{12}$$

If

$$(1-D_n)^{\left(1-\frac{1}{D_n}\right)} \leqslant \exp\left(\frac{1}{\beta}\right), \tag{13}$$

it implies $\frac{\partial f}{\partial V_{n,t}}(V_{n,e}, H_{n,e}) \geqslant -1$.

$$c \leqslant 1 \;\Rightarrow\; T \leqslant \left(1 - \left(\beta\left(\frac{1}{D_n}\right)[\ln(1-D_n)](1-D_n)\right)^{-1}\right) \cdot \left(\frac{2\beta(H_{n,e}-S_n)}{\gamma V_{n,e}}\right). \tag{14}$$

(b) *Sufficient condition*

According to Theorem 1 and 3, if the presented dynamical system is stable, it implies $|b| < 1 + c$ and $c < 1$

$$|b| < 1 + c \;\Rightarrow\; \frac{\partial f}{\partial V_{n,t}}(V_{n,e}, H_{n,e}) > -1. \tag{15}$$

If

$$(1-D_n)^{\left(1-\frac{1}{D_n}\right)} < \exp\left(\frac{1}{\beta}\right), \tag{16}$$

it implies $\frac{\partial f}{\partial V_{n,t}}(V_{n,e}, H_{n,e}) > -1$.

$$c < 1 \;\Rightarrow\; T < \left(1 - \left(\beta\left(\frac{1}{D_n}\right)[\ln(1-D_n)](1-D_n)\right)^{-1}\right) \cdot \left(\frac{2\beta(H_{n,e}-S_n)}{\gamma V_{n,e}}\right). \tag{17}$$

Both Eqs. (13) and (14) are the necessary conditions for linearized stability. Eqs. (16) and (17) are the sufficient conditions. □

From Eqs. (13), (14), (16), and (17) some traffic characteristics can be found.

1. Higher $\frac{V_{n,e}}{v_{n,d}}$ makes traffic stable, lower $\frac{V_{n,e}}{v_{n,d}}$ makes traffic unstable: When the desired speed of the following vehicle is close to the equilibrium speed of its lead vehicle (i.e. the speed is also its equilibrium speed), traffic will lead to equilibrium state, i.e. stable traffic. Otherwise, when the difference between the desired speed of the following vehicle and the equilibrium speed of its lead vehicle is great, traffic may be unstable. The unstable traffic is often observed under heavy traffic. From the proposed model, it can be explained that unstable heavy traffic may be due to the large difference between the driver's desired speed and equilibrium speed.
2. Lower T makes traffic stable, higher T makes traffic unstable: When the driver's reaction time is less, traffic will be stable. Otherwise, when the driver's reaction time is high, unstable traffic is likely to occur. The similar result of the influence of driver's reaction time on traffic stability is also found in GM model and other classical models [5–8]. Furthermore, under the same equilibrium speed and with a lower $\frac{V_{n,e}}{v_{n,d}}$, the reaction time should be less to make stable traffic possible. It implies when the desired speed of the following vehicle isn't close to equilibrium speed, drivers should react more frequently. Otherwise, traffic may be unstable.

4. Numerical examples

Examples for stable traffic and unstable traffic are presented in this section.The model parameters for these simulations are: $\lambda = 1$, $\alpha = 1$, $\beta = 1.1$, $\gamma = 1$ $L = 20$ (they have not been calibrated), $S_n = 5$ m, $T = 0.5$ s, $a_{\max} = 5$ m/s^2, and $a_{\min} = -5$ m/s^2.

4.1. Stable traffic

An example of movement process of four vehicles is illustrated below. The desired speeds of the first vehicle, the second one, the third one, and the last one are 50, 60, 70, and 80 km/h, respectively. The initial speeds of these vehicles are their desired speeds. The initial spacings between these vehicles are 100 m.

Fig. 3 shows car-following trajectories of these four vehicles. As there is no vehicle in front of the first vehicle, the first vehicle runs at its desired speed (i.e. 50 km/h). According to Section 3.2, following vehicles satisfy the necessary and sufficient conditions for linearized stability, therefore, they finally run at equilibrium state.

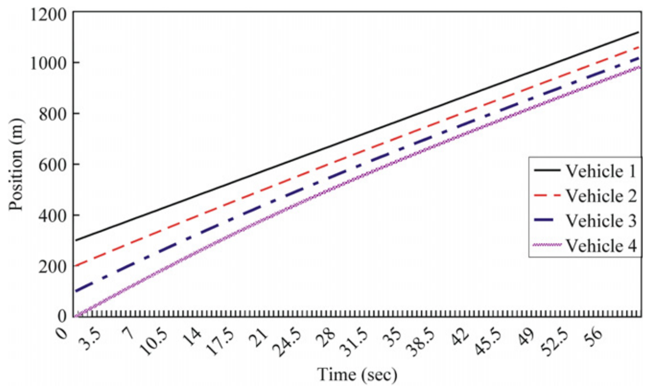

Fig. 3. Car-following trajectories.

Fig. 4 shows the spacing between these vehicles. All spacing reaches a particular value finally (i.e. equilibrium spacing). As mentioned in Section 2, Fig. 4. also reflects the model assumption that the driver with higher desired speed maintains a higher speed or a shorter spacing under the same condition. It is a common traffic phenomenon that different drivers may keep different spacing under the same condition.

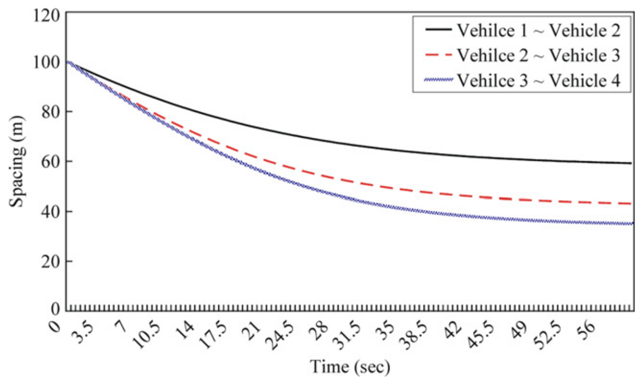

Fig. 4. Spacings between vehicles.

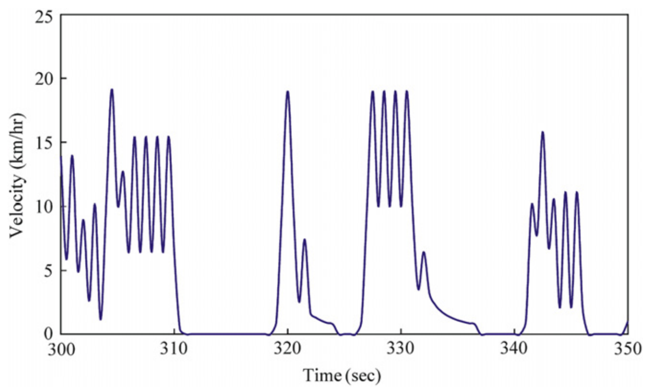

Fig. 5. Velocity profile under unstable traffic.

4.2. Unstable traffic

Traffic flow does not always lead to equilibrium state; sometimes unstable traffic occurs. The speed and spacing may change again and again over time. For example, when the traffic condition is heavy, vehicles sometimes fall into the stop-and-go situation. An example is shown to illustrate that the proposed model cannot only describe stable traffic, but also describe unstable traffic. In the following example, the desired speed of the first vehicle is assumed to be 5 km/h so that it will run at 5 km/h to simulate the heavy traffic condition. All the desired speeds of the following vehicles are 90 km/h. All of the initial spacings between a lead vehicle and a following one are 150 m. All following vehicles don't satisfy the necessary condition for stability because $(1-D_n)^{\left(1-\frac{1}{D_n}\right)} > \exp\left(\frac{1}{\beta}\right)$. Thus, traffic is unstable. Fig. 5 is the velocity profile for the 7th vehicle in the platoon. It shows the stop-and-go traffic condition that vehicles sometimes stop and sometimes move.

5. Conclusion

In this paper, traffic stability is discussed from the viewpoint of microscopic traffic flow. The stability analysis is based on a proposed car-following model. The model applies desired speed as a model variable so that it can reflect the difference between different drivers under the same condition. Unlike aggression or sensitivity factor applied in other car-following models, desired speed is not a parameter that should be calibrated, it can be measured directly. The necessary and sufficient conditions for stability are discussed, and they provide a new explanation from the viewpoint of drivers' behavior why some traffic is stable some is not, such as why heavy traffic is unstable. This approach provides an alternative way to modern traffic flow simulation. The model is currently being calibrated with the field data.

Acknowledgements

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