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Dynamic Output Feedback Control of Nonlinear Singularly Perturbed Systems

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ABSTRACT : *In this paper, the stabilization problem of two classes of nonlinear singularly perturbed systems via dynamic output feedback is investigated. First, we consider the nonlinear singularly perturbed systems in which the nonlinearities are continuously differentiable. The theoretical result demonstrates that, using the factorization approach, the dynamic output feedback controller designed for the reduced-order model of the linearized system is a stabilizing compensator for the original nonlinear singularly perturbed system, provided that ε is sufficiently small. Second, the nonlinear singularly perturbed systems in which the nonlinearities are not necessarily continuously differentiable but satisfy the global Lipschtz condition are examined. Combining the dynamic output feedback controller that stabilizes the reduced-order model of the linear part of the nonlinear singularly perturbed system with the quasi-stability result of Persidskii, a two-step compensating scheme is proposed to stabilize the original nonlinear singularly perturbed system under consideration for a sufficiently small ε.*

I. Introduction

Most physical systems contain some small parameters such as small time constants, masses, capacitances, etc. These small parameters increase the order of dynamic systems and then complicate the system analysis. Furthermore, they introduce the multi-time-scale property such that these systems simultaneously possess both slow states and fast states. Coupling of these states with each other makes the system analysis much more complex, and hence convenient methods to check stability have long been sought. Fortunately, singularly perturbed models for these systems give us a powerful tool for order reduction and separation of time scales. This singular perturbation approach, arising from an attempt to approximate a high-order system with another one of lower order, has proven to be a successful analytical tool that exploits directly the separation of system time scales, made explicit by the small singular perturbation parameter ε. When ε is small enough, approximations are obtained from the reduced-order models in separate time scales (**1**).

Singularly perturbed systems have been studied extensively in recent years; see, for example, Kokotovic *et al.* (**2**) and references therein. This is due not only to theoretical interest, but also to the relevance of this topic to control engineering applications. Indeed, the singular perturbation approach has been proven to be an effective tool for system analysis and control design (**3**). A fundamental feature of such a control theory is decomposition of the feedback design problem into two design subproblems for the slow and fast dynamics. The two designs are then combined to give a design for the full-order system (**4**). Moreover, if the fast subsystem is already stable, the only work we need to do is to design the stabilizing feedback control in the slow subsystem (i.e. reduced-order model). The stabilizing feedback control designed for the reduced-order model can stabilize the full-order system, provided that ε is sufficiently small.

The dynamic output feedback control problem of singularly perturbed systems has been addressed in many works. However, these contributions focused mainly on linear singularly perturbed systems; see, for example (**5–8**). In this work, the stabilization problem is investigated for two classes of nonlinear singularly perturbed systems via dynamic output feedback. First, we examine the nonlinear singularly perturbed systems in which the nonlinearities are continuously differentiable. Using the factorization approach, the dynamic output feedback controller is designed for the reduced-order model of the linearized system of the nonlinear singularly perturbed system. The theoretical result demonstrates that the above dynamic output feedback controller can stabilize the linearized system and then the original nonlinear singularly perturbed system as well, provided that ε is sufficiently small. Second, the nonlinear singularly perturbed systems in which the nonlinearities are not necessarily continuously differentiable but satisfy the global Lipschtz condition are investigated. Combining the dynamic output feedback controller that stabilizes the reduced-order model of the linear part of the nonlinear singularly perturbed system with the quasi-stability result of Persidskii, a two-step compensating scheme is proposed to stabilize the original nonlinear singularly perturbed system under consideration, provided that ε is sufficiently small.

This paper is organized as follows. In Section II, the factorization approach for designing the dynamic output feedback controllers in linear systems is introduced. In Section III, we consider a class of nonlinear singularly perturbed systems in which the nonlinearities are continuously differentiable. The factorization approach is used to design the dynamic output feedback controller for the reduced-order model of the linearized system such that the original nonlinear singularly perturbed systems is asymptotically stable at the origin. The class of nonlinear singularly perturbed systems in which the nonlinearities are not necessarily continuously differentiable but satisfy the global Lipschtz condition is examined in Section IV. A two-step compensating scheme

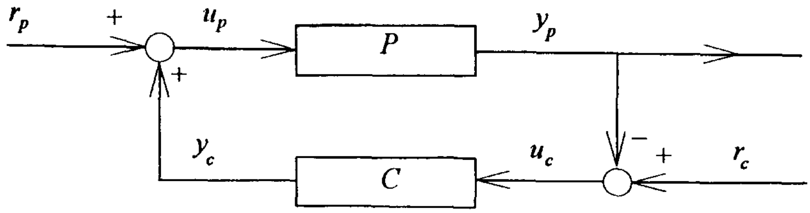


FIG. 1. One-parameter compensating scheme.

is proposed to stabilize the original nonlinear singularly perturbed system under consideration. Two illustrative examples are given in Section V. Finally, conclusions are provided in Section VI.

II. Factorization Approach

The factorization approach for designing the dynamic output feedback controllers in linear systems is reviewed in this section.

Consider the feedback system shown in Fig. 1: P represents the plant and C the compensator; r_p and r_c the external inputs; u_p and u_c the inputs to the plant and compensator, respectively; y_p and y_c the output of the plant and compensator, respectively.

Assume P is proper and has a state-space realization:

$$\begin{aligned} \dot{x} &= A_p x + B_p u_p, \\ y_p &= C_p x + E_p u_p. \end{aligned} \tag{1}$$

Clearly we have

$$P(s) = C_p(sI - A_p)^{-1} B_p + E_p.$$

Defining

$$u = \begin{bmatrix} u_c \\ u_p \end{bmatrix}, \quad r = \begin{bmatrix} r_c \\ r_p \end{bmatrix}$$

for the feedback connection of Fig. 1, we can get

$$u = H(P, C) r,$$

where

$$H(P, C) = \begin{bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{bmatrix}. \tag{2}$$

If the compensator C is so chosen that $\det(I+PC) \neq 0$ and $H(P, C) \in M(S(s))$†, then C stabilizes P (input–output stability). Assume that the pairs (A_p, B_p) and (A_p, C_p) are

† The set of matrices whose elements are proper stable rational functions in s with real coefficients.

stabilizable and detectable, respectively; the following theorem gives parametrization of the set of the stabilizing compensators C for P.

Theorem I (9)

Suppose that for the system (1) $P \in M(R(s))$† and (N_p, D_p), $(\tilde{D}_p, \tilde{N}_p)$ are any right coprime factorization and left coprime factorization of P, respectively. Select matrices U_p, V_p, $\tilde{U}_p$, $\tilde{V}_p$ in $M(S(s))$ such that

$$U_p N_p + V_p D_p = I \tag{3}$$

$$\tilde{N}_p \tilde{U}_p + \tilde{D}_p \tilde{V}_p = I. \tag{4}$$

Denote $S(P)$ the set of all Cs in $M(R(s))$ that stabilize P, i.e. the set of all Cs in $M(R(s))$ such that $H(P, C) \in M(S(s))$. Then

$$\begin{aligned} S(p) &= \{(V_p - R\tilde{N}_p)^{-1}(U_p + R\tilde{D}_p) : R \in M(S(s)), \det(V_p - R\tilde{N}_p) \neq 0\} \\ &= \{(\tilde{U}_p + D_p S)(\tilde{V}_p - N_p S)^{-1} : S \in M(S(s)), \det(\tilde{V}_p - N_p S) \neq 0\}. \end{aligned} \tag{5}$$

Note that all the matrices in (3)–(4) can be calculated by the following:

$$\begin{aligned} \tilde{N}_p &= C_p(sI - (A_p - FC_p))^{-1}(B_p - FE_p) + E_p, \\ \tilde{D}_p &= I - C_p(sI - (A_p - FC_p))^{-1}F, \\ N_p &= (C_p - E_p K)(sI - (A_p - B_p K))^{-1}B_p + E_p, \\ D_p &= I - K(sI - (A_p - B_p K))^{-1}B_p, \\ U_p &= K(sI - (A_p - FC_p))^{-1}F, \\ V_p &= I + K(sI - (A_p - FC_p))^{-1}(B_p - FE_p), \\ \tilde{U}_p &= K(sI - (A_p - B_p K))^{-1}F, \\ \tilde{V}_p &= I + (C_p - E_p K)(sI - (A_p - B_p K))^{-1}F, \end{aligned} \tag{6}$$

in which K and F are properly chosen such that both $A_p - B_p K$ and $A_p - FC_p$ are Hurwitz. ■

To discuss the internal stability of the feedback system in Fig. 1, the compensator C is realized in state-space form as

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c u_c, \\ y_c &= C_c x_c + E_c u_c \end{aligned} \tag{7}$$

with the transfer function

$$H_c(s) = C_c(sI - A_c)^{-1}B_c + E_c.$$

The following theorem states the relationship between internal stability and input–output stability for the feedback system of Fig. 1.

† The set of matrices whose elements are rational functions in s with real coefficients.

Theorem II (**9**)

Suppose that the two triples (A_p, B_p, C_p) and (A_c, B_c, C_c) are both stabilizable and detectable and that

$$\det(I+H_c(\infty)P(\infty)) = \det(I+E_cE_p) \neq 0.$$

Under these conditions, the closed-loop system of Fig. 1 is asymptotically stable if $H(P, C) \in M(S(s))$. ■

In the following sections, the factorization approach is applied to stabilize two different classes of nonlinear singularly perturbed systems.

III. Dynamic Output Feedback Controller

Consider the following nonlinear singularly perturbed system:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u) \\ \varepsilon\dot{x}_2 &= f_2(x_1, x_2, u) \\ y &= g(x_1, x_2), \end{aligned} \tag{8}$$

where the nonlinear functions f_1, f_2 and g are continuously differentiable and $f_1(0,0,0) = 0, f_2(0,0,0) = 0$ and $g(0,0) = 0$.

In this section, dynamic output feedback controllers are designed such that they can asymptotically stabilize the nonlinear singularly perturbed system (8) at the origin. Define

$$\begin{aligned} A_{11} &= \left.\frac{\partial f_1}{\partial x_1}\right|_{x_1=0,x_2=0,u=0}, \quad A_{12} = \left.\frac{\partial f_1}{\partial x_2}\right|_{x_1=0,x_2=0,u=0}, \quad B_1 = \left.\frac{\partial f_1}{\partial u}\right|_{x_1=0,x_2=0,u=0}, \\ A_{21} &= \left.\frac{\partial f_2}{\partial x_1}\right|_{x_1=0,x_2=0,u=0}, \quad A_{22} = \left.\frac{\partial f_2}{\partial x_2}\right|_{x_1=0,x_2=0,u=0}, \quad B_2 = \left.\frac{\partial f_2}{\partial u}\right|_{x_1=0,x_2=0,u=0}, \\ C_1 &= \left.\frac{\partial g}{\partial x_1}\right|_{x_1=0,x_2=0}, \quad C_2 = \left.\frac{\partial g}{\partial x_2}\right|_{x_1=0,x_2=0}. \end{aligned} \tag{9}$$

Therefore, the linearized system of (8) reads

$$\begin{aligned} \dot{x}_{1l} &= A_{11}x_{1l} + A_{12}x_{2l} + B_1u_l, \\ \varepsilon\dot{x}_{2l} &= A_{21}x_{1l} + A_{22}x_{2l} + B_2u_l, \\ y_l &= C_1x_{1l} + C_2x_{2l}. \end{aligned} \tag{10}$$

The matrix A_{22} is assumed here to be nonsingular and Hurwitz. Furthermore, the reduced-order model of system (10) can be obtained by setting $\varepsilon = 0$ in Eq. (10), i.e.

$$\dot{\bar{x}}_{1l} = A_{11}\bar{x}_{1l} + A_{12}\bar{x}_{2l} + B_1\bar{u}_l, \tag{11}$$

$$0 = A_{21}\bar{x}_{1l} + A_{22}\bar{x}_{2l} + B_2\bar{u}_l, \tag{12}$$

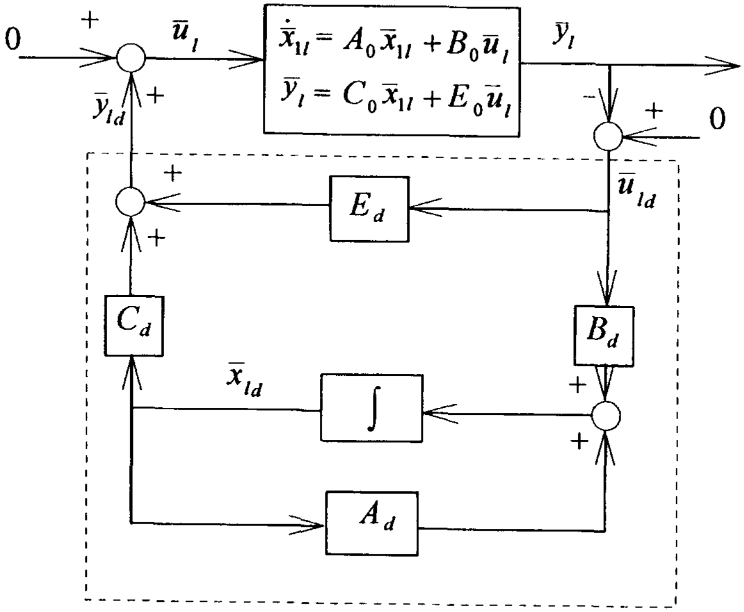


FIG. 2. Dynamic output feedback scheme for the reduced-order model (15).

$$\bar{y}_l = C_1\bar{x}_{1l} + C_2\bar{x}_{2l}. \tag{13}$$

Since A_{22} is nonsingular, we can get, from (12), that

$$\bar{x}_{2l} = -A_{22}^{-1}A_{21}\bar{x}_{1l} - A_{22}^{-1}B_2\bar{u}_l. \tag{14}$$

Substituting (14) into (11) and (13) yields

$$\begin{aligned} \dot{\bar{x}}_{1l} &= A_0\bar{x}_{1l} + B_0\bar{u}_l, \\ \bar{y}_l &= C_0\bar{x}_{1l} + E_0\bar{u}_l, \end{aligned} \tag{15}$$

in which

$$\begin{aligned} A_0 &= A_{11} - A_{12}A_{22}^{-1}A_{21}, \qquad B_0 = B_1 - A_{12}A_{22}^{-1}B_2, \\ C_0 &= C_1 - C_2A_{22}^{-1}A_{21}, \qquad E_0 = -C_2A_{22}^{-1}B_2. \end{aligned} \tag{16}$$

The system (15) is generally referred to as the reduced-order model of the linearized system (10). Suppose that the dynamic output feedback controller, which stabilizes the reduced-order model (15) under the feedback interconnection as depicted in Fig. 2, can be obtained from Theorem I and is realized as

$$\begin{aligned} \dot{\bar{x}}_{ld} &= A_d\bar{x}_{ld} + B_d\bar{u}_{ld} \\ \bar{y}_{ld} &= C_d\bar{x}_{ld} + E_d\bar{u}_{ld}. \end{aligned} \tag{17}$$

It will be shown that the dynamic output feedback controller (17) can also stabilize the

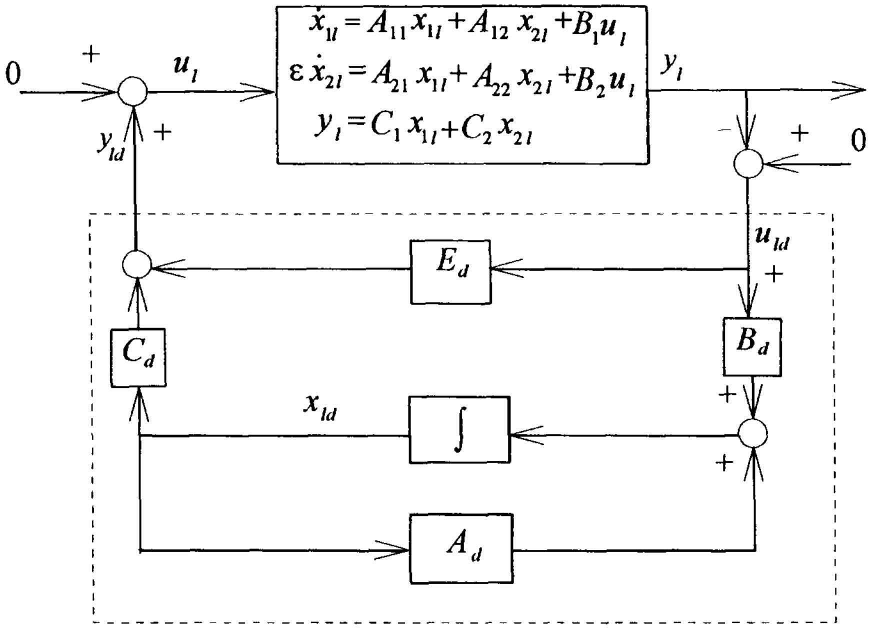


FIG. 3. Dynamic output feedback scheme for the linearized system (10).

linearized system (10) and then the original nonlinear singularly perturbed system (8) as well, provided that the singular perturbation parameter ε is sufficiently small.

Lemma 1

Suppose that the two triples (A_0, B_0, C_0) of (15) and (A_d, B_d, C_d) of (17) are both stabilizable and detectable and assume that (A_d, B_d, C_d, E_d) of (17) is a stabilizing compensator for (A_0, B_0, C_0, E_0). Let E_d be chosen such that the matrix $(I+E_dE_0)^{-1}$ exists and $B_2E_d = 0$ or $E_dC_2 = 0$. Then, for a sufficiently small ε, the linearized system (10) is stabilized by the dynamic output feedback controller (17) under the feedback scheme shown in Fig. 3.

Proof: From Fig. 2 it can be seen that

$$\begin{bmatrix}\dot{\bar{x}}_{1l}\\ \dot{\bar{x}}_{1d}\end{bmatrix} = \begin{bmatrix}A_0 & 0\\ 0 & A_d\end{bmatrix}\begin{bmatrix}\bar{x}_{1l}\\ \bar{x}_{ld}\end{bmatrix} + \begin{bmatrix}B_0 & 0\\ 0 & B_d\end{bmatrix}\begin{bmatrix}\bar{u}_l\\ \bar{u}_{ld}\end{bmatrix},$$

$$\begin{bmatrix}\bar{y}_l\\ \bar{y}_{ld}\end{bmatrix} = \begin{bmatrix}C_0 & 0\\ 0 & C_d\end{bmatrix}\begin{bmatrix}\bar{x}_{1l}\\ \bar{x}_{ld}\end{bmatrix} + \begin{bmatrix}E_0 & 0\\ 0 & E_d\end{bmatrix}\begin{bmatrix}\bar{u}_l\\ \bar{u}_{ld}\end{bmatrix},$$

$$\begin{bmatrix}\bar{u}_l\\ \bar{u}_{ld}\end{bmatrix} = \begin{bmatrix}0 & I\\ -I & 0\end{bmatrix}\begin{bmatrix}\bar{y}_l\\ \bar{y}_{ld}\end{bmatrix}.$$

After simplifying the above equation, we have

$$\begin{bmatrix} \dot{\bar{x}}_{1l} \\ \dot{\bar{x}}_{ld} \end{bmatrix} = \begin{bmatrix} A_0 - B_0(I+E_dE_0)^{-1}E_dC_0 & B_0(I+E_dE_0)^{-1}C_d \\ (-B_d+B_dE_0(I+E_dE_0)^{-1}E_d)C_0 & A_d - B_dE_0(I+E_dE_0)^{-1}C_d \end{bmatrix} \begin{bmatrix} \bar{x}_{1l} \\ \bar{x}_{ld} \end{bmatrix}. \tag{18}$$

Since the two triples (A_0, B_0, C_0) of (15) and (A_d, B_d, C_d) of (17) are both stabilizable and detectable, the system (18) is, according to Theorem II, asymptotically stable. Hence the matrix

$$H_0 \equiv \begin{bmatrix} A_0 - B_0(I+E_dE_0)^{-1}E_dC_0 & B_0(I+E_dE_0)^{-1}C_d \\ (-B_d+B_dE_0(I+E_dE_0)^{-1}E_d)C_0 & A_d - B_dE_0(I+E_dE_0)^{-1}C_d \end{bmatrix} \tag{19}$$

in (18) is Hurwitz.

Moreover, the dynamic output feedback controller (17), by changing the notations of variables, is rewritten here as:

$$\dot{x}_{ld} = A_dx_{ld} + B_du_{ld},$$
$$y_{ld} = C_dx_{ld} + E_du_{ld}.$$

From Fig. 3 it can be seen that

$$\begin{bmatrix} \dot{x}_{1l} \\ \dot{x}_{ld} \\ \varepsilon\dot{x}_{2l} \end{bmatrix} = \begin{bmatrix} A_{11} - B_1E_dC_1 & B_1C_d & A_{12} - B_1E_dC_2 \\ -B_dC_1 & A_d & -B_dC_2 \\ A_{21} - B_2E_dC_1 & B_2C_d & A_{22} - B_2E_dC_2 \end{bmatrix} \begin{bmatrix} x_{1l} \\ x_{ld} \\ x_{2l} \end{bmatrix}. \tag{20}$$

If the parameter E_d is chosen such that $B_2E_d = 0$ or $E_dC_2 = 0$, then (20) becomes

$$\begin{bmatrix} \dot{x}_{1l} \\ \dot{x}_{ld} \\ \varepsilon\dot{x}_{2l} \end{bmatrix} = \begin{bmatrix} A_{11} - B_1E_dC_1 & B_1C_d & A_{12} - B_1E_dC_2 \\ -B_dC_1 & A_d & -B_dC_2 \\ A_{21} - B_2E_dC_1 & B_2C_d & A_{22} \end{bmatrix} \begin{bmatrix} x_{1l} \\ x_{ld} \\ x_{2l} \end{bmatrix} \tag{21}$$

or

$$\begin{bmatrix} \dot{x}_{1l} \\ \dot{x}_{ld} \\ \dot{x}_{2l} \end{bmatrix} = \begin{bmatrix} A_{11} - B_1E_dC_1 & B_1C_d & A_{12} - B_1E_dC_2 \\ -B_dC_1 & A_d & -B_dC_2 \\ \dfrac{A_{21} - B_2E_dC_1}{\varepsilon} & \dfrac{B_2C_d}{\varepsilon} & \dfrac{A_{22}}{\varepsilon} \end{bmatrix} \begin{bmatrix} x_{1l} \\ x_{ld} \\ x_{2l} \end{bmatrix}. \tag{22}$$

Moreover, the system (21) can be rewritten in a more compact form as the following:

$$\begin{bmatrix} \dot{x}_{1l} \\ \dot{x}_{ld} \\ \cdots\cdots \\ \varepsilon\dot{x}_{2l} \end{bmatrix} = \left[\begin{array}{c:c} N_1 & N_2 \\ \hdashline N_3 & N_4 \end{array}\right] \begin{bmatrix} x_{1l} \\ x_{ld} \\ \cdots\cdots \\ x_{2l} \end{bmatrix}, \tag{23}$$

where

$$N_1 = \begin{bmatrix} A_{11}-B_1E_dC_1 & B_1C_d \\ -B_dC_1 & A_d \end{bmatrix}, \quad N_2 = \begin{bmatrix} A_{12}-B_1E_dC_2 \\ -B_dC_2 \end{bmatrix},$$

$$N_3 = [A_{21}-B_2E_dC_1 \quad B_2C_d], \quad N_4 = A_{22}.$$

Applying the following transformation to (21) (**2**):

$$\begin{bmatrix} x_{1l} \\ x_{ld} \\ \cdots\cdots \\ x_{2l} \end{bmatrix} = \left[\begin{array}{c:c} I & \varepsilon P_d \\ \hdashline -L_d & I-\varepsilon L_dP_d \end{array}\right] \begin{bmatrix} \xi_{d1} \\ \xi_{d2} \\ \cdots\cdots \\ \eta_d \end{bmatrix}, \tag{24}$$

i.e.

$$\begin{bmatrix} \xi_{d1} \\ \xi_{d2} \\ \cdots\cdots \\ \eta_d \end{bmatrix} = \left[\begin{array}{c:c} I-\varepsilon P_dL_d & -\varepsilon P_d \\ \hdashline L_d & I \end{array}\right] \begin{bmatrix} x_{1l} \\ x_{ld} \\ \cdots\cdots \\ x_{2l} \end{bmatrix}, \tag{25}$$

in which L_d and P_d satisfy

$$\begin{aligned} N_3-N_4L_d+\varepsilon L_dN_1-\varepsilon L_dN_2L_d &= 0, \\ \varepsilon(N_1-N_2L_d)P_d-P_d(N_4+\varepsilon L_dN_2)+N_2 &= 0, \end{aligned} \tag{26}$$

the closed-loop system (21), or equivalently (23), can then be converted into a decoupled form as the following:

$$\begin{bmatrix} \dot{\xi}_{d1} \\ \dot{\xi}_{d2} \\ \cdots\cdots \\ \varepsilon\dot{\eta} \end{bmatrix} = \left[\begin{array}{c:c} N_0+O(\varepsilon) & 0 \\ \hdashline 0 & N_4+O(\varepsilon) \end{array}\right] \begin{bmatrix} \xi_{d1} \\ \xi_{d2} \\ \cdots\cdots \\ \eta \end{bmatrix}, \tag{27}$$

where

$$\begin{aligned} N_0 &= N_1-N_2N_4^{-1}N_3 \\ &= \begin{bmatrix} A_{11}-B_1E_dC_1 & B_1C_d \\ -B_dC_1 & A_d \end{bmatrix} - \begin{bmatrix} A_{12}-B_1E_dC_2 \\ -B_dC_2 \end{bmatrix} A_{22}^{-1}[A_{21}-B_2E_dC_1 \quad B_2C_d] \\ &= \begin{bmatrix} N_0^1 & N_0^2 \\ N_0^3 & N_0^4 \end{bmatrix} \end{aligned} \tag{28}$$

with

$$N_0^1 = A_{11}-B_1E_dC_1-(A_{12}-B_1E_dC_2)A_{22}^{-1}(A_{21}-B_2E_dC_1),$$

$$N_0^2 = B_1C_d-(A_{12}-B_1E_dC_2)A_{22}^{-1}B_2C_d,$$

$$N_0^3 = -B_d C_1 + B_d C_2 A_{22}^{-1}(A_{21} - B_2 E_d C_1),$$

$$N_0^4 = A_d + B_d C_2 A_{22}^{-1} B_2 C_d.$$

Based on (16) and the fact that

$$(I+MN)^{-1} = I - M(I+NM)^{-1}N,$$

we have

$$(I+E_d E_0)^{-1} = (I - E_d C_2 A_{22}^{-1} B_2)^{-1} = I + E_d C_2 (I - A_{22}^{-1} B_2 E_d C_2)^{-1} A_{22}^{-1} B_2$$
$$= I + E_d C_2 A_{22}^{-1} B_2 (\because B_2 E_d C_2 = 0).$$

After lengthy manipulations, it can be shown that N_0 is identical to H_0 (in (19)) and thus Hurwitz. Since N_0 and N_4 (i.e. A_{22}) are both Hurwitz, the closed-loop system (27), or equivalently (21), is hence asymptotically stable for a sufficiently small ε. This completes the proof. ■

Remark 1

If the matrix E_d can not be properly chosen such that $B_2 E_d = 0$ or $E_d C_2 = 0$, then the following conditions (**2**) can be used to guarantee the result of Lemma 1:

(i) The pair (A_{22}, B_2) is weakly controllable†;
(ii) The pair (C_2, A_{22}) is weakly observable†.

The following theorem demonstrates that the dynamic output feedback controller, designed for the reduced-order model (15), can stabilize the original nonlinear singularly perturbed system (8).

Theorem III

Suppose that all the assumptions in Lemma 1 are available. Then, the pair (A_d, B_d, C_d, E_d) of (17), which is designed for the reduced-order model (15), is a stabilizing compensator for the original nonlinear singularly perturbed system (8) under the feedback configuration of Fig. 4, provided that ε is sufficiently small.

Proof: The dynamic output feedback controller (17), by changing the notations of variables, is rewritten here as

$$\dot{x}_d = A_d x_d + B_d u_d,$$
$$y_d = C_d x_d + E_d u_d. \tag{29}$$

From Fig. 4 we have

$$\dot{x}_1 = f_1(x_1, x_2, u),$$
$$\dot{x}_d = A_d x_d - B_d g(x_1, x_2),$$

† The weak controllability and weak observability of the fast eigenvalues (i.e. the eigenvalues of A_{22}) mean that they will not be affected by more than $O(\varepsilon)$ through output feedback.

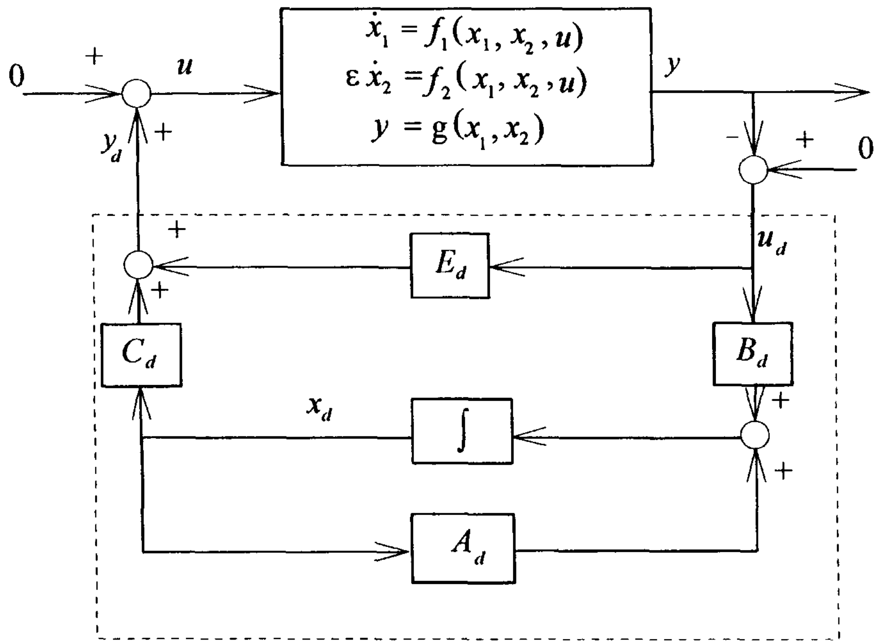


FIG. 4. Dynamic output feedback scheme for the original system (8).

$$\dot{x}_2 = \frac{1}{\varepsilon} f_2(x_1, x_2, u). \tag{30}$$

We rewrite (30) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_d \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_d & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_d \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -B_d \\ 0 \end{bmatrix} g(x_1, x_2) + \begin{bmatrix} f_1(x_1, x_2, u) \\ 0 \\ \frac{1}{\varepsilon} f_2(x_1, x_2, u) \end{bmatrix}. \tag{31}$$

Defining

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_d \\ x_2 \end{bmatrix} \tag{32}$$

and

$$\begin{aligned} g_z(z) &= g(x_1, x_2), \\ f_{1z}(z) &= f_1(x_1, x_2, u), \\ f_{2z}(z) &= f_2(x_1, x_2, u), \end{aligned} \tag{33}$$

then the system (31) can be rewritten as

$$\dot{z} = Z(z), \tag{34}$$

where

$$Z(z) \equiv \begin{bmatrix} Z_1(z) \\ Z_2(z) \\ Z_3(z) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_d & 0 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ -B_d \\ 0 \end{bmatrix} g_z(z) + \begin{bmatrix} f_{1z}(z) \\ 0 \\ \frac{1}{\varepsilon} f_{2z}(z) \end{bmatrix}. \tag{35}$$

In order to analyze the stability of system (34), we calculate the Jacobian matrix of $Z(z)$ at $z = 0$:

$$\left.\frac{\partial Z}{\partial z}\right|_{z=0} = \begin{bmatrix} \frac{\partial Z_1}{\partial z_1} & \frac{\partial Z_1}{\partial z_2} & \frac{\partial Z_1}{\partial z_3} \\ \frac{\partial Z_2}{\partial z_1} & \frac{\partial Z_2}{\partial z_2} & \frac{\partial Z_2}{\partial z_3} \\ \frac{\partial Z_3}{\partial z_1} & \frac{\partial Z_3}{\partial z_2} & \frac{\partial Z_3}{\partial z_3} \end{bmatrix}_{z=0} \tag{36}$$

with

$$\left.\frac{\partial Z_1}{\partial z_1}\right|_{z=0} = A_{11} - B_1 E_d C_1, \quad \left.\frac{\partial Z_1}{\partial z_2}\right|_{z=0} = B_1 C_d, \quad \left.\frac{\partial Z_1}{\partial z_3}\right|_{z=0} = A_{12} - B_1 E_d C_2,$$

$$\left.\frac{\partial Z_2}{\partial z_1}\right|_{z=0} = -B_d C_1, \quad \left.\frac{\partial Z_2}{\partial z_2}\right|_{z=0} = A_d, \quad \left.\frac{\partial Z_2}{\partial z_3}\right|_{z=0} = -B_d C_2,$$

$$\left.\frac{\partial Z_3}{\partial z_1}\right|_{z=0} = \frac{(A_{21} - B_2 E_d C_1)}{\varepsilon}, \quad \left.\frac{\partial Z_3}{\partial z_2}\right|_{z=0} = \frac{B_2 C_d}{\varepsilon}, \quad \left.\frac{\partial Z_3}{\partial z_3}\right|_{z=0} = \frac{A_{22}}{\varepsilon}.$$

Obviously the matrix in (36) is identical to the system matrix of (22). It is noted that since the closed-loop system (27), or equivalently system (22), is asymptotically stable for a sufficiently small ε, the Jacobian matrix of $Z(z)$ at $z = 0$ is Hurwitz, provided that ε is sufficiently small. Hence, we can see that if ε is small enough, then the dynamic output feedback controller (29) can stabilize the original nonlinear singularly perturbed system (8) under the feedback configuration of Fig. 4. This completes the proof. ■

Remark 2

The special cases

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u), \\ \varepsilon \dot{x}_2 &= f_2(x_1, x_2), \\ y &= g(x_1, x_2), \end{aligned} \tag{37}$$

or

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, u),\\ \varepsilon\dot{x}_2 &= f_2(x_1, x_2, u),\\ y &= g(x_1)\end{aligned} \tag{38}$$

will lead to $B_2 = 0$ or $C_2 = 0$, respectively. Hence, the assumption $B_2E_d = 0$ or $E_dC_2 = 0$ in Lemma 1 is always satisfied for any E_d.

So far, we have considered the nonlinear singularly perturbed system (8) in which the nonlinearities are continuously differentiable. The class of nonlinear singularly perturbed systems in which the nonlinearities are not necessarily continuously differentiable but satisfy the global Lipschtz condition is examined in the next section.

IV. Two-step Compensating Scheme

Consider the following nonlinear singularly perturbed system†:

$$\begin{aligned}\dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + f_1(x_1, x_2) + B_1u,\\ \varepsilon\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + f_2(x_1, x_2) + B_2u,\\ y &= C_1x_1 + C_2x_2 + g(x_1, x_2)\end{aligned} \tag{39}$$

where $x_1 \in \mathscr{R}^n$, $x_2 \in \mathscr{R}^m$, $u \in \mathscr{R}^{n_i}$, $y \in \mathscr{R}^{n_0}$. The functions f_1: $\mathscr{R}^n \times \mathscr{R}^m \to \mathscr{R}^n$, f_2: $\mathscr{R}^n \times \mathscr{R}^m \to \mathscr{R}^m$, g: $\mathscr{R}^n \times \mathscr{R}^m \to \mathscr{R}^{n_0}$; A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, C_1 and C_2 are the real matrices of appropriate dimensions. The system (39) can be rewritten in a more compact form as the following:

$$\begin{aligned}\dot{x} &= Ax + f(x) + Bu\\ y &= Cx + g_x(x),\end{aligned} \tag{40}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ \dfrac{A_{21}}{\varepsilon} & \dfrac{A_{22}}{\varepsilon} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \dfrac{B_2}{\varepsilon} \end{bmatrix}, \quad C = [C_1 \quad C_2],$$

$$f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ \dfrac{f_2(x_1, x_2)}{\varepsilon} \end{bmatrix}, \quad g_x(x) = g(x_1, x_2). \tag{41}$$

The following assumptions are made here:

(A1) $f(0) = 0$ (i.e. $f_1(0,0) = 0$ and $f_2(0,0) = 0$) and $g_x(0) = 0$ (i.e. $g(0,0) = 0$);
(A2) A_{22} is nonsingular and Hurwitz;
(A3) f and g_x satisfy the global Lipschtz condition, i.e. there exist constants k_f, k_g such that

† For the purpose of simplicity, the notations of the matrices in the linear part of system (39) are chosen to be the same as those in the linearized system (10).

$$\|f(\xi)-f(\eta)\| \leqslant k_f\|\xi-\eta\|, \qquad \forall(\xi,\eta)\in\mathscr{R}^n\times\mathscr{R}^m, \tag{42}$$

$$\|g_x(\xi)-g_x(\eta)\| \leqslant k_g\|\xi-\eta\|, \qquad \forall(\xi,\eta)\in\mathscr{R}^n\times\mathscr{R}^m. \tag{43}$$

Consider the linear part of (39):

$$\begin{aligned} x_{1l} &= A_{11}x_{1l}+A_{12}x_{2l}+B_1u_l,\\ \varepsilon\dot{x}_{2l} &= A_{21}x_{1l}+A_{22}x_{2l}+B_2u_l,\\ y_l &= C_1x_{1l}+C_2x_{2l}. \end{aligned} \tag{44}$$

The reduced-order model of system (44) can be obtained from (15) which is repeated here for convenience:

$$\begin{aligned} \dot{\bar{x}}_{1l} &= A_0\bar{x}_{1l}+B_0\bar{u}_l,\\ \bar{y}_l &= C_0\bar{x}_{1l}+E_0\bar{u}_l, \end{aligned} \tag{45}$$

in which

$$\begin{aligned} A_0 &= A_{11}-A_{12}A_{22}^{-1}A_{21}, \qquad B_0 = B_1-A_{12}A_{22}^{-1}B_2,\\ C_0 &= C_1-C_2A_{22}^{-1}A_{21}, \qquad E_0 = -C_2A_{22}^{-1}B_2. \end{aligned} \tag{46}$$

The stabilizing controller for the reduced-order model (45), which is described in (17), is also repeated here:

$$\begin{aligned} \dot{\bar{x}}_{ld} &= A_d\bar{x}_{ld}+B_d\bar{u}_{ld},\\ \bar{y}_{ld} &= C_d\bar{x}_{ld}+E_d\bar{u}_{ld}. \end{aligned} \tag{47}$$

In the preceding section we saw that the stabilizing controller (47), which is the realization of the compensator obtained from Theorem I, also stabilizes the full-order system (44) for a sufficiently small ε. The theoretical result demonstrates that combining the dynamic output feedback controller designed for the reduced-order model (45) with the quasi-stability result of Persidskii, a two-step compensating scheme will stabilize the original nonlinear singularly perturbed system (39), provided that ε is sufficiently small.

The two-step compensating scheme for stabilizing the system (39) is shown in Fig. 5, where C and H are controllers. The controller C is designed for stabilizing the reduced-order model (45) and is realized in (47); however, the variables $\bar{x}_{ld}$, $\bar{y}_{ld}$ and $\bar{u}_{ld}$ are replaced by x_d, y_d and u_d, respectively. The controller H will be chosen as the following form:

$$\begin{aligned} \dot{x}_h &= A_hx_h+B_hu_h\\ y_h &= O(\|x_h\|^k), \qquad \text{as}\|x_h\|\to 0, \end{aligned} \tag{48}$$

where A_h is Hurwitz, B_h is a real matrix and $k>1$. Moreover, the notation

$$y_h = O(\|x_h\|^k), \qquad \text{as}\|x_h\|\to 0,$$

means that exist positive constants q and δ, $\delta\to 0$, such that if $\|x_h\|\leqslant\delta$, then

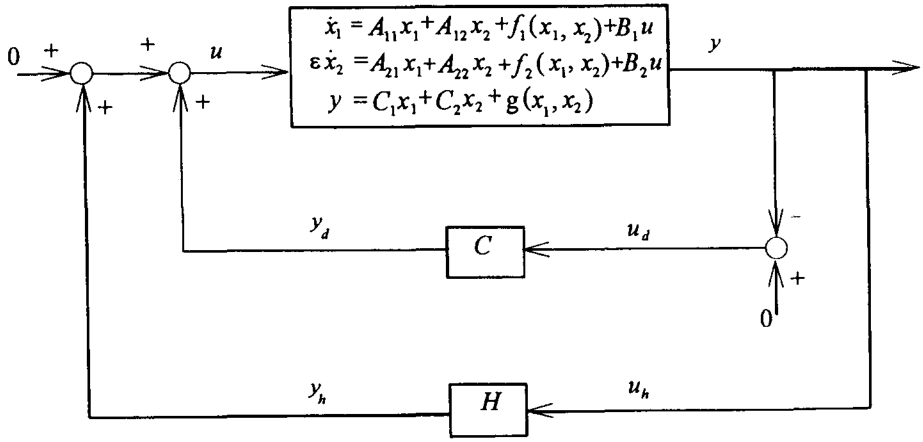


FIG. 5. Two-step compensating scheme for the original nonlinear singularly perturbed system (39).

$$\|y_h\| \leqslant q\|x_h\|^k.$$

Prior to examination of the stability of the closed-loop system under the two-step compensating scheme, the quasi-stability concept is first introduced.

Consider the nonlinear system which is partitioned into two subsystems of the forms

$$\dot{x} = X(z, x), \tag{49}$$

$$\dot{z} = Z(z, x). \tag{50}$$

Assume that the existence and uniqueness of the solutions of (49) and (50) are guaranteed and that $X(0,0) = Z(0,0) = 0$. Thus, the origin $(z,x) = (0,0)$ is an equilibrium point of the complete system. Let $y(t)$ be a continuous vector with compatible dimension and consider the associate system

$$\dot{x}(t) = X(y(t), x(t)). \tag{51}$$

Denote O_x, O_z as the origin $(z,x) = (0,0)$ of (49) and (50), respectively, and O the origin $(z,x) = (0,0)$ of the complete system. We say that the origin O_x is quasi-stable for (49) whenever, given any σ, there exists a positive $\delta(\sigma) \leqslant \sigma$ such that if $\|y(0)\| < \delta$ then any solution $x(t)$ of (51) with $\|x(0)\| \leqslant \delta$ has the property that in any time interval $0 \leqslant t \leqslant t_1$, in which $\|y(t)\| \leqslant \sigma$, we have $\|x(t)\| < \sigma$. Contrapositively, the origin O_x is said to be quasi-unstable for (49) whenever, given any δ, σ such that for any $y(t)$ with $\|y(0)\| < \delta$ and $y(t) < \sigma$ for $t \geqslant 0$, then for some solution $x(t)$ of (51) with $\|x(0)\| \leqslant \delta$, we have $\|x(t)\| \geqslant \sigma$ for some $t \geqslant 0$. The quasi-stability and quasi-instability for O_z are completely defined analogously.

The relationship between quasi-stability of the subsystems and stability of the complete system is stated in the following lemma established by Persidskii.

Lemma 2 (**10**)

If both O_x and O_z are quasi-stable for (49) and (50), respectively, O is then stable for the complete system. If O_x or O_z is quasi-unstable for the corresponding subsystem, O is then unstable for the complete system.

A type of quasi-stable systems is given in the following lemma.

Lemma 3 (**10**)
For the system

$$\dot{z} = Az + X(z, x) \tag{52}$$

under the assumption that the existence and uniqueness of the solution are guaranteed and $X(0,0) = 0$. If A is Hurwitz and

$$X(z,x) = O(\|z\| + \|x\|^k), \qquad k > 1$$

as $\|z\| + \|x\| \to 0$, i.e. there is a positive constant $\delta \to 0$ and a positive constant $\bar{K}$ such that

$$\|X(z,x)\| \leqslant \bar{K}(\|z\| + \|x\|^k), \qquad k > 1 \tag{53}$$

whenever $\|z\| + \|x\| \leqslant \delta$, the origin $(z, x) = (0, 0)$ of (52) is then quasi-stable.

Theorem IV
Suppose that (A1)–(A3) are satisfied and all the assumptions in Lemma 1 hold. Then, the original nonlinear singularly perturbed system (39) is stabilized by the controllers C and H under the feedback configuration of Fig. 5 for a sufficiently small ε.

Proof: The dynamic output feedback controller (47), by changing the notations of variables, is rewritten here as:

$$\begin{aligned} \dot{x}_d &= A_d x_d + B_d u_d, \\ y_d &= C_d x_d + E_d u_d. \end{aligned} \tag{54}$$

Combining (39), (48) and (54) under the configuration of Fig. 5 yields

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + f_1(x_1, x_2) + B_1 u \\ &= A_{11}x_1 + A_{12}x_2 + f_1(x_1, x_2) + B_1(y_h + y_d) \\ &= A_{11}x_1 + A_{12}x_2 + f_1(x_1, x_2) + B_1 y_h + B_1 C_d x_d \\ &\quad - B_1 E_d[C_1 x_1 + C_2 x_2 + g(x_1, x_2)] \\ &= (A_{11} - B_1 E_d C_1)x_1 + (A_{12} - B_1 E_d C_2)x_2 + B_1 C_d x_d \\ &\quad + f_1(x_1, x_2) - B_1 E_d g(x_1, x_2) + B_1 y_h, \end{aligned} \tag{55}$$

$$\begin{aligned} \dot{x}_d &= A_d x_d + B_d u_d \\ &= A_d x_d + B_d(-y) \\ &= -B_d C_1 x_1 + A_d x_d - B_d C_2 x_2 - B_d g(x_1, x_2), \end{aligned} \tag{56}$$

$$\begin{aligned}
\varepsilon\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + f_2(x_1, x_2) + B_2 u \\
&= A_{21}x_1 + A_{22}x_2 + f_2(x_1, x_2) + B_2(y_h + y_d) \\
&= A_{21}x_1 + A_{22}x_2 + f_2(x_1, x_2) + B_2 y_h + B_2 C_d x_d \\
&\quad - B_2 E_d[C_1 x_1 + C_2 x_2 + g(x_1, x_2)] \\
&= (A_{21} - B_2 E_d C_1)x_1 + A_{22}x_2 + B_2 C_d x_d \\
&\quad + f_2(x_1, x_2) - B_2 E_d g(x_1, x_2) + B_2 y_h, \qquad (57) \\
\dot{x}_h &= A_h x_h + B_h u_h \\
&= A_h x_h + B_h[C_1 x_1 + C_2 x_2 + g(x_1, x_2)] \\
&= A_h x_h + B_h C_1 x_1 + B_h C_2 x_2 + B_h g(x_1, x_2). \qquad (58)
\end{aligned}$$

The closed-loop system (55)–(58) can be rewritten in a more compact form as the following:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_d \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} - B_1 E_d C_1 & B_1 C_d & A_{12} - B_1 E_d C_2 \\ -B_d C_1 & A_d & -B_d C_2 \\ \dfrac{(A_{21} - B_2 E_d C_1)}{\varepsilon} & \dfrac{B_2 C_d}{\varepsilon} & \dfrac{A_{22}}{\varepsilon} \end{bmatrix} \begin{bmatrix} x_1 \\ x_d \\ x_2 \end{bmatrix}$$

$$+ \begin{bmatrix} f_1(x_1, x_2) \\ 0 \\ \dfrac{f_2(x_1, x_2)}{\varepsilon} \end{bmatrix} - \begin{bmatrix} B_1 E_d \\ B_d \\ \dfrac{B_2 E_d}{\varepsilon} \end{bmatrix} g(x_1, x_2) + \begin{bmatrix} B_1 \\ 0 \\ \dfrac{B_2}{\varepsilon} \end{bmatrix} y_h, \qquad (59a)$$

$$\dot{x}_h = A_h x_h + B_h C_1 x_1 + B_h C_2 x_2 + B_h g(x_1, x_2). \qquad (59b)$$

Define

$$z = \begin{bmatrix} x_1 \\ x_d \\ x_2 \end{bmatrix}, \qquad \|z\| = (\|x_1\|^2 + \|x_d\|^2 + \|x_2\|^2)^{1/2},$$

$$H_l = \begin{bmatrix} A_{11} + B_1 E_d C_1 & B_1 C_d & A_{12} - B_1 E_d C_2 \\ -B_d C_1 & A_d & -B_d C_2 \\ \dfrac{(A_{21} - B_2 E_d C_1)}{\varepsilon} & \dfrac{B_2 C_d}{\varepsilon} & \dfrac{A_{22}}{\varepsilon} \end{bmatrix},$$

$$f(x) = f_z(z)$$

and

$$g_x(x) = g_z(z),$$

such that

$$N(z,x_h) \equiv \begin{bmatrix} f_1(x_1,x_2) \\ 0 \\ \dfrac{f_2(x_1,x_2)}{\varepsilon} \end{bmatrix} - \begin{bmatrix} B_1E_d \\ B_d \\ \dfrac{B_2E_d}{\varepsilon} \end{bmatrix} g(x_1,x_2) + \begin{bmatrix} B_1 \\ 0 \\ \dfrac{B_2}{\varepsilon} \end{bmatrix} y_h$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} f(x) - \begin{bmatrix} B_1E_d \\ B_d \\ \dfrac{B_2E_d}{\varepsilon} \end{bmatrix} g_x(x) + \begin{bmatrix} B_1 \\ 0 \\ \dfrac{B_2}{\varepsilon} \end{bmatrix} y_h$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} f_z(z) - \begin{bmatrix} B_1E_d \\ B_d \\ \dfrac{B_2E_d}{\varepsilon} \end{bmatrix} g_z(z) + \begin{bmatrix} B_1 \\ 0 \\ \dfrac{B_2}{\varepsilon} \end{bmatrix} y_h$$

$$= Lf_z(z) - \begin{bmatrix} B_1E_d \\ B_d \\ \dfrac{B_2E_d}{\varepsilon} \end{bmatrix} g_z(z) + \begin{bmatrix} B_1 \\ 0 \\ \dfrac{B_2}{\varepsilon} \end{bmatrix} y_h,$$

where

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Consequently, the closed-loop system (59) can be rewritten as

$$\dot{z} = H_l z + N(z,x_h), \tag{60a}$$

$$\dot{x}_h = A_h x_h + [B_h C_1 \quad 0 \quad B_h C_2] z + B_h g_z(z). \tag{60b}$$

Since H_l is identical to the system matrix (22), by the same reason as that in the proof of Theorem III, H_l is Hurwitz for a sufficiently small ε.

Denote O_z as $(z,x_h) = (0,0)$ for the subsystem (60a). Then, O_z is an equilibrium point of (60a). Analogously, denote O_h as $(z,x_h) = (0,0)$ for the subsystem (60b). Then, O_h is an equilibrium point of (60b). Also O is denoted as $(z,x_h) = (0,0)$ for the complete system (60), then O is an equilibrium point for the complete system. The equilibrium point O is proven to be stable in the following.

First, it is shown that the subsystem (60a) is quasi-stable. From the definition of $N(z,x_h)$, we clearly have

$$\|N(z,x_h)\| = \left\| Lf_z(z) - \begin{bmatrix} B_1E_d \\ B_d \\ \dfrac{B_2E_d}{\varepsilon} \end{bmatrix} g_z(z) + \begin{bmatrix} B_1 \\ 0 \\ \dfrac{B_2}{\varepsilon} \end{bmatrix} y_h \right\|$$

$$= \left\| Lf(x) - \begin{bmatrix} B_1 E_d \\ B_d \\ \dfrac{B_2 E_d}{\varepsilon} \end{bmatrix} g_x(x) + \begin{bmatrix} B_1 \\ 0 \\ \dfrac{B_2}{\varepsilon} \end{bmatrix} y_h \right\|$$

$$\leqslant \|L\| \, \|f(x)\| + b_1 \|g_x(x)\| + b_2 \|y_h\|$$

$$\leqslant k_f \|L\| \, \|x\| + b_1 k_g \|x\| + b_2 \|y_h\|$$

$$= K\|x\| + b_2 q \|x_h\|^k$$

$$\leqslant K\|z\| + b_2 q \|x_h\|^k, \tag{61}$$

where

$$b_1 = \left\| \begin{bmatrix} B_1 E_d \\ B_d \\ \dfrac{B_2 E_d}{\varepsilon} \end{bmatrix} \right\|, \quad b_2 = \left\| \begin{bmatrix} B_1 \\ 0 \\ \dfrac{B_2}{\varepsilon} \end{bmatrix} \right\| \quad \text{and} \quad K = k_f \|L\| + b_1 k_g.$$

Let $\bar{K} \equiv \max(K, b_2 q)$, then

$$\|N(z, x_h)\| \leqslant \bar{K}(\|z\| + \|x_h\|^k), \tag{62}$$

i.e.

$$\|N(z, x_h)\| = O(\|z\| + \|x_h\|^k). \tag{63}$$

Hence, according to Lemma 3, the subsystem (60a) is quasi-stable for a sufficiently small ε. Subsequently, the subsystem (60b) is also proven to be quasi-stable in the following. From (60b), we have

$$x_h(t) = \exp(A_h t) x_h(0) + \int_0^t \exp(A_h(t-s))([B_h C_1 \quad 0 \quad B_h C_2] z(s) + B_h g_z(z(s)))\, \mathrm{d}s. \tag{64}$$

Since A_h is Hurwitz, there exists a positive constant r,

$$r \leqslant \min(-\mathrm{Re}(r_i)) \tag{65}$$

where r_i are the eigenvalues of A_h, and a positive constant $b \geqslant 1$ such that

$$\|\exp(A_h t)\| \leqslant b \exp(-rt).$$

Therefore,

$$\|x_h(t)\| \leqslant b \exp(-rt) \|x_h(0)\| + b \int_0^t \exp(-r(t-s))(\|[B_h C_1 \quad 0 \quad B_h C_2]\| \, \|z(s)\| + \|B_h\| \, \|g_z(z(s))\|)\, \mathrm{d}s$$

$$
\begin{aligned}
&\leqslant b\exp(-rt)\|x_h(0)\| \\
&\quad +b\int_0^t \exp(-r(t-s))((\|B_hC_1\|+\|B_hC_2\|)\|z(s)\|+k_g\|B_h\|\ \|z(s)\|)\,\mathrm{d}s \\
&\leqslant b\exp(-rt)\|x_h(0)\| \\
&\quad -b(\|B_hC_1\|+\|B_hC_2\|+k_g\|B_h\|)\int_0^t \exp(-r(t-s))\|z(s)\|\,\mathrm{d}s.
\end{aligned}
\tag{66}
$$

Consequently, it is clear that for $\|z(t)\| < \sigma$, there is a $\delta(\sigma) > 0$ such that if $\|x_h(0)\| < \delta$ then $\|x_h(t)\| < \sigma$. This proves the quasi-stability of the system (60b). According to Lemma 2, the equilibrium point O of the complete system (60) is stable for a sufficiently small ε. This completes the proof. ∎

Remark 3

According to the proof of Theorem IV, we have that if the dynamic output feedback controller (47) can asymptotically stabilize the linear part of the original nonlinear singularly perturbed system (39), i.e. system (44), for a sufficiently small ε, then the original system (39) can be stabilized by the two-step compensating scheme depicted in Fig. 5. Therefore, the merit of the design methodology proposed in this section is that the stability bound of ε can be deduced from the system (44) and hence is independent of the nonlinear part of the original system (39). The stability bound of ε of linear singularly perturbed systems has been extensively discussed in the literature (**11–14**).

V. Examples

Example 1

Consider the nonlinear singularly perturbed system:

$$
\begin{aligned}
\dot{x}_{s1} &= 0.1x_{s1}^2+x_{s2}x_f+\sin(x_{s2}+u), \qquad x_{s1}(0)=0.085, \\
\dot{x}_{s2} &= 2\sin(x_{s1})+0.1x_{s1}^2x_f-2x_f-5u, \qquad x_{s2}(0)=-0.1, \\
\varepsilon\dot{x}_f &= 5x_{s1}+0.2x_{s2}^2-5\sin(x_f)+u, \qquad x_2(0)=0.055, \\
y &= x_{s1}^2x_{s2}+2x_{s1}.
\end{aligned}
\tag{67}
$$

By comparing (67) with (8) we have

$$
x_1=\begin{bmatrix} x_{s1} \\ x_{s2} \end{bmatrix}, \qquad x_2=x_f, \qquad f_1(x_1,x_2,u)=\begin{bmatrix} 0.1x_{s1}^2+x_{s2}x_f+\sin(x_{s2}+u) \\ 2\sin(x_{s1})+0.1x_{s1}^2x_f-2x_f-5u \end{bmatrix},
$$

$$
f_2(x_1,x_2,u)=5x_{s1}+0.2x_{s2}^2-5\sin(x_f)+u, \qquad g(x_1,x_2)=x_{s1}^2x_{s2}+2x_{s1}. \tag{68}
$$

It is obvious that f_1, f_2 and g are continuously differentiable and $f_1(0,0,0)=0$,

$f_2(0,0,0) = 0$ and $g(0,0) = 0$. The matrices defined in (9) can be calculated as follows:

$$A_{11} = \frac{\partial f_1}{\partial x_1}\bigg|_{x_1=0,x_2=0,u=0} = \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix}, \quad A_{12} = \frac{\partial f_1}{\partial x_2}\bigg|_{x_1=0,x_2=0,u=0} = \begin{bmatrix} 0 \\ -2 \end{bmatrix},$$

$$A_{21} = \frac{\partial f_2}{\partial x_1}\bigg|_{x_1=0,x_2=0,u=0} = [5 \quad 0], \quad A_{22} = \frac{\partial f_2}{\partial x_2}\bigg|_{x_1=0,x_2=0,u=0} = -5,$$

$$B_1 = \frac{\partial f_1}{\partial u}\bigg|_{x_1=0,x_2=0,u=0} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \quad B_2 = \frac{\partial f_2}{\partial u}\bigg|_{x_1=0,x_2=0,u=0} = 1,$$

$$C_1 = \frac{\partial g}{\partial x_1}\bigg|_{x_1=0,x_2=0} = [2 \quad 0], \quad C_2 = \frac{\partial g}{\partial x_2}\bigg|_{x_1=0,x_2=0} = 0. \tag{69}$$

Therefore, the linearized system of (67) is written as

$$\begin{aligned} \dot{x}_{1l} &= \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix} x_{1l} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} x_{2l} + \begin{bmatrix} 1 \\ -5 \end{bmatrix} u_l, \\ \varepsilon \dot{x}_{2l} &= [5 \quad 0] x_{1l} - 5x_{2l} + u_l, \\ y_l &= [2 \quad 0] x_{1l}. \end{aligned} \tag{70}$$

It is noted that $A_{22} = -5$ is nonsingular and Hurwitz. Then, from (15) and (16), the reduced-order model of (70) is obtained as

$$\begin{aligned} \dot{\bar{x}}_{1l} &= A_0 \bar{x}_{1l} + B_0 \bar{u}_l, \\ \bar{y}_l &= C_0 \bar{x}_{1l} + E_0 \bar{u}_l, \end{aligned} \tag{71}$$

in which

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \quad B_0 = B_1 - A_{12}A_{22}^{-1}B_2 = \begin{bmatrix} 1 \\ -5.4 \end{bmatrix},$$

$$C_0 = C_1 - C_2 A_{22}^{-1} A_{21} = [2 \quad 0], \quad E_0 = -C_2 A_{22}^{-1} B_2 = 0.$$

It is obvious that the pairs (A_0, B_0) and (A_0, C_0) are stabilizable and detectable, respectively. Hence, we proceed to design the dynamic output feedback controller for the reduced-order model (71) such that it can asymptotically stabilize the system (67).

Choose $K = [-20 \quad -20]$ and $F = [5 \quad 0]^{\mathrm{T}}$ such that the matrices

$$A_0 - B_0 K = \begin{bmatrix} 20 & 21 \\ -108 & -110 \end{bmatrix} \quad \text{and} \quad A_0 - FC_0 = \begin{bmatrix} -10 & 1 \\ 0 & -2 \end{bmatrix}$$

are both Hurwitz, respectively. Replacing A_p, B_p, C_p and E_p in (6) with A_0, B_0, C_0 and E_0, respectively, the matrices V_p and U_p are then calculated as

$$U_p = K(sI-(A_0-FC_0))^{-1}F = \frac{s^2+100s+1168}{s^2+12s+20},$$

$$V_p = I+K(sI-(A_0-FC_0))^{-1}(B_0-FE_0) = \frac{-100s-200}{s^2+12s+20}. \tag{72}$$

Hence, if we set $R = 0$ in (5), then the stabilizing compensator for the reduced-order model (71) becomes

$$S(p) = V_p^{-1}U_p = \frac{-100s-200}{s^2+12s+20} \tag{73}$$

which can be realized as

$$\begin{bmatrix} \dot{x}_{d1} \\ \dot{x}_{d2} \end{bmatrix} = A_d \begin{bmatrix} x_{d1} \\ x_{d2} \end{bmatrix} + B_d u_d$$

$$y_d = C_d \begin{bmatrix} x_{d1} \\ x_{d2} \end{bmatrix} + E_d u_d, \tag{74}$$

where

$$A_d = \begin{bmatrix} 0 & 1 \\ -1168 & -100 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_d = [-200 \quad -100] \quad \text{and} \quad E_d = 0.$$

Therefore, we can see that $(I-E_dE_0)^{-1} = I$, $B_2E_d = E_dC_2 = 0$ and the pairs (A_d, B_d), (A_d, C_d) are stabilizable and detectable, respectively. Hence, all the assumptions in Lemma 1 are satisfied. Consequently, according to Theorem III, the dynamic output feedback controller (74) stabilizes the original nonlinear singularly perturbed system (67) for a sufficiently small ε. The simulation of the closed-loop system (67) and (74) under the feedback interconnection as depicted in Fig. 4 is shown in Fig. 6. It is obvious that the closed-loop system is asymptotically stable at the origin.

Example 2

Consider the nonlinear singularly perturbed system:

$$\dot{x}_s = 7x_s + [6 \quad -2]\begin{bmatrix} x_{f1} \\ x_{f2} \end{bmatrix} - 6|x_s| - 2u,$$

$$\varepsilon \begin{bmatrix} \dot{x}_{f1} \\ \dot{x}_{f2} \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} x_s + \begin{bmatrix} -2 & 5 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_{f1} \\ x_{f2} \end{bmatrix} + \begin{bmatrix} -5|x_{f1}| \\ -8\sin(x_{f1}) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$

$$y = 5x_s + [6 \quad 6]\begin{bmatrix} x_{f1} \\ x_{f2} \end{bmatrix} + x_s^2. \tag{75}$$

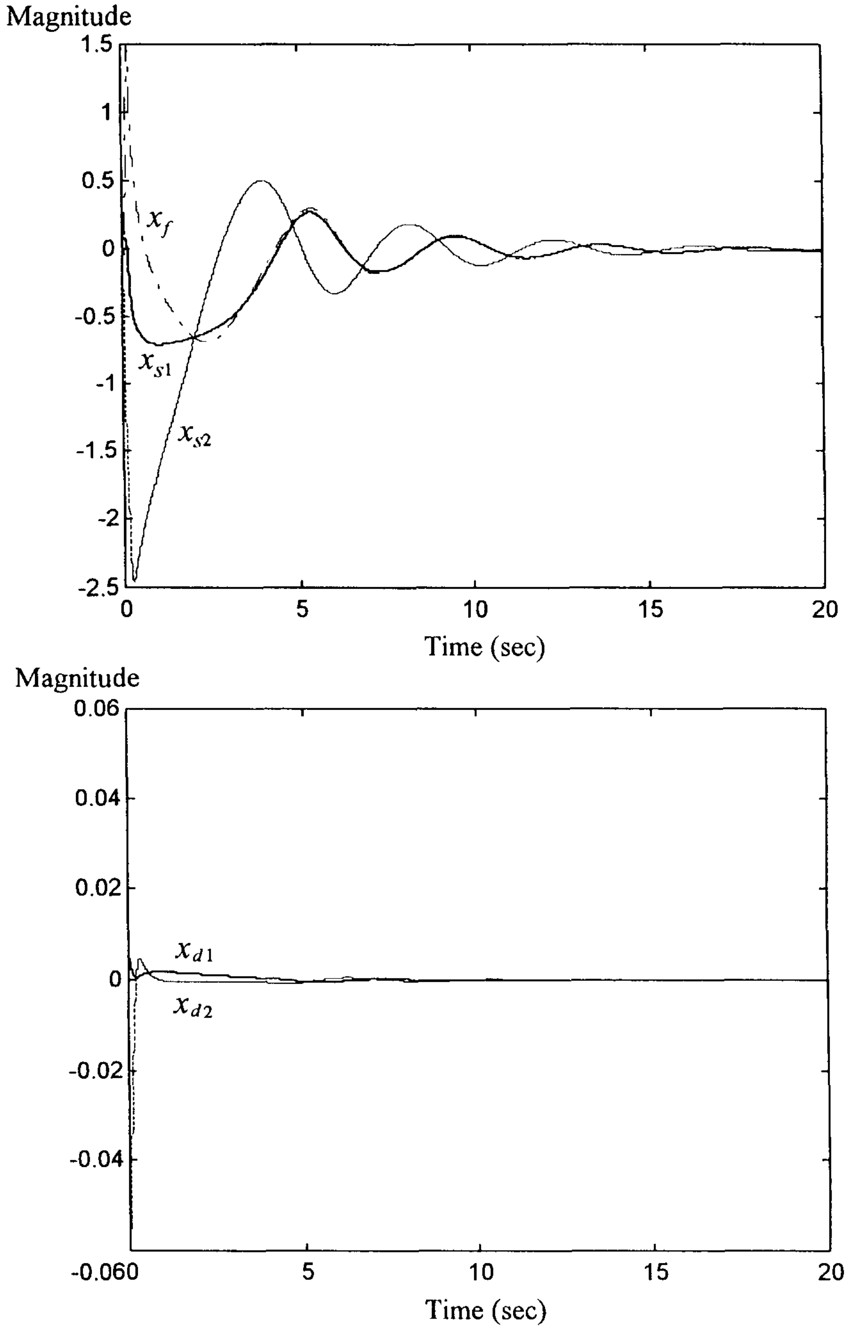


FIG. 6. The dynamics of the closed-loop system (67) and (74) with $\varepsilon = 0.02$ and initial condition $[x_{s1}(0)\ \ x_{s2}(0)\ \ x_{d1}(0)\ \ x_f(0)\ \ x_{d2}(0)]^T = [0.085\ \ -0.1\ \ 0.055\ \ 0.0055\ \ 0.05]^T$.

A comparison of system (75) with system (39) reveals

$$x_1 = x_s, \quad x_2 = \begin{bmatrix} x_{f1} \\ x_{f2} \end{bmatrix}, \quad A_{11} = 7, \quad A_{12} = [6 \quad -12],$$

$$A_{21} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -2 & 5 \\ 0 & -3 \end{bmatrix},$$

$$B_1 = -2, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_1 = 5, \quad C_2 = [6 \quad 6],$$

$$f_1(x_1, x_2) = -6|x_s|, \quad f_2(x_1, x_2) = \begin{bmatrix} -5|x_{f1}| \\ -8\sin(x_{f1}) \end{bmatrix}$$

$$\text{and} \quad g(x_1, x_2) = x_s^2. \tag{76}$$

Obviously A_{22} is nonsingular and Hurwitz. It is easy to check that the functions f_1, f_2 and g with $f_1(0,0) = 0, f_2(0,0) = 0$ and $g(0,0) = 0$ satisfy the global Lipschtz condition. Therefore, the assumptions (A1)–(A3) (in Section IV) are satisfied. The two-step compensating scheme for the system (75) can next be designed.

First, the corresponding matrices in (46) can be calculated as

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21} = 1, \quad B_0 = B_1 - A_{12}A_{22}^{-1}B_2 = 2,$$

$$C_0 = C_1 - C_2A_{22}^{-1}A_{21} = -1, \quad E_0 = -C_2A_{22}^{-1}B_2 = 9. \tag{77}$$

It can be seen that the pairs (A_0, B_0) and (A_0, C_0) are stabilizable and detectable, respectively. Therefore, we choose $K = 10$ and $F = -10$ such that

$$A_0 - B_0K = -19 \quad \text{and} \quad A_0 - FC_0 = -9$$

are both Hurwitz. Replacing A_p, B_p, C_p and E_p in (6) with A_0, B_0, C_0 and E_0, respectively, we have

$$\bar{N}_p = C_0(sI - (A_0 - FC_0))^{-1}(B_0 - FE_0) + E_0 = \frac{9s - 11}{s + 9},$$

$$\bar{D}_p = I - C_0(sI - (A_0 - FC_0))^{-1}F = \frac{s - 1}{s + 9},$$

$$N_p = (C_0 - E_0K)(sI - (A_0 - B_0K))^{-1}B_0 + E_0 = \frac{9s - 11}{s + 19},$$

$$D_p = I - K(sI - (A_0 - B_0K))^{-1}B_0 = \frac{s - 1}{s + 19},$$

$$U_p = K(sI - (A_0 - FC_0))^{-1}F = \frac{-100}{s + 9},$$

$$V_p = I + K(sI - (A_0 - FC_0))^{-1}(B_0 - FE_0) = \frac{s + 929}{s + 9},$$

$$\bar{U}_p = K(sI-(A_0-B_0K))^{-1}F = \frac{-100}{s+19},$$

$$\bar{V}_p = I+(C_0-E_0K)(sI-(A_0-B_0K))^{-1}F = \frac{s+929}{s+19}.$$

Hence, if we set $R = 1/(s+10)$ in (5), then the stabilizing compensator for the reduced-order model (45) becomes

$$S(p) = \left(\frac{s+929}{s+9} - \frac{1}{s+10}\frac{9s-11}{s+9}\right)^{-1}\left(\frac{-100}{s+9} + \frac{1}{s+10}\frac{s-1}{s+9}\right) = \frac{-99s-1001}{s^2+930s+9301} \tag{78}$$

which can be realized as

$$\begin{bmatrix}\dot{x}_{d1}\\ \dot{x}_{d2}\end{bmatrix} = A_d\begin{bmatrix}x_{d1}\\ x_{d2}\end{bmatrix} + B_d u_d,$$

$$y_d = C_d\begin{bmatrix}x_{d1}\\ x_{d2}\end{bmatrix} + E_d u_d, \tag{79}$$

where

$$A_d = \begin{bmatrix}0 & 1\\ -9301 & -930\end{bmatrix}, \quad B_d = \begin{bmatrix}0\\ 1\end{bmatrix}, \quad C_d = [-1001 \quad -99] \quad \text{and} \quad E_d = 0.$$

It can be shown that all the assumptions in Lemma 1 are satisfied. Furthermore, the stable linear compensator H can be chosen, according to (48), as the following:

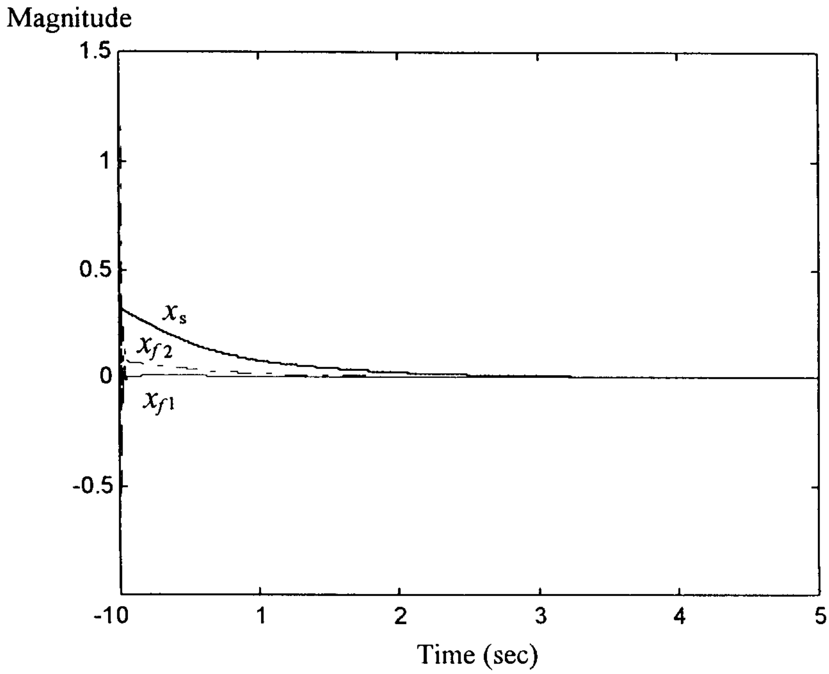
$$\dot{x}_h = -5x_h + u_h,$$

$$y_h = x_h^2. \tag{80}$$

The dynamics of the closed-loop system (75), (79) and (80) under the interconnection shown in Fig. 5 are illustrated in Fig. 7. These figures clearly indicate that the closed-loop system is asymptotically stable at the origin.

VI. Conclusions

In this paper, the stabilization problem of two classes of nonlinear singularly perturbed systems is discussed. In the first class of systems, the nonlinear functions describing the system dynamics need to be continuously differentiable. While this condition is not necessarily needed in the second class of systems, instead it is assumed that the nonlinear functions describing the system dynamics only satisfy the global Lipschtz condition. The set $S(p)$ of all linear stabilizing compensators given in Theorem 1 plays an important role in our control synthesis, since it can stabilize the reduced-order model of the linearized (or linear) part of the two classes of nonlinear singularly perturbed systems under consideration. We have proved that this can guarantee the stabilization of the first class of systems for a sufficiently small ε. However, this is not



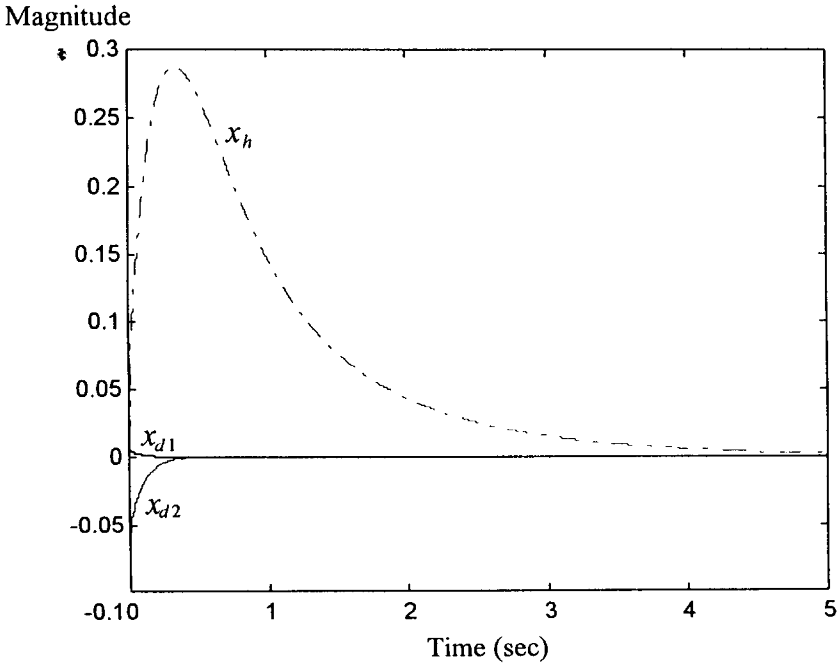


FIG. 7. The dynamics of the closed-loop (75), (79) and (80) with $\varepsilon = 0.02$ and initial conditions $[x_s(0) \ x_{f1}(0) \ x_{f2}(0) \ x_{d1}(0) \ x_{d2}(0) \ x_h(0)]^T = [0.35 \quad -0.42 \ 0.5 \ 0.005 \ 0.03 \ 0.01]^T$.

the case for the other class of systems. The quasi-stability result by Persidskii is then introduced to combine the set $S(p)$ obtained from the factorization theory to establish a two-step compensating scheme for stabilizing this class of nonlinear singularly perturbed systems, provided that ε is sufficiently small. The theoretical result demonstrates that, under this two-step compensating scheme, the closed-loop system is asymptotically stable.

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