

Short Communication

Note on “Wash criterion in analytic hierarchy process”

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Abstract

The paper of Finan and Hurley – published in *Computers and Operations Research* (2002) – was re-examined, where they discussed a seemingly contradictory phenomenon resulting from the ignoring of wash criteria in the analytic hierarchy process (AHP). With that, they raised a serious challenge to the AHP methodology. However, by reviewing their arguments and example data, analyses regarding to their propositions and numerical example are presented in this paper to counter their challenge.

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1. Introduction

The analytic hierarchy process (AHP) methodology utilizes pair-wise comparisons to derive the weights for multiple criteria and subsequently the rank-order of the alternatives for decision making. Belton and Gear (1983) identified a rank reversal phenomenon in AHP and generated a substantive amount of discussion in the discipline. Researchers including the creator of AHP, Saaty (1977), all consented to the rank reversal phenomenon, though still debating its impact. The rank reversal is a phenomenon associated with the resulting of a different alternative rank order when new alternatives are added to

an existing hierarchy. Saaty (1987) suggested that once a new alternative is added, one should introduce new functional criteria or use structural information to modify the weights of the existing functional criteria to avoid reversal. In contrast, there are also numerous researches aiming at resolving the phenomenon of rank reversal by modifying AHP theory or principles. Saaty (1995) demonstrated that rank reversal will not be a flaw and can be resolved in four ways – absolute measurement, relative measurement, the criteria depending on the alternatives but not on their numbers, and the criteria depending on both the criteria and the alternatives – depending on the characteristics of the criteria. Definitely, rank reversal is not a mathematical or theoretical predicament but a practical phenomenon in the decision making process for various problems.

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This paper focuses on Finan and Hurley’s (2002) paper where they utilized the notion of wash (non-discriminating) criteria to establish two propositions with respect to AHP. They proved that for a two-level AHP model when the comparison matrix is perfectly consistent, ignoring a wash criterion would not change the rank order for the alternatives. They stated that they could not prove or disprove for the case of an imperfectly consistent decision maker (DM). They then constructed a three-level example to illustrate that rank reversal does occur when a wash criterion is ignored. By demonstrating the occurrence of rank reversal with a counter-example, they tried to lay out a contradiction in the AHP methodology and thus a flaw in the methodology. They claimed that their discovery posted a serious challenge to the AHP methodology.

In a later research, Liberatore and Nydick (2004) claimed that after removing wash criteria, the relative weights should be reevaluated and hence no rank reversal problems. Furthermore, Saaty and Vargas (2006) asserted that wash criteria could not be blindly deleted.

In this paper, the open question for the case of an imperfect consistent DM posted by Finan and Hurley (2002) is solved and thus a more general case is shown. In addition, the “counter-example” by Finan and Hurley is in fact based on the parameter values which violate the basic assumption of the AHP methodology.

2. Review of Finan and Hurley

In the paper by Finan and Hurley, they first assumed that the DM begins with $n + 1$ criteria indexed by the set $J = \{0, 1, \dots, n\}$ and m choice alternatives indexed by $I = \{1, 2, \dots, m\}$. The DM’s problem is to decide a rank-order of the m alternatives. The wash criterion is indexed by 0 and the reduced set of criteria by $\bar{J} = \{1, \dots, n\}$.

Finan and Hurley (2002) assumed the DM to be perfectly consistent and denoted the set of weights by c_j for the full criteria set J and by \bar{c}_j for the reduced criteria set, such that

$$c_j = (1 - c_0)\bar{c}_j \quad \text{for } j = 1, 2, \dots, n. \tag{1}$$

To see this, they assumed that the elements of the pairwise comparison matrix for the full criteria set has elements a_{ij} and noted that

$$\frac{c_i}{c_j} = a_{ij} = \frac{\bar{c}_i}{\bar{c}_j} \quad \text{for all } i, j \geq 1. \tag{2}$$

It can be deduced from the above equation that Finan and Hurley (2002) must have first determined c_0, c_1, \dots, c_n , with $c_j > 0$ and $\sum_{j=0}^n c_j = 1$ in order to derive a_{ij} . When the DM is perfectly consistent, Eq. (1) does hold, and can be expressed as

$$\bar{c}_j = \frac{c_j}{1 - c_0} \quad \text{for } j = 1, \dots, n. \tag{3}$$

They denoted the AHP hierarchy with t levels of criteria as $H(t)$. With the AHP model being $H(1)$, for Saaty or the additive method (SAHP), they derived Proposition 1 to show that the rank order is unaffected by eliminating the wash criterion. For the multiplicative procedure (MAHP), they derived Proposition 2 and established the same conclusion. However, an $H(2)$ example was then used to demonstrate that ignoring a wash criterion does result in rank reversal. They claimed and we quote: “In view of the fact that every hierarchy with multiple levels of criteria can, in principle, be modeled as a hierarchy with a single level of criteria, it must be that our methods for collapsing a hierarchy with multiple levels of criteria are incorrect. In sum, we view our results as a serious challenge to the AHP methodology”. In the following sections, contrary to their claim, the validity of their Proposition 1 is first discussed. Then, the “counter-example” by Finan and Hurley is examined and dismissed.

3. Limitation of the propositions in Finan and Hurley

The open question raised by Finan and Hurley (2002) on pp. 1028, line 27–28, stated: “We cannot prove the same result in the case of an imperfectly consistent DM”. In the following, the analysis for an imperfectly consistent DM is established. It is clear that if the equality of

$$\begin{bmatrix} 1 & b_1 & \dots & b_n \\ 1/b_1 & & & \\ \dots & & a_{ij} & \\ 1/b_n & & & \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{bmatrix} = \lambda_{n+1} \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{bmatrix} \tag{4}$$

holds, then so does

$$\begin{bmatrix} & & & \\ & a_{ij} & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} \frac{c_1}{1-c_0} \\ \dots \\ \frac{c_n}{1-c_0} \end{bmatrix} = \lambda_n \begin{bmatrix} \frac{c_1}{1-c_0} \\ \dots \\ \frac{c_n}{1-c_0} \end{bmatrix}, \tag{5}$$

where λ_{n+1} and λ_n are the maximum eigenvalues for the $(n + 1) \times (n + 1)$ and $(n \times n)$ pairwise comparison matrices, respectively.

From Eq. (4), we know that

$$c_0 + \sum_{j=1}^n b_j c_j = \lambda_{n+1} c_0, \quad \text{and} \quad (6)$$

$$\frac{c_0}{b_i} + \sum_{j=1}^n a_{ij} c_j = \lambda_{n+1} c_i \quad \text{for } i = 1, 2, \dots, n. \quad (7)$$

For the time being, Eq. (5) is assumed to hold in order to derive the necessary condition, which yields that

$$\sum_{j=1}^n a_{ij} \frac{c_j}{1 - c_0} = \lambda_n \frac{c_i}{1 - c_0} \quad \text{for } i = 1, 2, \dots, n. \quad (8)$$

By Eqs. (7) and (8), it can be established that $(\lambda_{n+1} - \lambda_n) b_j c_j = c_0$ for $j = 1, 2, \dots, n$ and Eq. (6) can be used to derive the following relation

$$\lambda_{n+1}^2 - (1 + \lambda_n) \lambda_{n+1} + \lambda_n - n = 0. \quad (9)$$

From Eq. (9), if $[a_{ij}]$ is perfectly consistent, by Saaty (1977, 1980), then $\lambda_n = n$ such that $\lambda_{n+1} = n + 1$, again by Saaty (1977, 1980), and the $(n + 1) \times (n + 1)$ pairwise comparison matrix is perfectly consistent, too.

Based on the above discussion, we may construct a counter-example where a $(n + 1) \times (n + 1)$ pairwise comparison matrix, $M_{(n+1) \times (n+1)}$, is not perfectly consistent, and $[a_{ij}]$, where $[a_{ij}]$ is $M_{(n+1) \times (n+1)}$ after deleting the first row and the first column, is perfectly consistent, such that Finan and Hurley’s assertion with Eq. (3) is invalid. Consequently, the proof of Proposition 1 in Finan and Hurley (2002) is not applicable for the more general case where the DM is not perfectly consistent. In other words, the wash criterion can only be ignored in a very special case as employed in their Proposition 1.

The counter-example for the case of an imperfect consistent DM assumes

$$[a_{ij}] = I_3 \quad \text{and} \quad M_{4 \times 4} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1/2 & 1 & 1 & 1 \\ 1/3 & 1 & 1 & 1 \\ 1/4 & 1 & 1 & 1 \end{bmatrix}. \quad (10)$$

As the eigenvalues for $M_{4 \times 4}$ are 4.0458, $-0.0229 \pm 0.4299i$ and 0, the normalized principal eigenvector is therefore $[0.4912, 0.1865, 0.1662, 0.1561]^T$. By the method of Finan and Hurley (2002), i.e., Eq. (3), it follows that

$$\begin{bmatrix} 0.1865 & 0.1662 & 0.1561 \\ 1 - 0.4912 & 1 - 0.4912 & 1 - 0.4912 \end{bmatrix}^T = [0.3665, 0.3267, 0.3068]^T. \quad (11)$$

By the method of Finan and Hurley (2002), the normalized principal eigenvector of $[a_{ij}]$ with $a_{ij} = 1$ for $1 \leq i, j \leq 3$, will be $[0.3665, 0.3267, 0.3068]^T$. However, the normalized principal eigenvector for this $[a_{ij}]$ should be $[1/3, 1/3, 1/3]^T$, and hence, contradicting what Propositions 1 was trying to establish for the case of an imperfectly consistent DM.

In addition, Finan and Hurley assumed (as given in the above Eq. (2)) that

$$a_{ij} = \frac{c_i}{c_j} \quad \text{for all } i, j \geq 1, \quad (12)$$

which is questionably odd. From Saaty (1977, 1980), the entries in the pairwise comparison matrix should be in the set $\{1, 2, \dots, 9, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{9}\}$. However, the computational result of $\frac{c_i}{c_j}$ cannot be guaranteed to lie in the set $\{1, 2, \dots, 9, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{9}\}$. Consequently, their Propositions 1 and 2 are based on variables with questionable values.

4. Questionable numerical example by Finan and Hurley

Consider the same numerical example of Finan and Hurley (2002) with the following data:

Goal	G				
Main criteria	J		J'		
Main criteria weights	0.55		0.45		
Subcriteria	J ₀	J ₁	J ₂	J' ₁	J' ₂
Subcriteria weights	0.6	0.2	0.2	0.5	0.5
Option A1	0.5	0.8	0.4	0.2	0.6
Option A2	0.5	0.2	0.6	0.8	0.4

In Finan and Hurley (2002), they assumed that the comparison matrix is perfectly consistent. Based on that, three questionable sets of data exist in their table: (a) main criteria weights 0.55 and 0.45, (b) the weights for alternatives A1 and A2 for subcriteria J_2 being 0.4 and 0.6, and (c) the weights for alternatives A1 and A2 for subcriteria J'_2 being 0.6 and 0.4.

The perfectly consistent comparison matrix with principal eigenvector $[0.55, 0.45]^T$ will be $\begin{bmatrix} 1 & 11/9 \\ 9/11 & 1 \end{bmatrix}$. However, the entries $a_{12} = 11/9$ and $a_{21} = 9/11$ are not in the set $\{1, 2, \dots, 9, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{9}\}$. This means that 0.55 and 0.45 are chosen arbitrarily rather than derived from the required set of data. Similarly, $[0.4, 0.6]^T$ and $[0.6, 0.4]^T$ are not principal eigenvectors for perfectly consistent

comparison matrices with entries in $\{1, 2, \dots, 9, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{9}\}$.

To better examine the issue of rank reversal, the three sets of relative weights of numeral example in Finan and Hurley (2002) are modified to reflect the correct AHP. For the values 11/9 and 9/11 that models a DM's view, the closest values from the allowable entry set $\{1, 2, \dots, 9, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{9}\}$ will be 1. The eigenvector $[0.55, 0.45]^T$ therefore becomes $[0.5, 0.5]^T$. For the ratios 0.4/0.6 and 0.6/0.4, the closest ones to be changed to are: 1 and 1, or 1/2 and 2. Since the ratio 0.6/0.4 is of an equal distance to 1 and 2, the ratio 0.4/0.6 is analyzed where

$$\frac{0.4}{0.6} - \frac{1}{2} = \frac{1}{6} < 1 - \frac{0.4}{0.6} = \frac{1}{3}. \quad (13)$$

Hence, the weight vector $[0.4, 0.6]^T$ is changed to $[1/3, 2/3]^T$ accordingly, and similarly, $[0.6, 0.4]^T$ to $[2/3, 1/3]^T$.

Based on our above analysis for modification, by the Saaty method, it can be found that the final weights of A_1 and A_2 , J_0 should be changed from $\begin{pmatrix} 0.477 \\ 0.523 \end{pmatrix}$ to $\begin{pmatrix} 0.48 \\ 0.52 \end{pmatrix}$ if with wash criteria and should be changed from $\begin{pmatrix} 0.51 \\ 0.49 \end{pmatrix}$ to $\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ if without wash criteria. With the modification of data to fit the requirement of AHP, the counter-example constructed by Finan and Hurley (2002) does not result in rank reversal for alternatives A_1 and A_2 .

5. Conclusion

Rank reversal phenomena which were used by Finan and Hurley (2002) to construct a series of

arguments to challenge the AHP methodology are shown to be flawed both in the process and in data. Though there may still be issues that need to be addressed about the AHP, however, not with a wrongful case.

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