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Fault-tolerant hamiltonian connectedness of cycle composition networks

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Abstract

It is important for a network to tolerate as many faults as possible. With the graph representation of an interconnection network, a k-regular hamiltonian and hamiltonian connected network is super fault-tolerant hamiltonian if it remains hamiltonian after removing up to $k-2$ vertices and/or edges and remains hamiltonian connected after removing up to $k-3$ vertices and/or edges. Super fault-tolerant hamiltonian networks have an optimal flavor with regard to the fault-tolerant hamiltonicity and fault-tolerant hamiltonian connectivity. For this reason, a cycle composition framework was proposed to construct a $(k+2)$-regular super fault-tolerant hamiltonian network based on a collection of n k-regular super fault-tolerant hamiltonian networks containing the same number of vertices for $n \geqslant 3$ and $k \geqslant 5$. This paper is aimed to emphasize that the cycle composition framework can be still applied even when $k=4$.

Keywords: Hamiltonian; Hamiltonian connected; Fault tolerance; Super fault-tolerant hamiltonian

1. Introduction

The architecture of an interconnection network is usually represented by a graph whose vertices and edges represent processors and communication links, respectively. Thus, we use the terms graph and network interchangeably. Throughout this paper, we concentrate on loopless undirected graphs. For the graph definitions and notations we follow the ones given by Bondy and Murty [1]. A graph G consists of a nonempty set $V(G)$ and a subset $E(G)$ of $\{(u,v)|(u,v)$ is an unordered pair of $V(G)\}$. The set $V(G)$ is called the *vertex set* of G and $E(G)$ is called the *edge set*. Two vertices u and v of G are adjacent if $(u,v) \in E(G)$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let S be a nonempty subset of $V(G)$. The subgraph *induced* by S is the subgraph of G with its vertex set S and with its edge set which consists of those edges joining any two vertices

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in S. We use $G - S$ to denote the subgraph of G induced by $V(G) - S$. Analogously, the subgraph generated by a nonempty subset $F \subseteq E(G)$ is the subgraph of G with its edge set F and its vertex set consisting of those vertices of G incident with at least one edge of F. We use $G - F$ to denote the subgraph of G with vertex set $V(G)$ and edge set $E(G) - F$. The degree of a vertex u in G, denoted by $\deg_G(u)$, is the number of edges incident to u. A graph G is *k-regular* if all its vertices have the same degree k. A *matching* of size k in a graph G is a set of k edges with no shared endpoints. The vertices belonging to the edges of a matching are *saturated* by the matching; the others are unsaturated. A *perfect matching* is a matching that saturates every vertex of G.

A *path* P of length k from a vertex x to a vertex y in a graph G is a sequence of distinct vertices $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ such that $x = v_0$, $y = v_k$, and $(v_{i-1}, v_i) \in E(G)$ for every $1 \leqslant i \leqslant k$. More precisely, we write $P = \langle v_0, e_1, v_1, e_2, v_2, \ldots v_{k-1}, e_k, v_k \rangle$, in which $e_i = (v_{i-1}, v_i) \in E(G)$ for every i. For convenience, we write P as $\langle v_0, \ldots, v_i, Q, v_j, \ldots, v_k \rangle$ where $Q = \langle v_i, \ldots, v_j \rangle$. Note that we allow Q to be a path of length zero. Moreover, we use P^{-1} to denote the path $\langle v_k, v_{k-1}, \ldots, v_1, v_0 \rangle$. To emphasize the beginning and ending vertices of P, we also write P as $P[x, y]$. A path of a graph G is a *hamiltonian path* if it spans G. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A cycle of G is a *hamiltonian cycle* if it traverses all vertices of G. A graph G is hamiltonian if it has a hamiltonian cycle, and G is hamiltonian connected if there exists a hamiltonian path joining any two vertices of G.

A suitable network is generally designed to satisfy some specified requirements. For example, the hamiltonian property is one of the major concerns for designing the network topology and fault tolerance is desirable in massive parallel systems. So these two properties can be concerned in the network topology as follows. A graph G is called l-fault-tolerant hamiltonian (resp. l-fault-tolerant hamiltonian connected) if it remains hamiltonian (resp. hamiltonian connected) after removing at most l vertices and/or edges. The fault-tolerant hamiltonicity of G, $\mathscr{H}_f(G)$, is defined to be the maximum integer l such that $G - F$ remains hamiltonian for every $F \subset V(G) \cup E(G)$ with $|F| \leqslant l$ if G is hamiltonian, and undefined otherwise. Obviously, $\mathscr{H}_f(G) \leqslant \delta(G) - 2$, where $\delta(G) = \min\{\deg_G(v) | v \in V(G)\}$. A regular graph G is *optimal fault-tolerant hamiltonian* if $\mathscr{H}_f(G) = \delta(G) - 2$. The fault-tolerant hamiltonian connectivity of G, $\mathscr{H}_f^\kappa(G)$, is defined to be the maximum integer l such that $G - F$ remains hamiltonian connected for every $F \subset V(G) \cup E(G)$ with $|F| \leqslant l$ if G is hamiltonian connected, and undefined otherwise. Obviously, $\mathscr{H}_f^\kappa(G) \leqslant \delta(G) - 3$. A regular graph G is *optimal fault-tolerant hamiltonian connected* if $\mathscr{H}_f^\kappa(G) = \delta(G) - 3$. A regular graph is *super fault-tolerant hamiltonian* if $\mathscr{H}_f(G) = \delta(G) - 2$ and $\mathscr{H}_f^\kappa(G) = \delta(G) - 3$. For instance, twisted-cubes, crossed-cubes, móbius cubes and recursive circulant graphs are all super fault-tolerant hamiltonian [2,4–6,8].

A network will have higher fault tolerance if it is super fault-tolerant hamiltonian. For this reason, Chen et al. [3] proposed a systematic framework to recursively construct super fault-tolerant hamiltonian graphs as follows. Let $G_0, G_1, \ldots, G_{n-1}$ be n k-regular super fault-tolerant hamiltonian graphs with the same number of vertices. The *cycle composition network* $H = G(G_0, G_1, \ldots, G_{n-1}; M_{0,1}, M_{1,2}, \ldots, M_{n-2,n-1}, M_{n-1,0})$ is defined to

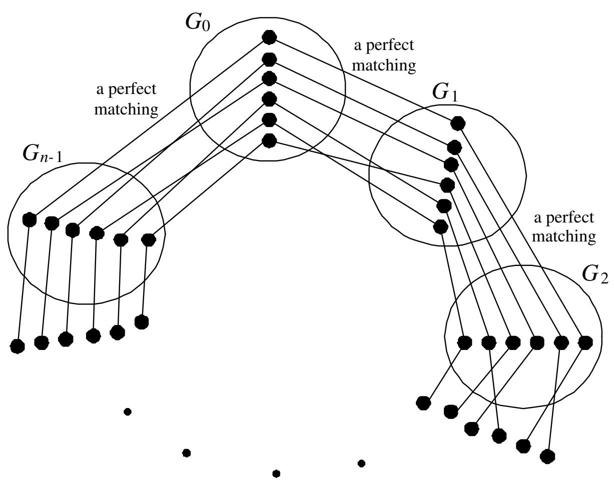

Fig. 1. Illustration for $G_{\langle 0,1,\ldots,n-1,0 \rangle}$.

be the graph with the vertex set $V(H) = \bigcup_{i=0}^{n-1} V(G_i)$ and the edge set $E(H) = \bigcup_{i=0}^{n-1}(E(G_i) \cup M_{i,i+1})$ where $M_{i,j}$ is an arbitrary perfect matching between the vertices of G_i and those of G_j. See Fig. 1. Then Chen et al. [3] proved that $G(G_0, G_1, \ldots, G_{n-1}; M_{0,1}, M_{1,2}, \ldots, M_{n-2,n-1}, M_{n-1,0})$, abbreviated as $G_{\langle 0,1,\ldots,n-1,0\rangle}$, is super fault-tolerant hamiltonian for $n \geqslant 3$ and $k \geqslant 5$.

Theorem 1 [3]. *Assume $n \geqslant 3$ and $k \geqslant 5$. Let $G_0, G_1, \ldots, G_{n-1}$ be n k-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leqslant i \leqslant n-1$, let $M_{i,i+1}$ be a perfect matching between the vertices of G_i and those of G_{i+1}. Then $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is $(k+2)$-regular super fault-tolerant hamiltonian.*

For instance, the recursive circulant graph, which was proposed by Park and Chwa [7], is essentially constructed as a special case in this way, and it is shown to be super fault-tolerant hamiltonian under a certain condition [8]. Similarly, k-ary n-cubes are also recursively constructed using the same framwork [9]. In this paper, we shall extend Theorem 1 by showing that $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is still super fault-tolerant hamiltonian even when $k = 4$. Such an extension is significant because only the remaining case of $k = 3$ needs to be concerned carefully or to be checked by computer while the topological properties of cycle composition networks are investigated.

2. Fault-tolerant hamiltonicity

For the ease of exposition, the notations we used in this paper are described as follows. We denote the graph $G(G_i, G_{i+1}, \ldots, G_j; M_{i,i+1}, M_{i+1,i+2}, \ldots, M_{j-1,j})$ by $G_{\langle i,i+1,\ldots,j\rangle}$. Let u be a vertex of G_i with some i. We use $(u)^-$ to denote the vertex of G_{i-1} such that $((u)^-, u) \in M_{i-1,i}$, and use $(u)^+$ to denote the vertex of G_{i+1} such that $(u, (u)^+) \in M_{i,i+1}$. Hence, we have $u = ((u)^-)^+ = ((u)^+)^-$. Moreover, all additions and subtractions are considered modulo n. In order to prove the main results, we need the following lemmas.

Lemma 1. *Assume $n \geqslant 1$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leqslant i \leqslant n-2$, let $M_{i,i+1}$ be a perfect matching between the vertices of G_i and those of G_{i+1}. Moreover, let $F_i \subseteq V(G_i) \cup E(G_i)$ with $|F_i| \leqslant 1$ for every $0 \leqslant i \leqslant n-1$ and let $X_{i,i+1} \subseteq M_{i,i+1}$ with $|X_{i,i+1}| \leqslant 1$ such that $|F_i| + |F_{i+1}| + |X_{i,i+1}| \leqslant 2$ is satisfied for all $0 \leqslant i \leqslant n-2$. Let u and v be two vertices of $G_0 - F_0$. Then there is a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - ((\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1}))$ joining u to v.*

Proof. For convenience, let $F = (\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1})$. We prove this lemma by induction on n. Obviously, the statement is trivial when $n = 1$. For any $n \geqslant 2$, we suppose that the statement holds for $n-1$. Depending on $|V(G_0)|$, two cases are distinguished.

Case 1: Suppose that $|V(G_0)| = 5$. Thus, G_0 is isomorphic to the complete graph K_5. First assume $|F_0| = 0$. Since $|F_0| + |F_1| + |X_{0,1}| \leqslant 2$, we can choose two vertices x, y of G_0 such that $|\{x, y\} \cap \{u, v\}| \leqslant 1$ and $|F \cap \{(x)^+, (y)^+, (x, (x)^+), (y, (y)^+)\}| = 0$. Accordingly, we can construct a hamiltonian path $P = \langle u, P_1, x, y, P_2, v\rangle$ of G_0, in which P_1 or P_2 may be a path of length zero. On the other hand, assume that $|F_0| = 1$. Since G_0 is 4-regular super fault-tolerant hamiltonian, there is a hamiltonian path P of $G_0 - F_0$ joining u to v. Since $|F_0| + |F_1| + |X_{0,1}| \leqslant 2$ and $|F_0| = 1$, there exists an edge (x, y) on P such that $|F \cap \{(x)^+, (y)^+, (x, (x)^+), (y, (y)^+)\}| = 0$. Accordingly, we write $P = \langle u, P_1, x, y, P_2, v\rangle$, in which P_1 or P_2 may be a path of length zero. By induction hypothesis, there is a hamiltonian path T of $G_{\langle 1,\ldots,n-1\rangle} - ((\bigcup_{i=1}^{n-1} F_i) \cup (\bigcup_{i=1}^{n-2} X_{i,i+1}))$ joining $(x)^+$ to $(y)^+$. Then $\langle u, P_1, x, (x)^+, T, (y)^+, y, P_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1\rangle} - F$ joining u to v. See Fig. 2 for illustration.

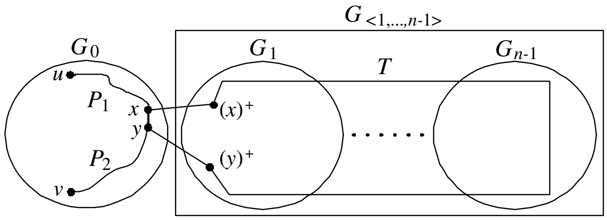

Fig. 2. Illustration for Lemma 1.

Case 2: Suppose that $|V(G_0)| \geqslant 6$. Since G_0 is super fault-tolerant hamiltonian, there is a hamiltonian path P of $G_0 - F_0$ joining u to v. Since $|F_0| + |F_1| + |X_{0,1}| \leqslant 2$, there exists an edge (x, y) on P such that $|F \cap \{(x)^+, (y)^+, (x, (x)^+), (y, (y)^+)\}| = 0$. Accordingly, we write $P = \langle u, P_1, x, y, P_2, v\rangle$, in which P_1 or P_2 may be a path of length zero. By induction hypothesis, there is a hamiltonian path T of $G_{\langle 1,\ldots,n-1\rangle} - ((\bigcup_{i=1}^{n-1} F_i) \cup (\bigcup_{i=1}^{n-2} X_{i,i+1}))$ joining $(x)^+$ to $(y)^+$. Then $\langle u, P_1, x, (x)^+, T, (y)^+, y, P_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1\rangle} - F$ joining u to v. $\square$

Lemma 2. *Assume $n \geqslant 1$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leqslant i \leqslant n-2$, let $M_{i,i+1}$ be a perfect matching between the vertices of G_i and those of G_{i+1}. Moreover, let $F_i \subseteq V(G_i) \cup E(G_i)$ with $|F_i| \leqslant 1$ for every $0 \leqslant i \leqslant n-1$ and let $X_{i,i+1} \subseteq M_{i,i+1}$ with $|X_{i,i+1}| \leqslant 1$ for every $0 \leqslant i \leqslant n-2$ such that $|F_i| + |F_{i+1}| + |X_{i,i+1}| \leqslant 2$ is satisfied for all $0 \leqslant i \leqslant n-2$. Let u be a vertex of $G_0 - F_0$ and v be a vertex of $G_t - F_t$ with $t \geqslant 0$. Then there is a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - ((\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1}))$ joining u to v.*

Proof. For convenience, let $F = (\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1})$. When $t = 0$, the statement follows from Lemma 1. Hence, we suppose $t > 0$ in the following. Since G_t is 4-regular, we have $|V(G_t)| \geqslant 5$. Moreover, since $|F_{t-1}| + |F_t| + |X_{t-1,t}| \leqslant 2$, we can choose a vertex w of $G_t - (F_t \cup \{v\})$ such that $|F \cap \{w, (w)^-, (w, (w)^-)\}| = 0$ and $(w)^- \neq u$.

Let $y_0 = u$ and $x_{t-1} = (w)^-$. Since every G_i, $0 \leqslant i \leqslant t-1$, is 4-regular and $|F_i| + |F_{i+1}| + |X_{i,i+1}| \leqslant 2$, we sequentially choose a vertex x_i of $G_i - F_i$ and denote $(x_i)^+$ by y_{i+1} such that $x_i \neq y_i$ and $|F \cap \{x_i, y_{i+1}, (x_i, y_{i+1})\}| = 0$ from $i = 0$ to $i = t-3$ while $t \geqslant 3$. Next, we choose a vertex x_{t-2} of $G_{t-2} - (F_{t-2} \cup \{y_{t-2}\})$ and denote $(x_{t-2})^+$ by y_{t-1} such that $|F \cap \{x_{t-2}, y_{t-1}, (x_{t-2}, y_{t-1})\}| = 0$ and $y_{t-1} \neq x_{t-1}$ while $t \geqslant 2$. Since every G_i, $0 \leqslant i \leqslant t-1$, is super fault-tolerant hamiltonian, there is a hamiltonian path P_i of $G_i - F_i$ joining y_i to x_i. By Lemma 1, there is a hamiltonian path T of $G_{\langle t,\ldots,n-1\rangle} - ((\bigcup_{i=t}^{n-1} F_i) \cup (\bigcup_{i=t}^{n-2} X_{i,i+1}))$ joining w to v. Then $\langle u = y_0, P_0, x_0, (x_0)^+ = y_1, \ldots, x_{t-2}, (x_{t-2})^+ = y_{t-1}, P_{t-1}, x_{t-1} = (w)^-, w, T, v\rangle$ is a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - F$ joining u to v. See Fig. 3 for illustration. $\square$

Using Lemma 2, we prove the following result.

Theorem 2. *Assume $n \geqslant 3$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leqslant i \leqslant n-1$, let $M_{i,i+1}$ be a perfect matching between the vertices of G_i and those of G_{i+1}. Then $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is optimal fault-tolerant hamiltonian.*

Proof. Obviously, $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is 6-regular. Thus, we are going to show that it is 4-fault-tolerant hamiltonian. Let F be a faulty set of $G_{\langle 0,1,\ldots,n-1,0\rangle}$ with $|F| \leqslant 4$. For convenience, let $F_i = F \cap (V(G_i) \cup E(G_i))$ for $0 \leqslant i \leqslant n-1$. Without loss of generality, we assume that $|F_0| \geqslant |F_i|$ for all $1 \leqslant i \leqslant n-1$. Depending on $|F_0|$, five cases are distinguished.

Case 1: Suppose that $|F_0| = 4$. Let $F_0 = \{f_1, f_2, f_3, f_4\}$. Since G_0 is 2-fault-tolerant hamiltonian, there is a hamiltonian cycle C in $G_0 - \{f_3, f_4\}$.

Subcase 1.1: Suppose that both f_1 and f_2 are on C but they are not adjacent. Thus, we can write $C = \langle x_1, f_1, y_1, H_1, x_2, f_2, y_2, H_2, x_1\rangle$, in which H_1 or H_2 may be a path of length zero. By Lemma 2, there is a hamiltonian path $S_1[(x_1)^-, (y_1)^-]$ in G_{n-1} and there is a hamiltonian path $S_2[(x_2)^+, (y_2)^+]$ in $G_{\langle 1,\ldots,n-2\rangle}$. Then

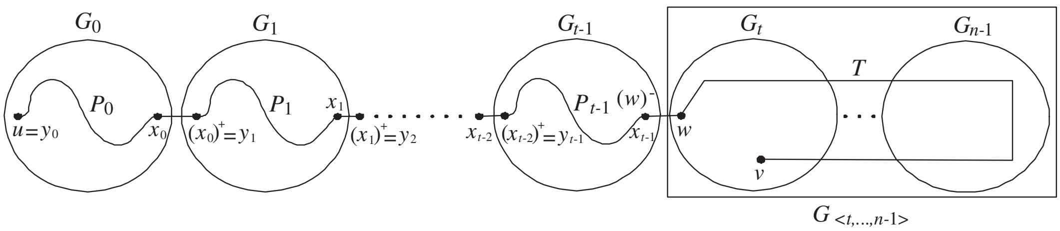

Fig. 3. Illustration for Lemma 2.

$\langle x_1, (x_1)^-, S_1, (y_1)^-, y_1, H_1, x_2, (x_2)^+, S_2, (y_2)^+, y_2, H_2, x_1\rangle$ is a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. See Fig. 4a for illustration.

Subcase 1.2: Suppose that both f_1 and f_2 are on C and they are adjacent. Thus, we write $C = \langle x, R, y, f_1, f_2, x\rangle$. By Lemma 2, there is a hamiltonian path H of $G_{\langle 1,\ldots,n-1\rangle}$ joining $(y)^+$ to $(x)^+$. Then $\langle x, R, y, (y)^+, H, (x)^+, x\rangle$ is a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. See Fig. 4b for illustration.

Subcase 1.3: Suppose that either f_1 or f_2 is on C. Without loss of generality, we assume that f_1 is on C. Thus, we write C as $\langle x, R, y, f_1, x\rangle$. Then a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ can be formed in the same way as that used in Subcase 1.2.

Subcase 1.4: Suppose that neither f_1 nor f_2 is on C. Thus, we write C as $\langle x, R, y, x\rangle$ with any edge $(x, y) \in E(C)$. Then a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ can be formed in the same way as that used in Subcase 1.2.

Case 2: Suppose that $|F_0| = 3$. Let $F_0 = \{f_1, f_2, f_3\}$. Since G_0 is 2-fault-tolerant hamiltonian, there is a hamiltonian cycle C in $G_0 - \{f_2, f_3\}$. Thus, we have either $f_1 \notin V(C) \cup E(C)$ or $f_1 \in V(C) \cup E(C)$. Accordingly, we write $C = \langle x, R, y, x\rangle$ by picking any edge (x, y) on C if $f_1 \notin V(C) \cup E(C)$; we write $C = \langle x, R, y, f_1, x\rangle$ if f_1 is on C. Let $F' = F - F_0$. Since $|F| \leqslant 4$ and $|F_0| = 3$, $|F'| \leqslant 1$. Moreover, one can see either $|\{(x)^+, (y)^+, (x, (x)^+), (y, (y)^+)\} \cap F| = 0$ or $|\{(x)^-, (y)^-, (x, (x)^-), (y, (y)^-)\} \cap F| = 0$. With symmetry, we assume that $|\{(x)^+, (y)^+, (x, (x)^+), (y, (y)^+)\} \cap F| = 0$. By Lemma 2, there is a hamiltonian path H of $G_{\langle 1,\ldots,n-1\rangle} - F'$ joining $(y)^+$ to $(x)^+$. Then $\langle x, R, y, (y)^+, H, (x)^+, x\rangle$ is a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$.

Case 3: Suppose that $|F_0| = 2$ and $|F_i| = 2$ with any $1 \leqslant i \leqslant n-1$. Since both G_0 and G_i are 2-fault-tolerant hamiltonian, there is a hamiltonian cycle C in $G_0 - F_0$ and there is a hamiltonian cycle T in $G_i - F_i$. Since every G_j, $0 \leqslant j \leqslant n-1$, is 4-regular, $|V(G_j)| \geqslant 5$.

Subcase 3.1: Suppose that $i \in \{1, n-1\}$. With symmetry, we assume that $i = 1$. Apparently, there is a vertex u in $G_0 - F_0$ such that $(u)^+$ is in $G_1 - F_1$. Without loss of generality, we write $C = \langle u, R_1, x, u\rangle$ and $T = \langle (u)^+, y, R_2, (u)^+\rangle$ so that $(y)^+$ is different from $(x)^-$. By Lemma 2, there is a hamiltonian path H of $G_{\langle 2,\ldots,n-1\rangle} - F$ joining $(x)^-$ to $(y)^+$. Then $\langle u, R_1, x, (x)^-, H, (y)^+, y, R_2, (u)^+, u\rangle$ is a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. See Fig. 5a.

Subcase 3.2: Suppose that $i \notin \{1, n-1\}$. Obviously, there is a vertex u in $G_0 - F_0$ and a vertex v in $G_i - F_i$ such that $(u)^+ \neq (v)^-$. Without loss of generality, we write $C = \langle u, x, R_1, u\rangle$ and $T = \langle v, R_2, y, v\rangle$ so that $(y)^+$ is different from $(x)^-$. By Lemma 2, there is a hamiltonian path P_1 of $G_{\langle 1,\ldots,i-1\rangle}$ joining $(u)^+$ to $(v)^-$. Similarly, there is a hamiltonian path P_2 of $G_{\langle i+1,\ldots,n-1\rangle}$ joining $(y)^+$ to $(x)^-$. Then $\langle u, (u)^+, P_1, (v)^-, v, R_2, y, (y)^+, P_2, (x)^-, x, R_1, u\rangle$ is a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. See Fig. 5b for illustration.

Case 4: Suppose that $|F_0| = 2$ and $|F_i| \leqslant 1$ for every $1 \leqslant i \leqslant n-1$. Since G_0 is 2-fault-tolerant hamiltonian, there is a hamiltonian cycle C in $G_0 - F_0$. Since G_0 is 4-regular, we have $|V(G_0 - F_0)| \geqslant 3$. For convenience, let $m = |V(G_0 - F_0)|$. Accordingly, we write $C = \langle u_0, u_1, u_2, \ldots, u_{m-1}, u_0\rangle$. Without loss of generality, we assume that $|F \cap \{(u_0)^+, (u_1)^-, (u_0, (u_0)^+), (u_1, (u_1)^-)\}| = 0$. Let $F' = F - F_0$. By Lemma 2, there is a hamiltonian path T of $G_{\langle 1,\ldots,n-1\rangle} - F'$ joining $(u_0)^+$ to $(u_1)^-$. Then $\langle u_0, (u_0)^+, T, (u_1)^-, u_1, \ldots, u_{m-1}, u_0\rangle$ is a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. See Fig. 6a for illustration.

Case 5: Suppose that $|F_0| \leqslant 1$. That is, $|F_i| \leqslant 1$ for all $0 \leqslant i \leqslant n-1$. For convenience, let $X_{i,i+1} = F \cap M_{i,i+1}$ for $0 \leqslant i \leqslant n-1$. Suppose that there exists an integer t of $\{0, 1, \ldots, n-1\}$ such that $|F_t| + |F_{t+1}| + |X_{t,t+1}| \geqslant 3$. Without loss of generality, t can be assumed to be $n-1$. Otherwise, t is fixed to be $n-1$. Accordingly, we have $|F_i| + |F_{i+1}| + |X_{i,i+1}| \leqslant 2$ for $0 \leqslant i \leqslant n-2$. Since $|F_{n-1}| + |F_0| + |X_{n-1,0}| \leqslant 4$, we can choose a vertex x of $G_{n-1} - F_{n-1}$ such that $|F \cap \{(x)^+, (x, (x)^+)\}| = 0$. Let $F' = F - X_{n-1,0}$. By Lemma 2, there

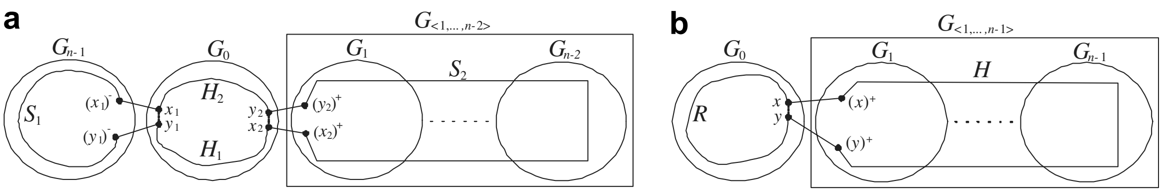

Fig. 4. Illustration for case 1 of Theorem 2.

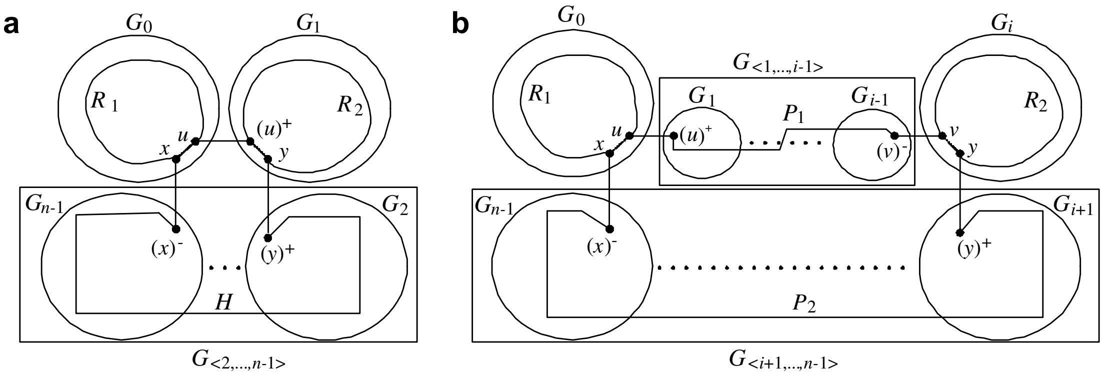

Fig. 5. Illustration for case 3 of Theorem 2.

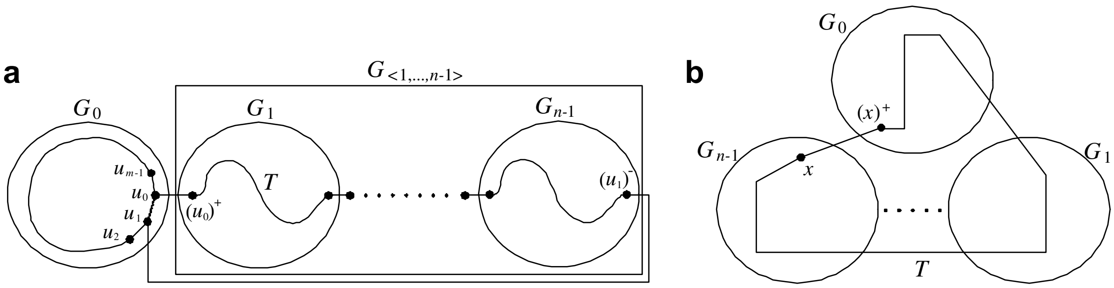

Fig. 6. Illustration for case 4 and case 5 of Theorem 2.

is a hamiltonian path T of $G_{\langle 0,1,\ldots,n-1\rangle} - F'$ joining x to $(x)^+$. Then $\langle x, T, (x)^+, x\rangle$ is a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. See Fig. 6b for illustration. □

3. Fault-tolerant hamiltonian connectedness

In this section, we are going to show that the cycle composition network is optimal fault-tolerant hamiltonian connected. This result is divided into three propositions.

Proposition 1. *Assume $n \geqslant 1$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leqslant i \leqslant n-1$, let $M_{i,i+1}$ be a perfect matching between the vertices of G_i and those of G_{i+1}. Let F be a subset of $V(G_0) \cup E(G_0)$ with $|F| = 3$. Then $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ is hamiltonian connected.*

Proof. Let $F = \{f_1, f_2, f_3\}$. Since G_0 is 2-fault-tolerant hamiltonian, there is a hamiltonian cycle C in $G_0 - \{f_2, f_3\}$. Since G_0 is 4-regular, $|V(C)| \geqslant 3$. Let u and v be two vertices of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. Then we have to construct a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. The following cases are distinguished.

Case 1: Suppose that u and v are in $G_0 - F$. Since G_0 is 1-fault-tolerant hamiltonian connected, there is a hamiltonian path H of $G_0 - \{f_3\}$ joining u to v. Suppose that f_1 and f_2 are exclusive from H. Thus, we write $H = \langle u, P_1, x, y, P_2, v\rangle$ with any edge $(x, y) \in E(H)$. Suppose that either f_1 or f_2 is exclusive from H. Without loss of generality, we assume that f_2 is exclusive from H. Thus, we may write $H = \langle u, P_1, x, f_1, y, P_2, v\rangle$. Suppose that both f_1 and f_2 are on H and they are adjacent. Thus, we may write $H = \langle u, P_1, x, f_1, f_2, y, P_2, v\rangle$. By Lemma 2, there is a hamiltonian path T of $G_{\langle 1,\ldots,n-1\rangle}$ joining $(x)^+$ to $(y)^+$. Then $\langle u, P_1, x, (x)^+, T, (y)^+, y, P_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Fig. 7a for illustration.

Suppose that both f_1 and f_2 are on H but they are not adjacent. Thus, we may write $H = \langle u, A_1, x_1, f_1, y_1, A_2, x_2, f_2, y_2, A_3, v\rangle$. Using Lemma 2, we can find a hamiltonian path D_1 of $G_{\langle 1,\ldots,n-2\rangle}$ joining $(x_1)^+$ to $(y_1)^+$. Similarly, there is a hamiltonian path D_2 of G_{n-1} joining $(x_2)^-$ to $(y_2)^-$. Hence, $\langle u, A_1, x_1,$

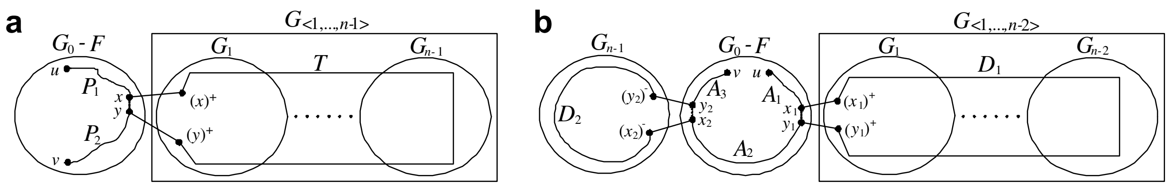

Fig. 7. Illustration for case 1 of Proposition 1.

$(x_1)^+, D_1, (y_1)^+, y_1, A_2, x_2, (x_2)^-, D_2, (y_2)^-, y_2, A_3, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Fig. 7b for illustration.

Case 2: Suppose that u and v are in G_i for some $1 \leqslant i \leqslant n-1$. With symmetry, we assume that $i \neq n-1$. Suppose that f_1 is on the hamiltonian cycle C of $G_0 - \{f_2, f_3\}$. Since $|V(C)| \geqslant 3$, we write $C = \langle x, P, y, f_1, x\rangle$. Otherwise, we write $C = \langle x, P, y, x\rangle$ with any edge $(x, y) \in E(C)$.

Subcase 2.1: Suppose that $(x)^+ \neq u$ and $(x)^+ \neq v$. Thus, either $(y)^- \neq (u)^+$ or $(y)^- \neq (v)^+$. Without loss of generality, we assume that $(y)^- \neq (v)^+$. By Lemma 2, there is a hamiltonian path T_1 of $G_{\langle 1,\ldots,i\rangle} - \{v\}$ joining u to $(x)^+$. Similarly, there is a hamiltonian path T_2 of $G_{\langle i+1,\ldots,n-1\rangle}$ joining $(y)^-$ to $(v)^+$. Then $\langle u, T_1, (x)^+, x, P, y, (y)^-, T_2, (v)^+, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Fig. 8a for illustration.

Subcase 2.2: Suppose that $(x)^+ = u$ or $(x)^+ = v$. Without loss of generality, we assume that $(x)^+ = u$. By Lemma 2, there is a hamiltonian path T of $G_{\langle 1,\ldots,n-1\rangle} - \{u\}$ joining $(y)^-$ to v. Then $\langle u = (x)^+, x, P, y, (y)^-, T, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Fig. 8b for illustration.

Case 3: Suppose that u is in $G_0 - F$ and v is in G_i with any $i > 0$. Since $i \neq 1$ or $i \neq n-1$, we may assume that $i \neq 1$. Since $|V(C)| \geqslant 3$, we write $C = \langle u, T, z, u\rangle$ with $z \neq u$. Moreover, T can be written as $\langle u, P_1, x, f_1, y, P_2, z\rangle$ if f_1 is on T, or T can be written as $\langle u, P_1, x, y, P_2, z\rangle$ otherwise.

Subcase 3.1: Suppose that $(z)^- \neq v$. Since G_1 is 1-fault-tolerant hamiltonian connected, there is a hamiltonian path H of G_1 joining $(x)^+$ to $(y)^+$. By Lemma 2, there is a hamiltonian path R of $G_{\langle 2,\ldots,n-1\rangle}$ joining $(z)^-$ to v. Then $\langle u, P_1, x, (x)^+, H, (y)^+, y, P_2, z, (z)^-, R, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Fig. 9a.

Subcase 3.2: Suppose that $(z)^- = v$. By Lemma 2, there is a hamiltonian path H of $G_{\langle 1,\ldots,n-1\rangle} - \{v\}$ joining $(x)^+$ to $(y)^+$. Then $\langle u, P_1, x, (x)^+, H, (y)^+, y, P_2, z, (z)^- = v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Fig. 9b for illustration.

Case 4: Suppose that u is in G_i and v is in G_j for any $1 \leqslant i < j \leqslant n-1$. Suppose that f_1 is on C. Then we write $C = \langle x, P, y, f_1, x\rangle$. Otherwise, we write $C = \langle x, P, y, x\rangle$ with any $(x, y) \in E(C)$. Since $(x)^+ \neq u$ or $(y)^+ \neq u$, we may assume that $(x)^+ \neq u$.

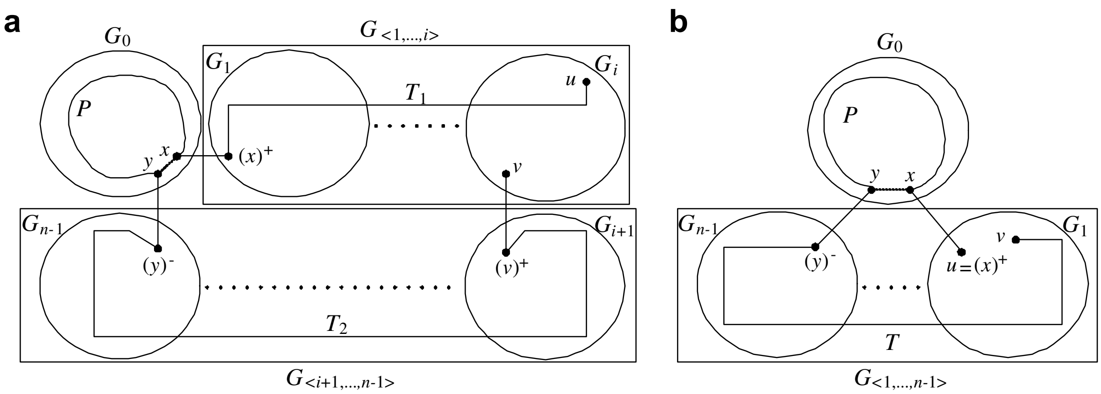

Fig. 8. Illustration for case 2 of Proposition 1.

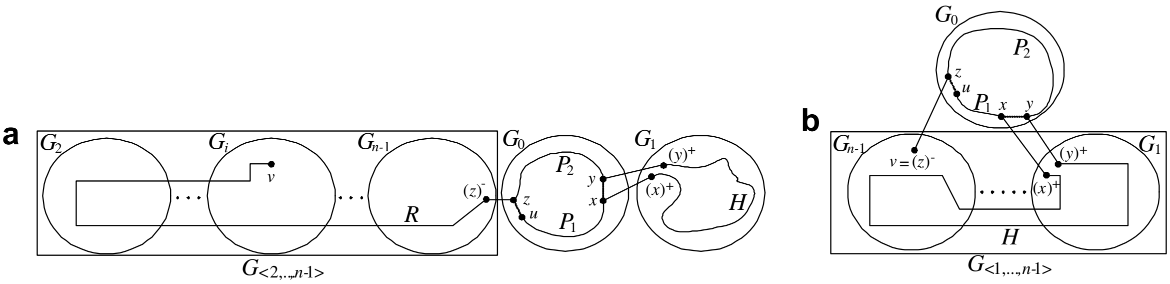

Fig. 9. Illustration for case 3 of Proposition 1.

Subcase 4.1: Suppose that $(y)^- \neq v$. By Lemma 2, there is a hamiltonian path T_1 of $G_{\langle 1,\ldots,i\rangle}$ joining u to $(x)^+$. Similarly, there is a hamiltonian path T_2 of $G_{\langle i+1,\ldots,n-1\rangle}$ joining $(y)^-$ to v. Then $\langle u, T_1, (x)^+, x, P, y, (y)^-, T_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Fig. 10a for illustration.

Subcase 4.2: Suppose that $(y)^- = v$. By Lemma 2, there is a hamiltonian path H of $G_{\langle 1,\ldots,n-1\rangle} - \{v\}$ joining u to $(x)^+$. Then $\langle u, H, (x)^+, x, P, y, (y)^- = v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Fig. 10(b) for illustration. □

Proposition 2. *Assume $n \geqslant 1$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leqslant i \leqslant n-1$, let $M_{i,i+1}$ be a perfect matching between the vertices of G_i and those of G_{i+1}. Let F be a faulty set of $G_{\langle 0,1,\ldots,n-1,0\rangle}$ such that $|F| = 3$ and $|F \cap (V(G_0) \cup E(G_0))| = 2$. Then $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ is hamiltonian connected.*

Proof. For convenience, let $F_i = F \cap (V(G_i) \cup E(G_i))$ and $X_{i,i+1} = F \cap M_{i,i+1}$ for every $0 \leqslant i \leqslant n-1$. Moreover, let $F' = F - F_0$. Obviously, we have $|F_0| = 2$, $|F'| = 1$, and $|F_i| \leqslant 1$ for all $1 \leqslant i \leqslant n-1$. Since G_0 is 4-regular, $|V(G_0)| \geqslant 5$ and $|V(G_0 - F_0)| \geqslant 3$. Moreover, since G_0 is 2-fault-tolerant hamiltonian, there is a hamiltonian cycle C in $G_0 - F_0$. Let u and v be any two vertices of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. Then we have to construct a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v.

Case 1: Suppose that u and v are in $G_0 - F_0$. Since $|V(G_0 - F_0)| \geqslant 3$, we may write $C = \langle u, P, y, u\rangle$ in which $y \neq u$. Moreover, we may write $P = \langle u, H_1, x, v, H_2, y\rangle$. Note that the length of H_1 becomes zero if $u = x$. Since $|F'| = 1$, we have $|X_{0,1}| + |F_1| = 0$ or $|X_{n-1,0}| + |F_{n-1}| = 0$. With symmetry, we assume that $|X_{0,1}| + |F_1| = 0$. By Lemma 2, there is a hamiltonian path T of $G_{\langle 1,\ldots,n-1\rangle} - F'$ joining $(x)^+$ to $(y)^+$. Then $\langle u, H_1, x, (x)^+, T, (y)^+, y, H_2^{-1}, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Fig. 11 for illustration.

Case 2: Suppose that u and v are in either $G_1 - F_1$ or $G_{n-1} - F_{n-1}$. With symmetry, we assume that u and v are in $G_1 - F_1$.

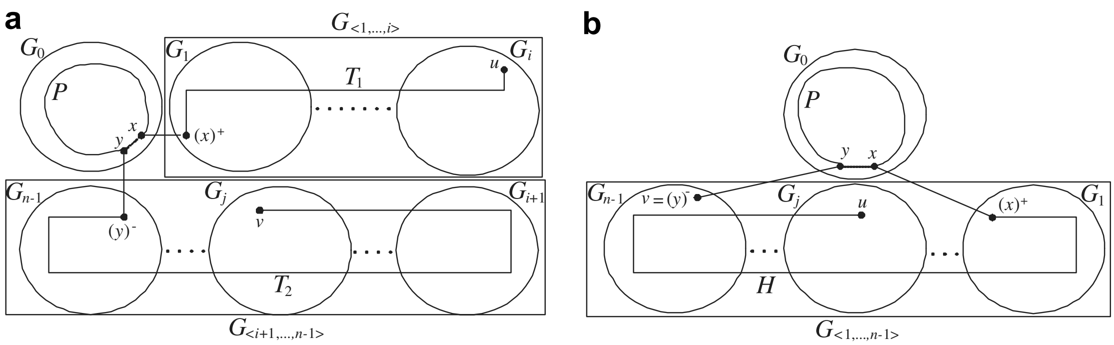

Fig. 10. Illustration for case 4 of Proposition 1.

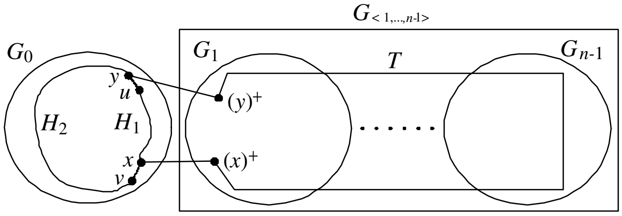

Fig. 11. Illustration for case 1 of Proposition 2.

Subcase 2.1: Suppose that $|X_{0,1}| + |F_1| = 1$. Since $|V(G_0 - F_0)| \geqslant 3$, we choose a vertex x of the hamiltonian cycle C such that $|F' \cap \{(x)^+, (x,(x)^+)\}| = 0$. Hence, C can be written as $C = \langle y, x, z, P, y \rangle$. Since $(x)^+ \neq u$ or $(x)^+ \neq v$, we assume that $(x)^+ \neq v$. Since G_1 is 1-fault-tolerant hamiltonian connected, there is a hamiltonian path $Q[u,v]$ of $G_1 - F_1$. Since $(x)^+ \neq v$, we write $Q = \langle u, T_1, (x)^+, w, T_2, v \rangle$. Note that T_1 or T_2 may be a path of length zero. Moreover, we select a vertex from $\{y, z\}$, say y, such that $(y)^- \neq (w)^+$. By Lemma 2, there is a hamiltonian path H of $G_{\langle 2,\ldots,n-1 \rangle}$ joining $(y)^-$ to $(w)^+$. Then $\langle u, T_1, (x)^+, x, z, P, y, (y)^-, H, (w)^+, w, T_2, v \rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0 \rangle} - F$ joining u to v. See Fig. 12a for illustration.

Subcase 2.2: Suppose that $|X_{0,1}| + |F_1| = 0$. Thus, we can choose a vertex x of C such that $|F' \cap \{(x)^+, (x,(x)^+)\}| = 0$ and $(x)^+ \notin \{u, v\}$. Hence, the hamiltonian cycle C of $G_0 - F_0$ can be written as $C = \langle y, x, z, P, y \rangle$.

Subcase 2.2.1: Suppose that $|\{(y)^+, (z)^+\} \cap \{u, v\}| \geqslant 1$. Without loss of generality, we assume that $(z)^+ = u$. By Lemma 2, there is a hamiltonian path T of $G_{\langle 1,\ldots,n-1 \rangle} - (F' \cup \{u\})$ joining $(x)^+$ to v. Then $\langle u = (z)^+, z, P, y, x, (x)^+, T, v \rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0 \rangle} - F$ joining u to v. See Fig. 12b for illustration.

Subcase 2.2.2: Suppose that $|\{(y)^+, (z)^+\} \cap \{u, v\}| = 0$. Since $|F' \cap \{(y)^-, (y,(y)^-)\}| = 0$ or $|F' \cap \{(z)^-, (z,(z)^-)\}| = 0$, we assume that $|F' \cap \{(y)^-, (y,(y)^-)\}| = 0$. Since G_1 is 1-fault-tolerant hamiltonian connected, there is a hamiltonian path Q of $G_1 - \{((x)^+, ((y)^-)^-)\}$. Since $(x)^+ \notin \{u, v\}$, Q can be represented by $\langle u, T_1, w_1, (x)^+, w_2, T_2, v \rangle$. Note that T_1 or T_2 may be a path of length zero. Accordingly, we have that $|F' \cap \{(w_1)^+, (w_1,(w_1)^+)\}| = 0$ or $|F' \cap \{(w_2)^+, (w_2,(w_2)^+)\}| = 0$. Without loss of generality, we assume that $|F' \cap \{(w_2)^+, (w_2,(w_2)^+)\}| = 0$. By Lemma 2, there is a hamiltonian path H of $G_{\langle 2,\ldots,n-1 \rangle} - F'$ joining $(y)^-$ to $(w_2)^+$. Then $\langle u, T_1, w_1, (x)^+, x, z, P, y, (y)^-, H, (w_2)^+, w_2, T_2, v \rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0 \rangle} - F$ joining u to v. See Fig. 12c.

Case 3: Suppose that u and v are in $G_i - F_i$ with any $1 < i < n-1$. Without loss of generality, we assume that $\sum_{j=1}^{i-1} |F_j| + \sum_{j=0}^{i-1} |X_{j,j+1}| = 0$. Since $|V(G_0 - F_0)| \geqslant 3$, we first choose a vertex x of C such that $|F' \cap \{(x)^-, (x,(x)^-)\}| = 0$. Thus, we can write $C = \langle z, x, y, P, z \rangle$. Next, we choose a vertex t of $G_i - (F_i \cup \{u\})$ such that $|F' \cap \{(t)^+, (t,(t)^+)\}| = 0$ and $(t)^+ \neq (x)^-$. Since G_i is 1-fault-tolerant hamiltonian connected, there is a hamiltonian path H in $G_i - F_i$ joining u to t. Then H can be represented by $\langle u, R_1, w, v, R_2, t \rangle$, in which R_1 or R_2 may be a path of length zero. Since $(y)^+ \neq (w)^-$ or $(z)^+ \neq (w)^-$, we assume that $(y)^+ \neq (w)^-$. By Lemma 2, there is a hamiltonian path T_1 of $G_{\langle 0,\ldots,i-1 \rangle} - F'$ joining $(w)^-$ to $(y)^+$. Similarly, there is a hamiltonian path T_2 of $G_{\langle i+1,\ldots,n-1 \rangle} - F'$ joining $(x)^-$ to $(t)^+$. As a result, $\langle u, R_1, w, (w)^-, T_1, (y)^+, y, P, z, x, (x)^-, T_2, (t)^+, t, R_2^{-1}, v \rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0 \rangle} - F$ joining u to v. See Fig. 13a for illustration.

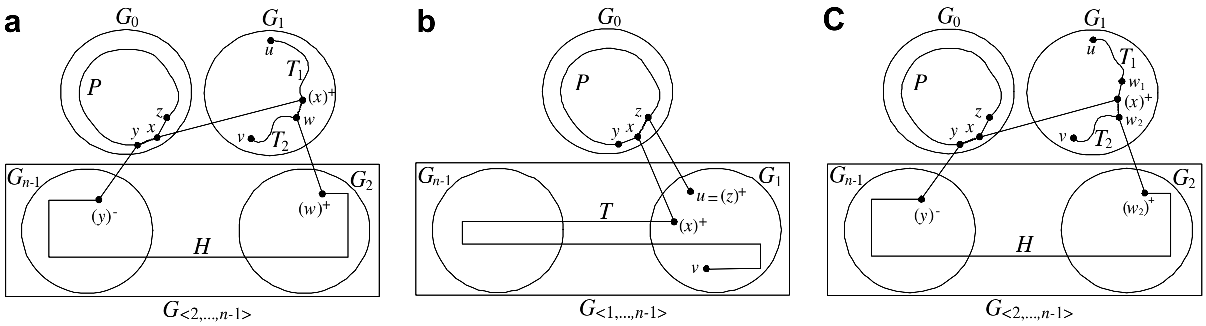

Fig. 12. Illustration for case 2 of Proposition 2.

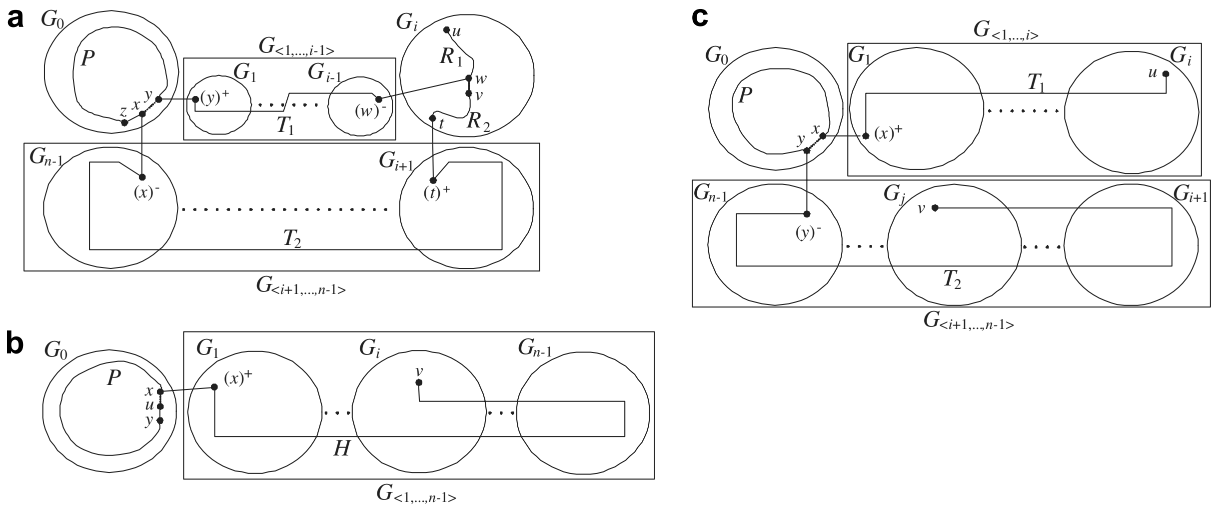

Fig. 13. Illustration for case 3, case 4 and case 5 of Proposition 2.

Case 4: Suppose that u is in $G_0 - F_0$ and v is in $G_i - F_i$ with any $i > 0$. Since $|V(G_0 - F_0)| \geqslant 3$, we can write $C = \langle x, u, y, P, x\rangle$. Since $|F'| = 1$, we have $|X_{0,1}| + |F_1| = 0$ or $|X_{n-1,0}| + |F_{n-1}| = 0$. Without loss of generality, we assume $|X_{0,1}| + |F_1| = 0$. Hence, we have $(x)^+ \neq v$ or $(y)^+ \neq v$. Without loss of generality, we assume $(x)^+ \neq v$. By Lemma 2, there is a hamiltonian path H of $G_{\langle 1,\dots,n-1\rangle} - F'$ joining $(x)^+$ to v. Then $\langle u, y, P, x, (x)^+, H, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\dots,n-1,0\rangle} - F$ joining u to v. See Fig. 13b for illustration.

Case 5: Suppose that u is in $G_i - F_i$ and v is in $G_j - F_j$ for any $1 \leqslant i < j \leqslant n-1$. Since $|F'| = 1$, we have $|X_{0,1}| + |F_1| = 0$ or $|X_{n-1,0}| + |F_{n-1}| = 0$. Without loss of generality, we assume $|X_{n-1,0}| + |F_{n-1}| = 0$. Since $|V(G_0 - F_0)| \geqslant 3$, we can choose a vertex x of C such that $(x)^+ \neq u$ and $|F' \cap \{(x)^+, (x, (x)^+)\}| = 0$. Moreover, at least one neighbor of x on C, namely y, satisfies $(y)^- \neq v$. Accordingly, we can write $C = \langle\, x, P, y, x\rangle$. By Lemma 2, there is a hamiltonian path T_1 of $G_{\langle 1,\dots,i\rangle} - F'$ joining u to $(x)^+$. Similarly, there is a hamiltonian path T_2 of $G_{\langle i+1,\dots,n-1\rangle} - F'$ joining $(y)^-$ to v. Then $\langle u, T_1, (x)^+, x, P, y, (y)^-, T_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\dots,n-1,0\rangle} - F$ joining u to v. See Fig. 13c for illustration. $\square$

Lemma 3. *Assume $n \geqslant 3$. Let $G_0, G_1, \dots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leqslant i \leqslant n-2$, let $M_{i,i+1}$ be a perfect matching between the vertices of G_i and those of G_{i+1}. Moreover, let $F_i \subseteq V(G_i) \cup E(G_i)$ with $|F_i| \leqslant 1$ for every $0 \leqslant i \leqslant n-1$ and let $X_{i,i+1} \subseteq M_{i,i+1}$ with $|X_{i,i+1}| \leqslant 1$ for every $0 \leqslant i \leqslant n-2$ such that $|F_i| + |F_{i+1}| + |F_{i+2}| + |X_{i,i+1}| + |X_{i+1,i+2}| \leqslant 2$ is satisfied for all $0 \leqslant i \leqslant n-3$. Let u and v be two vertices of $G_t - F_t$ with $0 < t < n-1$. Then there is a hamiltonian path of $G_{\langle 0,\dots,n-1\rangle} - ((\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1}))$ joining u to v.*

Proof. For convenience, let $F = (\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1})$. Since G_t is 4-regular super fault-tolerant hamiltonian, there is a hamiltonian path P of $G_t - F_t$ joining u to v. Depending on $|F_t|$, we distinguish the following two cases.

Case 1: Suppose that $|F_t| = 1$. Thus, one can see that $|V(G_t - F_t)| \geqslant 4$. Let $w_1 = u$. Thus, we write P as $\langle u = w_1,\ w_2,\ w_3,\ w_4,\ R,\ v\rangle$. Since $|F_t| = 1$, one can see that $|F_{t-1}| + |F_{t+1}| + |X_{t-1,t}| + |X_{t,t+1}| \leqslant 1$. Hence, we select a vertex w_i from $\{w_2, w_3\}$ such that $|F \cap \{(w_i)^-, (w_i)^+, (w_i, (w_i)^-), (w_i, (w_i)^+)\}| = 0$. Accordingly, one can see that either $|F \cap \{(w_{i-1})^+, (w_{i+1})^-, (w_{i-1}, (w_{i-1})^+), (w_{i+1}, (w_{i+1})^-)\}| = 0$ or $|F \cap \{(w_{i-1})^-, (w_{i+1})^+, (w_{i-1}, (w_{i-1})^-), (w_{i+1}, (w_{i+1})^+)\}| = 0$. Without loss of generality, we assume $|F \cap \{(w_{i-1})^+, (w_{i+1})^-, (w_{i-1}, (w_{i-1})^+), (w_{i+1}, (w_{i+1})^-)\}| = 0$. Hence, we can further write P as $\langle\, u = w_1, P_1, w_{i-1}, w_i, w_{i+1}, P_2, v\rangle$. By Lemma 2, there is a hamiltonian path T of $G_{\langle 0,\dots,t-1\rangle} - ((\bigcup_{i=0}^{t-1} F_i) \cup (\bigcup_{i=0}^{t-2} X_{i,i+1}))$ joining $(w_i)^-$ to $(w_{i+1})^-$. Similarly, there is a hamiltonian path Q of $G_{\langle t+1,\dots,n-1\rangle} - ((\bigcup_{i=t+1}^{n-1} F_i) \cup (\bigcup_{i=t+1}^{n-2} X_{i,i+1}))$ joining $(w_{i-1})^+$ to $(w_i)^+$. Then $\langle u = w_1, P_1, w_{i-1}, (w_{i-1})^+, Q, (w_i)^+, w_i, (w_i)^-, T, (w_{i+1})^-, w_{i+1}, P_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,\dots,n-1\rangle} - F$ joining u to v.

Case 2: Suppose that $|F_t| = 0$. First, assume that $|V(G_t)| \geqslant 6$. Hence, we can select two adjacent edges (x, y), $(y, z) \in E(P)$ such that $|F \cap \{(x)^+, (y)^+, (y)^-, (z)^-, (x, (x)^+), (y, (y)^+), (y, (y)^-), (z, (z)^-)\}| = 0$ or $|F \cap \{(x)^-, (x, (x)^-), (y)^-, (y, (y)^-), (y, (y)^+), (z, (z)^+), (y)^+, (z)^+\}| = 0$. Without loss of generality, we assume that $|F \cap \{(x)^+, (y)^+, (y)^-, (z)^-, (x, (x)^+), (y, (y)^+), (y, (y)^-), (z, (z)^-)\}| = 0$. Accordingly, P can be written as $\langle u, P_1, x, y, z, P_2, v\rangle$, in which P_1 or P_2 may be a path of length zero. By Lemma 2, there is a hamiltonian path T of $G_{\langle 0,\ldots,t-1\rangle} - ((\bigcup_{i=0}^{t-1} F_i) \cup (\bigcup_{i=0}^{t-2} X_{i,i+1}))$ joining $(y)^-$ to $(z)^-$. Similarly, there is a hamiltonian path Q of $G_{\langle t+1,\ldots,n-1\rangle} - ((\bigcup_{i=t+1}^{n-1} F_i) \cup (\bigcup_{i=t+1}^{n-2} X_{i,i+1}))$ joining $(x)^+$ to $(y)^+$. Then $\langle u, P_1, x, (x)^+, Q, (y)^+, y, (y)^-, T, (z)^-, z, P_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - F$ joining u to v.

Next, assume that $|V(G_t)| = 5$. Thus, G_t is isomorphic to the complete graph K_5. Let $V(G_t) = \{u = w_1, w_2, w_3, w_4, w_5 = v\}$. First of all, we choose a vertex from $\{w_2, w_3, w_4\}$, say w_2, such that $|F \cap \{(w_2)^-, (w_2)^+, (w_2, (w_2)^-), (w_2, (w_2)^+)\}| = 0$. Secondly, we choose two vertices x, y from $\{w_3, w_4, w_5\}$ such that $|F \cap \{(x)^+, (x, (x)^+), (y)^-, (y, (y)^-)\}| = 0$. Accordingly, a hamiltonian path of G_t can be written as $\langle u = w_1, P_1, x, w_2, y, P_2, w_5 = v\rangle$. Then a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - F$ joining u to v can be formed in a way similar to that mentioned above. □

Lemma 4. *Assume $n \geqslant 3$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leqslant i \leqslant n-2$, let $M_{i,i+1}$ be a perfect matching between the vertices of G_i and those of G_{i+1}. Moreover, let $F_i \subseteq V(G_i) \cup E(G_i)$ with $|F_i| \leqslant 1$ for every $0 \leqslant i \leqslant n-1$ and let $X_{i,i+1} \subseteq M_{i,i+1}$ with $|X_{i,i+1}| \leqslant 1$ for every $0 \leqslant i \leqslant n-2$ such that $|F_i| + |F_{i+1}| + |F_{i+2}| + |X_{i,i+1}| + |X_{i+1,i+2}| \leqslant 2$ is satisfied for all $0 \leqslant i \leqslant n-3$. Let u be a vertex of $G_s - F_s$ and v be a vertex of $G_t - F_t$ with $0 \leqslant s \leqslant t \leqslant n-1$. Then there is a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - ((\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1}))$ joining u to v.*

Proof. For convenience, let $F = (\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1})$. When $s = 0$, the statement follows from Lemma 2. When $0 < s = t < n-1$, the statement follows from Lemma 3. So, we consider the case when $0 < s < t$ in the following. Since G_s is 4-regular, we have $|V(G_s)| \geqslant 5$. Moreover, since $|F_s| + |F_{s+1}| + |X_{s,s+1}| \leqslant 2$, we can choose a vertex x of $G_s - (F_s \cup \{u\})$ such that $|F \cap \{x, (x)^+, (x, (x)^+)\}| = 0$ and $(x)^+ \neq v$. By Lemma 2, there is a hamiltonian path P of $G_{\langle 0,\ldots,s\rangle} - ((\bigcup_{i=0}^{s} F_i) \cup (\bigcup_{i=0}^{s-1} X_{i,i+1}))$ joining u to x. Similarly, there is a hamiltonian path T of $G_{\langle s+1,\ldots,n-1\rangle} - ((\bigcup_{i=s+1}^{n-1} F_i) \cup (\bigcup_{i=s+1}^{n-2} X_{i,i+1}))$ joining $(x)^+$ to v. Then $\langle u, P, x, (x)^+, T, v\rangle$ is a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - F$ joining u to v. □

Proposition 3. *Assume $n \geqslant 1$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leqslant i \leqslant n-1$, let $M_{i,i+1}$ be a perfect matching between the vertices of G_i and those of G_{i+1}. Let F be a faulty set of $G_{\langle 0,1,\ldots,n-1,0\rangle}$ such that $|F| = 3$ and $|F \cap (V(G_i) \cup E(G_i))| \leqslant 1$ for $0 \leqslant i \leqslant n-1$. Then $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ is hamiltonian connected.*

Proof. Let u be a vertex of $G_a - F_a$ and let v be a vertex of $G_b - F_b$ for any $0 \leqslant a \leqslant b \leqslant n-1$. For convenience, let $F_i = F \cap (V(G_i) \cup E(G_i))$ and $X_{i,i+1} = F \cap M_{i,i+1}$ for every $0 \leqslant i \leqslant n-1$. Obviously, we have $|F_i| \leqslant 1$. Moreover, let t be the integer such that $|X_{t,t+1}| = \max\{|X_{i,i+1}|\ |0 \leqslant i \leqslant n-1\}$. Depending on $|X_{t,t+1}|$, two cases are distinguished.

Case 1: Suppose that $|X_{t,t+1}| \geqslant 1$. Without loss of generality, t can be assumed to be $n-1$. Accordingly, we have $|X_{i,i+1}| \leqslant 1$ for every $0 \leqslant i \leqslant n-2$. Let $F' = F - X_{n-1,0}$. Hence, we have $|F'| \leqslant 2$ and $|F_i| + |F_{i+1}| + |F_{i+2}| + |X_{i,i+1}| + |X_{i+1,i+2}| \leqslant 2$ for all $0 \leqslant i \leqslant n-3$. By Lemma 4, $G_{\langle 0,1,\ldots,n-1\rangle} - F'$ is hamiltonian connected.

Case 2: Suppose that $|X_{t,t+1}| = 0$. Then we set t to be $a-1$. Obviously, we have $|F_i| + |F_{i+1}| + |X_{i,i+1}| \leqslant 2$ for all $0 \leqslant i \leqslant n-2$. By Lemma 2, $G_{\langle a,a+1,\ldots,n-1,0,\ldots,a-1\rangle} - F$ is hamiltonian connected.

Finally, $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ is concluded to be hamiltonian connected. □

Theorem 3. *Assume $n \geqslant 3$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leqslant i \leqslant n-1$, let $M_{i,i+1}$ be a perfect matching between the vertices of G_i and those of G_{i+1}. Then $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is optimal fault-tolerant hamiltonian connected.*

Proof. Obviously, $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is 6-regular. Thus, we are going to show that $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is 3-fault-tolerant hamiltonian connected. Let F be a faulty set of $G_{\langle 0,1,\ldots,n-1,0\rangle}$ with $|F| \leqslant 3$. For convenience, let $F_i =$

$F \cap (V(G_i) \cup E(G_i))$ for $0 \leqslant i \leqslant n-1$. Without loss of generality, we assume that $|F_0| \geqslant |F_i|$ for all $1 \leqslant i \leqslant n-1$. Depending on $|F_0|$, three cases are distinguished. The first case that $|F_0| = 3$ is proved by Proposition 1. The second case when $|F_0| = 2$ is proved by Proposition 2. Finally, the case for $|F_0| \leqslant 1$ follows from Proposition 3. □

According to Theorem 1–3, we have the following corollary.

Corollary 1. *Assume $n \geqslant 3$ and $k \geqslant 4$. Let $G_0, G_1, \ldots, G_{n-1}$ be n k-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leqslant i \leqslant n-1$, let $M_{i,i+1}$ be a perfect matching between the vertices of G_i and those of G_{i+1}. Then $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is $(k+2)$-regular super fault-tolerant hamiltonian.*

4. Conclusion

In this paper, we improve the result of Chen et al. [3] by showing that on the basis of n 4-regular super fault-tolerant hamiltonian networks $G_0, \ldots, G_{n-1}$, $n \geqslant 3$, the cycle composition network $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is super fault-tolerant hamiltonian. However, we conjecture that this result may not be true based on n cubic networks. Therefore, such an extension is significant because only the remaining case for 3-regular graphs needs to be checked with brute force while the topological properties of the cycle composition network is investigated.

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