行政院國家科學委員會專題研究計畫成果報告

從 Boltzmann 微觀車流模式推導巨觀車流理論模式之研究

The Study of Derivation Macroscopic Traffic Flow Model from the Microscopic Boltzmann Vehicular Traffic Flow Model

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一、中文摘要

本研究提出一由波茲曼方程推倒出巨觀車流模式之完整架構，波茲曼方程由 Prigogine 與 Herman [1]引用於描述動態車流行為，調整行為與互動行為之模式構建. 由 Prigogine 與 Herman 所發展之車流波茲曼方程僅考慮了隨時間與空間變化的相空間速度機率分佈，但忽略了加速度的影響，Paveri-Fontana [2]修正其模式，而 Helbing [3,4]與 Hoogendoorn 及 Bovy [5,6]等人則進一步延伸其方法堆導出一系列之巨觀動態車流模式. 然而其所引用之二街動差函數並無交通上之意義，引此本研究以個別速度變異為動差函數代入模式推導巨觀車流模式，並以描述車輛互動之交通場描述多車道行為，在系統趨於均衡狀態假設下構建自洽系統方程.

關鍵詞：波茲曼方程; 泊松方程; 巨觀車流模式; 多車道車流模式.

Abstract

In this study, we present a systematic self-consistent multiclass multilane traffic model derived from the vehicular Boltzmann equation and the traffic dispersion model. The multilane domain is considered as a two-dimensional space and the interaction among vehicles in the domain is described by a dispersion model. The dispersion model, which is a nonlinear Poisson equation, is derived from the car-following theory and the equilibrium assumption. In addition, the dynamic evolution of the traffic flow is determined by the systematic gas-kinetic model derived from the Boltzmann equation. Multiplying Boltzmann equation by the zeroth, first and second order moment functions, integrating both side of the equation and using chain rules, we can derive continuity, motion and variance equation respectively. However, the second order moment function, which is the square of the individual velocity, is employed by previous researches does not have physical meaning in traffic flow. The velocity variance equation we propose is derived from multiplying Boltzmann equation by the individual velocity variance. It modifies the previous model and presents a new gas-kinetic traffic flow model. By coupling the gas-kinetic model and the dispersion model, a self-consistent system is presented.

Keywords: Boltzmann equation; Poisson equation; Macroscopic traffic equations; Multilane traffic flow

二、緣由與目的

During the recent five decades by developing kinetic traffic flow model, it is possible to model more realistic traffic phenomena for traffic scientists in laboratories. Kinetic traffic flow models describe and forecast the time variant traffic variables, such as density, traffic volume and velocity. In addition, the performance of the traffic-control alternatives and the network design can be evaluated by traffic simulation.

Since Lighthill, Whitham [7] and Richards [8] firstly proposed their kinetic model, the related subjects are broadly researched and debated. The LWR model was extended to second order model, which includes the continuity equation and a phenomenological velocity equation. The second order model was named PW model [9, 10]. However, this kind of models has a lot of arguments, so families of gas-kinematic models [3-6] are presented. The development of gas-kinetic models includes the discussion of multilane, multiclass users and overtaking, lane-changing, relaxation and interaction maneuvers. As the review of Boltzmann equation, Boltzmann equation is a phase-plane distribution. The macroscopic quantities are derived as follows. The first step is multiplying Boltzmann equation by the moment functions. The second step is integrating both sides of the equations and using the chain rules. At last, the macroscopic quantities are obtained. Therefore, the resulting macroscopic quantity and the moment function must have physical meanings. From the previous researches, the second order moment function multiplied to Boltzmann equation is the square of individual velocity [3-6]. Nevertheless, the square of individual velocity, which denotes the individual kinetic energy in physics, is meaningless in traffic. Although the second order expansion results in the velocity variance equation, additional terms may be generated. For this reason, we multiply Boltzmann equation by the individual velocity variance in order to modify the derivation of velocity variance equation herein.

A complete dynamic system should include motion equations and state equations. The state equation considered in this study is the vehicular dispersion model [11]. The model is derived from the car-following theory and the equilibrium assumption. Under a specific macroscopic situation, the most possible microscopic combination is defined as the equilibrium state. And the system is assumed to tend toward the equilibrium state. According to the dispersion model, density is distributed on the road. By coupling the dispersion model to the kinetic system, a self-consistent system is obtained. Furthermore, we consider the multilane model in a two-dimensional space because the driving behavior of road users may not be restricted to drive one by one, especially motorcyclists.

三、Results and Discussion

Since macroscopic models derived from Boltzmann equation can aggregate the microscopic behavior to be group behavior, this study proposes a Boltzmann equation and derives it to macroscopic models. There are three main differences between this study and previous studies. The first one is that a multilane road is considered as a two-dimensional domain. The second one is that the acceleration effect of Boltzmann equation is considered as the influence of traffic field. The last one is that the second order moment function considered herein is individual velocity variance. Traffic field is derived from car-following theory, which describes the interaction between vehicles. The detail of derivation of traffic field is illustrated in section 3.2. The concept of traffic field not only describes the interaction between vehicles, but also makes the macroscopic system consistent. Before introducing traffic field, definitions of variables and the relations among variables should be mentioned first.

3.1 Definitions

By reason of some driving behavior of road users cannot be restricted in one lane or even they derive in one lane they still not be restricted to drive one after one, such as, driving behavior of motorcycles. Therefore, a multilane highway is regarded as a two-dimensional space in this study. We assume that there exists a phase-plane distribution function $f(\mathbf{x},\mathbf{v},t)$ at a given time and at a given point, where $\mathbf{x}=(x,y)$ denotes position, $\mathbf{v}=(v_x,v_y)$ denotes individual velocity and t denotes time. Since $\mathbf{v}$ denotes individual velocity, it is impossible to restrict a specific velocity at a specific time and place.

In addition, $d\mathbf{x}/dt=\mathbf{v}$ and $d\mathbf{v}/dt=e\mathbf{E}$, where $\mathbf{E}$ denotes traffic field and is going to derive in detail in section 3.2. By Taylor's expansion or total derivative of *f*, the changing of *f* is shown by

$$\frac{df(\mathbf{x},\mathbf{v},t)}{dt}=\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{x}}f(\mathbf{x},\mathbf{v},t)+e\mathbf{E}\cdot\nabla_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t)=\left(\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\right)_{coll}, \quad (1)$$

$$f(\mathbf{x},\mathbf{v},t)\big|_{\partial\Omega_{\mathbf{v}}}=0, \quad (2)$$

where *f* is defined on Ω and $\partial\Omega$ is the boundary of Ω. $\partial\Omega_{\mathbf{v}}$ is the boundary of individual velocity. Since *f* is a distribution function, it is reasonable to assume that *f* is equal to 0 at the extreme value (i.e. boundary $\partial\Omega_{\mathbf{v}}$). Thus, density is given by

$$k(\mathbf{x},t)=\int_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t)d\mathbf{v} \quad (3)$$

and flow density, which is defined by Cho and Lo [41], is given by

$$\mathbf{q}(\mathbf{x},t)=\int_{\mathbf{v}}\mathbf{v}f(\mathbf{x},\mathbf{v},t)d\mathbf{v}=k(\mathbf{x},t)\mathbf{u}(\mathbf{x},t), \quad (4)$$

where $\mathbf{u}(\mathbf{x},t)$ denotes average velocity (or so-called group velocity), which is defined as

$$\mathbf{u}(\mathbf{x},t)=\frac{\int_{\mathbf{v}}\mathbf{v}f(\mathbf{x},\mathbf{v},t)d\mathbf{v}}{\int_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t)d\mathbf{v}}. \quad (5)$$

Next, three kinds of velocity variance are defined. The first one is total velocity variance, which is velocity variance between individual velocity and equilibrium velocity. Total velocity variance is defined by

$$\grave{E}(\mathbf{x},t)=\frac{\int_{\mathbf{v}}\|\mathbf{v}-\mathbf{u}_e\|^2 f(\mathbf{x},\mathbf{v},t)d\mathbf{v}}{\int_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t)}. \quad (6)$$

The second one is individual velocity variance, which is the velocity variance between individual velocity and group velocity. Furthermore, equilibrium average velocity and equilibrium average variance are denoted by $\mathbf{u}_e(k,\mathbf{u},\Theta)$ and $\Theta_e(k,\mathbf{u},\Theta)$, respectively. An equilibrium state is defined as the most possible microscopic state under a specific macroscopic state. Since $\mathbf{u}_e$ depends upon *k*, **u** and Θ only, $\nabla_{\mathbf{x}}\mathbf{u}_e=0$, $\nabla_{\mathbf{x}}\cdot\mathbf{u}_e=0$, $\partial\mathbf{u}_e/\partial t=0$. Another variable, which appears in the derivation, is skewness. It is defined by

$$\tilde{\mathbf{A}}(\mathbf{x},t)=\frac{\int_{\mathbf{v}}(\mathbf{v}-\mathbf{u}(k))\|\mathbf{v}-\mathbf{u}(k)\|^2 f(\mathbf{x},\mathbf{v},t)d\mathbf{v}}{\int_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t)}, \quad (7)$$

which is the bias of the distribution. From the empirical study by Helbing [3, 4], $\tilde{\mathbf{A}}(\mathbf{x},t)\approx\mathbf{0}$. Since the influences of the third or higher order moment functions are negligible, we do not have to expand the higher order conservation laws. The basic variables *k*, *Q*, Θ are scalar and **q**, **u**, $\tilde{\mathbf{A}}(\mathbf{x},t)$ are vectors.

The basic idea of deriving macroscopic models from Boltzmann equation is the same as finding the expectation of a random variable. Therefore, finding the individual variables that are meaningful and multiplying them to distribution *f* will obtain macroscopic variables (average or group quantities). The individual variables are named as moment functions and denoted as $\varphi(\mathbf{x},\mathbf{v},t)$. The moment functions chosen by related researches are 1, $\mathbf{v}$, and $\mathbf{v}^2$. However, $\mathbf{v}^2$ doesn't have physical meaning in traffic. For this reason, the moment functions chosen herein are 1, $\mathbf{v}$, and $\|\mathbf{v}-\mathbf{u}_e\|^2$, where $\|\mathbf{v}-\mathbf{u}_e\|^2$ is individual velocity variance. Thus, multiplying Eq.(1)

by $\phi(\mathbf{x},\mathbf{v},t)$ and integrating the function, we have

The integration form of Eq.(1) is illustrated as

$$\int_{\mathbf{v}} \frac{df(\mathbf{x},\mathbf{v},t)}{dt}\phi(\mathbf{x},\mathbf{v},t)d\mathbf{v}$$
$$=\int_{\mathbf{v}} \frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\phi(\mathbf{x},\mathbf{v},t)d\mathbf{v}+\int_{\mathbf{v}}[\mathbf{v}\cdot\nabla_{\mathbf{x}} f(\mathbf{x},\mathbf{v},t)]\phi(\mathbf{x},\mathbf{v},t)d\mathbf{v}+\int_{\mathbf{v}}[e\mathbf{E}\cdot\nabla_{\mathbf{v}} f(\mathbf{x},\mathbf{v},t)]\phi(\mathbf{x},\mathbf{v},t)d\mathbf{v}$$
$$=\int_{\mathbf{v}}\left(\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\right)_{coll}\phi(\mathbf{x},\mathbf{v},t)d\mathbf{v}$$

. (8)

By substituting 1, $\mathbf{v}$, and $\|\mathbf{v}-\mathbf{u}_e\|^2$ for $\phi(\mathbf{x},\mathbf{v},t)$ in Eq.(8), the macroscopic system will be obtained. The derivation is shown from section 3.3 to 3.5. Before deriving the macroscopic model from Boltzmann equation, the concept of traffic field [11] should be mentioned in brief first.

3.2 Traffic field

Traffic field is employed to describe the traffic pressure and the accelerated effect in this study. Since the traffic field distributes density on a road, the relation between the traffic field and the density is named as the dispersion model. The derivation of the dispersion model starts with the discussion of the interaction between a single vehicle and other vehicles by car-following theory. Two assumptions are made. The first one is that the influence of cars in the same lane is M times larger than that in the adjacent lanes, where M is a scalar. The second one is that the traffic field is independent of velocity. For the sake of safety, one vehicle on a road adjusts its velocity and spacing according to the relative position between other vehicles so as to avoid the accident. It is assumed that each vehicle has its own field. Vehicles exclude each other by their own field. Thus, the interaction (in terms of traffic force or traffic pressure, which is denoted by **F**), which is produced by the traffic field ($\tilde{\mathbf{E}}$), among vehicles is a resistance. To simplify the complication of the problem, $\tilde{\mathbf{E}}$ is assumed to depend on spacing and to satisfy the inverse-square law (the gravity model), which means the influence of other vehicles is larger when the spacing is smaller. In the continuous space, the traffic field is represented as

$$\mathbf{E}=\frac{e}{å}\int_{\Omega}\left((k-k_s)/\|\mathbf{X}\|^2\right)d\Omega, \quad (9)$$

where e denotes the passanger car equivalent and $£$ denotes the interacting parameter, if vehicles and driving behavior on the road are the same. k is the actual density and k_s is the unrestrained density that is the density which vehicles do not interfere with each other. The transformed traffic field is a conservative field. Then, a potential function ¤ exists by the potential theory. The potential function ¤ satisfies $\mathbf{E}=-\nabla_{\mathbf{x}}¤$. Thus, the magnitude of traffic field is illustrated as

$$div\mathbf{E}=-\Delta ¤=e(k-k_s)/£+K_a, \quad (10)$$

where $div\mathbf{E}$ denotes the magnitude of traffic field, $K_a=K_a(\mathbf{x})$, which depends on the position $\mathbf{x}$, is the adjust term of the road condition if the road condition is ideal .

3.3 Continuity Equation

After introducing traffic field and Poisson equation, derivation from Boltzmann equation to the macroscopic system is presented. Firstly, let $\phi=1$, so we have the conversation of vehicle numbers:

$$\frac{\partial k(\mathbf{x},t)}{\partial t}+\nabla_{\mathbf{x}}\cdot\mathbf{q}(\mathbf{x},t)=0. \quad (11)$$

3.4 Motion Equation

The motion equation is derived by substituting the first order moment function, $\mathbf{v}$, for ϕ in Eq. (8). Then the expectation function of velocity is illustrated as

$$\frac{\partial(k\mathbf{u})}{\partial t}+\nabla_{\mathbf{x}}\cdot\left(k\mathbf{è}_a+k\mathbf{uu}\right)=ek\mathbf{E}-\frac{k\mathbf{u}-k\mathbf{u}_e}{2_m}. \quad (12)$$

3.5 Variance Equation (Conservation of Energy)

The last equation considered herein is the variance equation, which is obtained by substituting the second moment function, $\|\mathbf{v}-\mathbf{u}_e\|^2$, for ϕ in Eq.(8). Then the expectation function of velocity variance is illustrated as

$$\frac{\partial(k\Theta)}{\partial t}+\nabla_{\mathbf{x}}\cdot[(k\Theta\mathbf{u})+2k(\mathbf{u}-\mathbf{u}_e)\mathbf{è}_a]=-2ek\mathbf{E}\cdot(\mathbf{u}-\mathbf{u}_e)-\frac{k\Theta-k\Theta_e}{2_e} \quad (13)$$

is obtained.

The system equations developed above includes three conservation laws derived from Boltzmann equation. These three equations are transient equations, which describe the changing of variables. However, a complete dynamic system not only needs transient equations, but also needs state equations, which describe the state of variables. A state equation is needed so as to make the system self-consistent. In this study, the dynamic system is assumed to become the equilibrium state gradually. Therefore, the equilibrium distribution

$$k=K_0\exp\left((e\cdot-e¤)/\Theta_e\right), \quad (14)$$

is employed to be the state equation in the system. K_0 is the essential density, Θ_e is the equilibrium velocity variance, $\cdot$ is the potential equivalent of the velocity variance threshold. $\cdot$ is named as the potential barrier here. By coupling Eqs (10) and (14), the nonlinear dispersion model is obtained. If a set of boundary conditions of the traffic potential is applied, vehicles are forced to drive through the road according to the path, which has the least resistance. Therefore, vehicles on the two-dimensional research domain will not move forward and backward or circle round. They will try to pass through the road as soon as possible.

3.7 Closure Relations

The system presented herein also needs closure relations so as to determine the equilibrium velocity $\mathbf{u}_e(k,\mathbf{u},\Theta)$, equilibrium variance $\Theta_e(k,\mathbf{u},\Theta)$, and relaxation time 2_m and 2_e in Eqs (12) and (13). There are a variety of possible closure relations, which could be adopted from previous studies. [3-6, 12-14]. The $\mathbf{u}_e$ and Θ_e proposed in study are represented by

(15)

and

$$\Theta_e(k,\mathbf{u},\Theta)=\tau_e \mathbf{p}_p(k)\cdot k\Theta\mathbf{u}\,, \tag{16}$$

respectively. $\mathbf{u}_0$ is the average desired velocity, $\mathbf{p}_b(k)\in[0,1]$ is the braking probability vector, and $\mathbf{p}_p(k)\in[0,1]$ is the passing probability vector. The explicit forms are obtained by specifying expressions for $\mathbf{p}_b(k)$ and $\mathbf{p}_p(k)$. Equation (15) means that the equilibrium velocity decreases as $\mathbf{p}_b(k)$ increases. On the other hand, the equilibrium variance increases as $\mathbf{p}_p(k)$ increases. Since $\mathbf{u}_e(k)$ and $\Theta_e(k,\mathbf{u})$ are equilibrium equations, the functions suggested in this study are

$$\mathbf{u}_e(k)=\frac{\int_{\mathbf{v}}\mathbf{v}f_e(\mathbf{x},\mathbf{v},t)d\mathbf{v}}{\int_{\mathbf{v}}f_e(\mathbf{x},\mathbf{v},t)d\mathbf{v}} \tag{17}$$

and

$$\Theta_e(k)=\frac{\int_{\mathbf{v}}\|\mathbf{v}-\mathbf{u}_e\|^2 f_e(\mathbf{x},\mathbf{v},t)d\mathbf{v}}{\int_{\mathbf{v}}f_e(\mathbf{x},\mathbf{v},t)d\mathbf{v}} \tag{18}$$

respectively. f_e denotes the steady state homogeneous solution of Boltzmann equation. The relaxation time τ_m and τ_e are shown as

$$\tau_m(k)=\frac{T_m}{g(k)} \quad \text{and} \quad \tau_e(k)=\frac{T_e}{g(k)}\,, \tag{19}$$

which are modified from the suggestion of Helbing [3-4]. *g(k)* is the proportion of freely moving vehicles, T_m is the reaction time of velocity and T_e is the reaction time of variance. As f_e, *g(k)*, T_m and T_e are determined, the closure relations are expressed specifically. Then, the self-consistent system is complete.

四、結論與建議

In this paper we have derived a macroscopic multilane traffic model for multiple classes users. The system is a self-consistent system; it can be solved with proper initial conditions and boundary conditions. Our consideration is based on the following assumptions:(a) a multilane road is considered as a two-dimensional domain. (b) the whole system will tend toward equilibrium. (c) the individual velocity variance is employed as the second moment function.

Considering a multilane road as a two-dimensional domain allows us to handle the driving behaviors, which are not restricted to drive one by one in a single lane. Another advantage of this consideration is to avoid modeling complicated lane-changing behavior. Lane-changing behavior is controlled by the nonlinear Poisson equation. If the research area is only a single lane road, the system can be reduced to a one-dimensional model.

This study derives a dynamic macroscopic traffic flow system from Boltzmann equation. Boltzmann equation employed herein includes accelerated effect, which is controlled by Poisson equation. This study modifies the second moment function as $\|\mathbf{v}-\mathbf{u}_e\|^2$, which is individual velocity variance, and reformulates velocity variance equation to be more reasonable. This study exposes three moment functions. If there still exists the other meaningful moment, it should be considered as its influence is significant.

The system equations can be simplified to adapt different traffic condition because not all variables are significant in each traffic condition. For example, in uniform and equilibrium traffic flow, the influence of velocity variance equation may be ignored. The simplification is needed because computation of the whole system takes a lot of time. According to different traffic conditions, simplification should be discussed and validated further. Also, numerical methods should be developed to solve the system.。

五、計劃自評

In this study, we have derived a macroscopic multilane traffic model from the microscopic traffic flow model-Boltzmann equation successfully. The system is a self-consistent system; it can be solved with proper initial conditions and boundary conditions. The result is accepted by the *Physica A*, “Modeling of self-consistent multi-class dynamic traffic flow model”.

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