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Chaos, Solitons and Fractals 35 (2008) 705–717
CHAOS
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Chaos synchronization of fractional order modified duffing systems with parameters excited by a chaotic signal

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Accepted 22 May 2006

Abstract

In this paper, chaos synchronizations of two uncoupled fractional order chaotic modified Duffing systems are obtained. By replacing their corresponding parameters by the same function of chaotic state variables of a third chaotic system, chaos synchronization can be obtained. The method is named parameter excited chaos synchronization which can be successfully obtained for very low total fractional order 0.2. Numerical simulations are illustrated by phase portrait, Poincaré map and state error plots.

1. Introduction

Since the pioneering work by Pecora and Carroll [1], various effective methods for chaos synchronization have been reported [2–36]. However, most of synchronizations can only be realized under the hypotheses that there exists coupling between two chaotic systems. In practice, such as in physical and electrical systems, sometimes it is difficult even impossible to couple two chaotic systems, In comparison with coupled chaotic systems, for synchronization between the uncoupled chaotic systems, there are many advantages [8,9]. In this paper, synchronization of two fractional Duffing systems whose corresponding parameters are excited by a chaotic signal of a third system is studied.

Fractional calculus is a 300-year-old mathematical topic [37–40]. Although it has a long history, the applications of fractional calculus to physics and engineering are just a recent focus of interest [41–61].

The system studied in this paper is a modified form of nonlinear damped Duffing system. The chaos synchronizations of two uncoupled fractional order modified Duffing systems are obtained by replacing their corresponding parameters by the same function of chaotic state variables of a third chaotic system. The method is named parameter excited chaos synchronization which can be successfully obtained for very low total fractional order 0.2. Numerical simulations are illustrated by phase portraits, Poincaré maps and state error plots.

The rest of this paper is organized as follows. In Section 2 the fractional derivative and its approximation are introduced. In Section 3 the system under study is described both in its integer and fractional forms. In Section 4 numerical simulations of synchronization scheme based on driving the corresponding parameters of two chaotic systems by a chaotic signal of a third system are presented. In Section 5 conclusions are drawn.

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0960-0779/$ - see front matter
doi:10.1016/j.chaos.2006.05.101

2. Fractional derivative and its approximation

Two commonly used definitions for the general fractional differintegral are the Grunwald definition and the Riemann–Liouville definition. The Riemann–Liouville definition of the fractional integral is given here as [62]:

$$\frac{d^q f(t)}{dt^q} = \frac{1}{\Gamma(-q)} \int_0^1 \frac{f(\tau)}{(t-\tau)^{q+1}} d\tau, \quad q < 0 \tag{1}$$

where q can have noninteger values, and thus the name fractional differintegral. Notice that the definition is based on integration and more importantly is a convolution integral for $q < 0$. When $q > 0$, then the usual integer nth derivative must be taken of the fractional $(q - n)$th integral, and yields the fractional derivative of order q as

$$\frac{d^q f}{dt^q} = \frac{d^n}{dt^n}\left[\frac{d^{q-n} f}{dt^{q-n}}\right], \quad q > 0 \quad \text{and} \quad n \text{ an integer} > q. \tag{2}$$

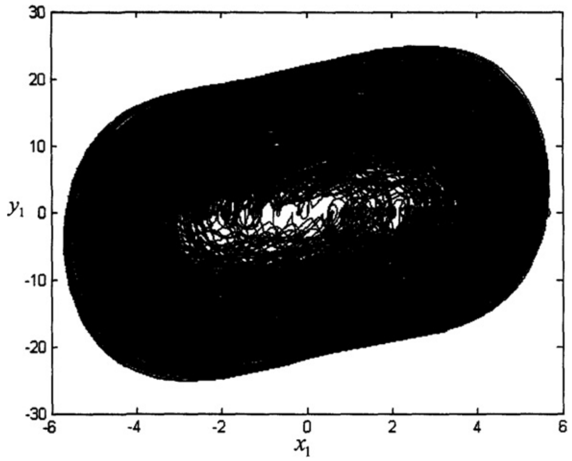

Fig. 1. The phase portrait and Poincaré map of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.9$ for *Case 1*.

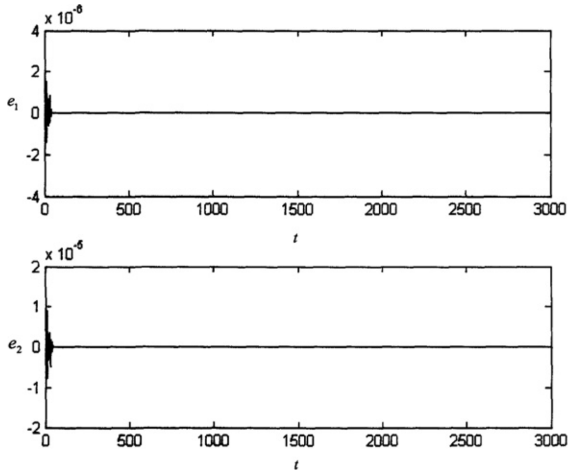

Fig. 2. The time histories of the errors of the states of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.9$ for *Case 1*.

This appears so vastly different from the usual intuitive definition of derivative and integral that the reader must abandon the familiar concepts of slope and area and attempt to get some new insight. Fortunately, the basic engineering tool for analyzing linear systems, the Laplace transform, is still applicable and works as one would expect, i.e.,

$$L\left\{\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q}\right\} = s^q L\{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{\mathrm{d}^{q-1-k} f(t)}{\mathrm{d}t^{q-1-k}}\right]_{t=0}, \text{ for all } q, \tag{3}$$

where n is an integer such that $n-1<q<n$. If the initial conditions are considered to be zero, this formula reduces to the more expected and comforting form:

$$L\left\{\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q}\right\} = s^q L\{f(t)\}. \tag{4}$$

An efficient method is to approximate fractional operators by using standard integer order operators. In [62], an effective algorithm is developed to approximate fractional order transfer functions. Basically, the idea is to approximate the system behavior in the frequency domain. By utilizing frequency domain techniques based on Bode diagrams, one can obtain a linear approximation of fractional order integrator, the order of which depends on the desired bandwidth

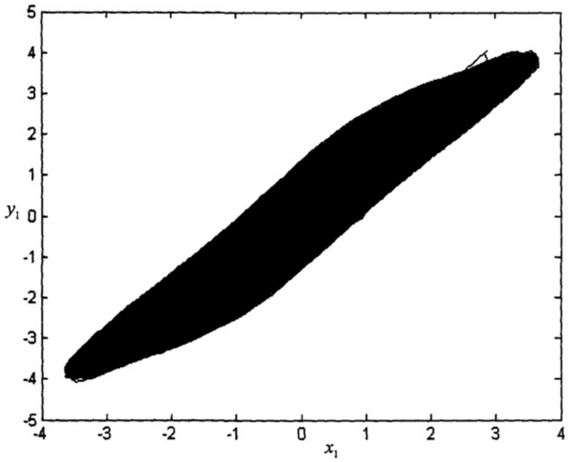

Fig. 3. The phase portrait and Poincaré map of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.1$ for *Case 1*.

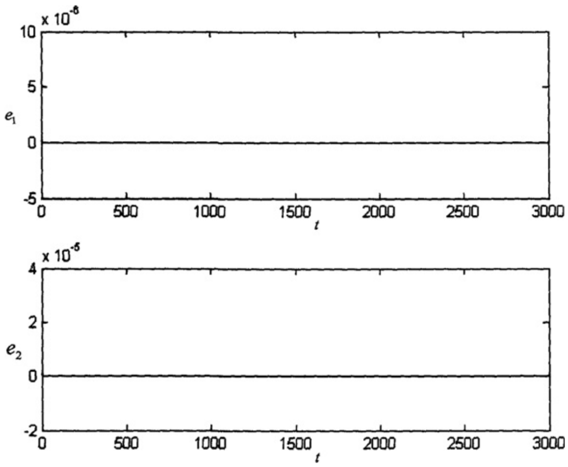

Fig. 4. The time histories of the errors of the states of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.1$ for *Case 1*.

and discrepancy between the actual and the approximate magnitude Bode diagrams. In Table 1 of [49], approximations for $1/s^q$ with $q = 0.1$–0.9 in steps 0.1 are given, with errors of approximately 2 dB. These approximations are used in following simulations.

3. A fractional order modified duffing system

The famous Duffing system is:

$$\ddot{x} + a\dot{x} + x + x^3 = b\cos\omega t, \tag{5}$$

where a, b are constant parameters.

It can be written as two first order ordinary differential equations:

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x - x^3 - ay + b\cos\omega t. \end{cases} \tag{6}$$

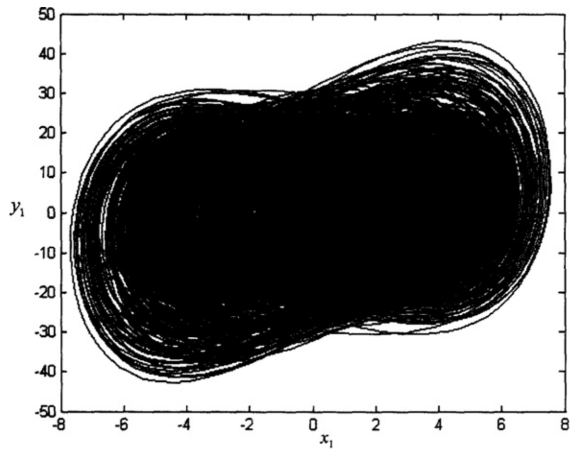

Fig. 5. The phase portrait and Poincaré map of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.9$ for *Case 2*.

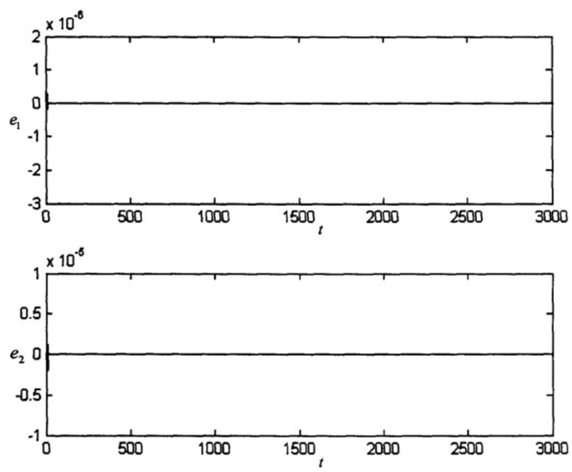

Fig. 6. The time histories of the errors of the states of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.9$ for *Case 2*.

Consider the following modified Duffing system:

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x - x^3 - ay + bz, \\ \frac{dz}{dt} = w, \\ \frac{dw}{dt} = -cz - dz^3. \end{cases} \tag{7}$$

It becomes an autonomous system with four states where a, b, c, and d are constant parameters of the system. System (7) can be divided into two parts:

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x - x^3 - ay + bz \end{cases} \tag{8}$$

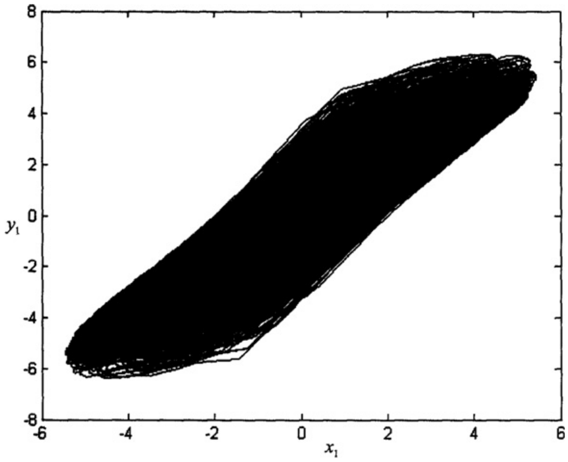

Fig. 7. The phase portrait and Poincaré map of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.1$ for *Case 2*.

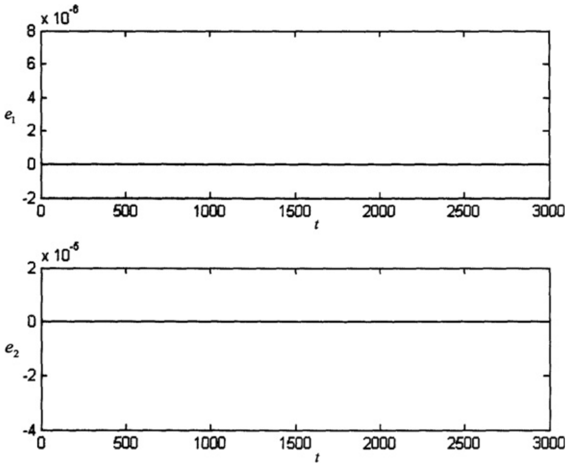

Fig. 8. The time histories of the errors of the states of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.1$ for *Case 2*.

and

$$\begin{cases} \frac{dz}{dt} = w, \\ \frac{dw}{dt} = -cz - dz^3. \end{cases} \tag{9}$$

As a nonlinear oscillator, system (9) provide the periodic time function bz to system (8) as an excitation which produces the chaos in system (8). To sum up, system (8) can be considered as a nonautonomous system with two states x, y with bz as an excitation which is a given periodic function of time, while system (8) and system (9) together can be considered as an autonomous system with four states x, y, z, w. We focus on system (8), while system (9) remains an integral order system.

Now, consider a fractional order modified Duffing system. Here, the conventional derivatives in Eq. (8) are replaced by the fractional derivatives as follows:

$$\begin{cases} \frac{d^{q_1}x}{dt^{q_1}} = y, \\ \frac{d^{q_2}y}{dt^{q_2}} = -x - x^3 - ay + bz, \\ \frac{dz}{dt} = w, \\ \frac{dw}{dt} = -cz - dz^3, \end{cases} \tag{10}$$

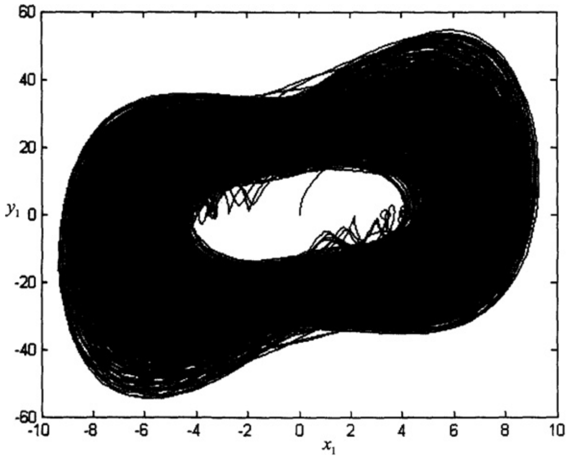

Fig. 9. The phase portrait and Poincaré map of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.9$ for *Case 3*.

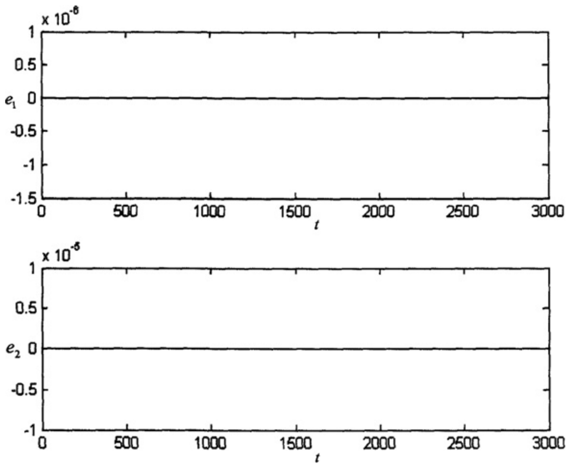

Fig. 10. The time histories of the errors of the states of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.9$ for *Case 3*.

where system parameter b is allowed to be varied, and q_1, q_2 are two fractional order numbers. Simulations are then performed using $q_i (i = 1, 2)$ varied from 0.1 to 0.9, respectively. The approximations from Table 1 of [49] are used for the simulations of the appropriate q_ith integrals. When $q_i < 1$, the approximations are used directly. It should further be noted that approximations used in the simulations for $1/s^{q_i}$, when $q_i > 1$, are obtained by using $1/s$ times the approximation for $1/s^{q_i-1}$ from Table 1.

4. Numerical simulations for chaos synchronization with parameter driven by a chaotic signal

In this section, two chaotic fractional order modified Duffing systems:

$$\begin{cases} \frac{d^{q_1} x_1}{dt^{q_1}} = y_1, \\ \frac{d^{q_2} y_1}{dt^{q_2}} = -x_1 - x_3^3 - a y_1 + b z_1, \\ \frac{dz_1}{dt} = w_1, \\ \frac{dw_1}{dt} = -c z_1 - d z_3^3 \end{cases} \tag{11}$$

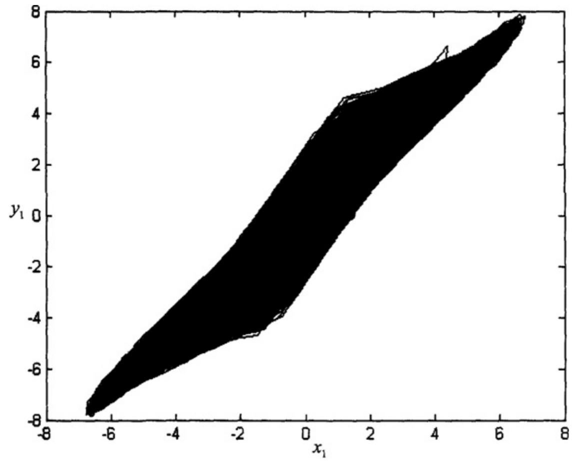

Fig. 11. The phase portrait and Poincaré map of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.1$ for *Case 3*.

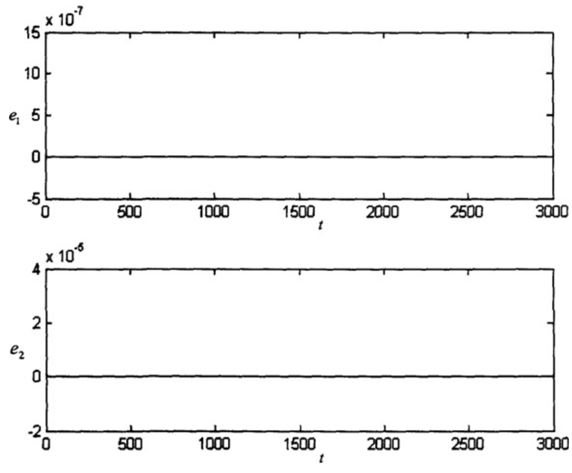

Fig. 12. The time histories of the errors of the states of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.1$ for *Case 3*.

and

$$\begin{cases} \frac{d^{q_1}x_2}{dt^{q_1}} = y_2, \\ \frac{d^{q_2}y_2}{dt^{q_2}} = -x_2 - x_3^3 - ay_2 + bz_2, \\ \frac{dz_2}{dt} = w_2, \\ \frac{dw_2}{dt} = -cz_2 - dz_2^3, \end{cases} \tag{12}$$

where q_1 and q_2 are the fractional orders, are synchronized by replacing corresponding parameters by the same function of chaotic states of a third chaotic modified Duffing system:

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x - x^3 - ay + bz, \\ \frac{dz}{dt} = w \\ \frac{dw}{dt} = -cz - dz^3, \end{cases} \tag{13}$$

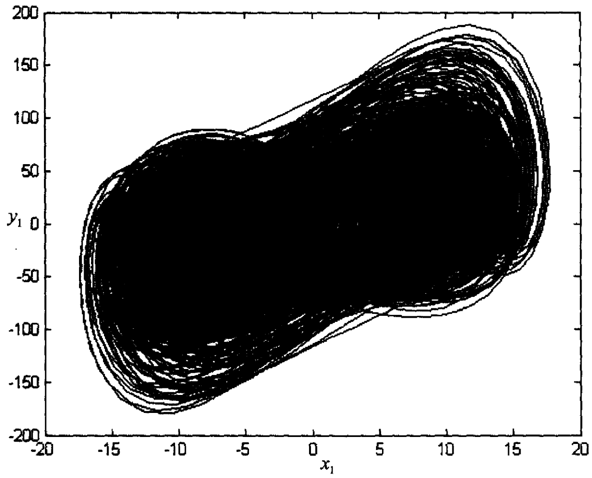

Fig. 13. The phase portrait and Poincaré map of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.9$ for *Case 4*.

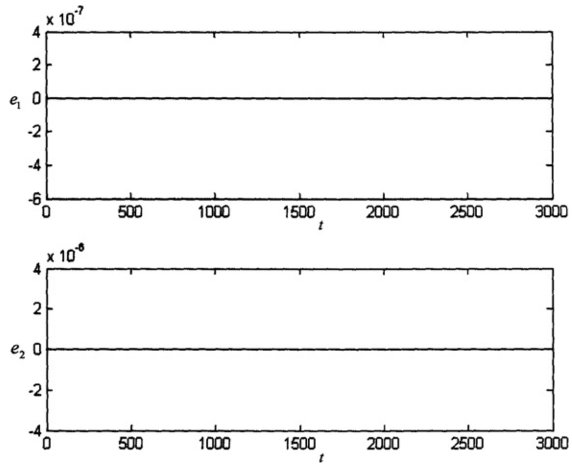

Fig. 14. The time histories of the errors of the states of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.9$ for *Case 4*.

where $a = 0.05$, $b = 53$, $c = 1$, and $d = 0.3$ are constant parameters of the system. Define the error states as $e_1 = x_1 - x_2$ and $e_2 = y_1 - y_2$ in system (11) and (12). The synchronization scheme is to replace the corresponding parameters b in system (11) and (12) by the same function of chaotic states of system (13) such that $\|e(t)\| \to 0$ as $t \to \infty$. In following simulations, for various derivative orders q_1 and q_2, we replace the system parameter b in system (11) and (12) by x, y, x^2, y^2, xy where x and y are state variables in system (13). Simulations are performed under $q_1 = q_2 = 0.1$–0.9 in steps of 0.1. In our numerical simulations, four parameters $a = 0.05$, $b = 53$, $c = 1$ and $d = 0.3$ of system (13) are fixed. The initial states of system (13) are $x(0) = 3$, $y(0) = 4$, $z(0) = 1$ and $w(0) = 0$. The numerical simulations are carried out by MATLAB.

Case 1: The parameters $a = 0.05$, $c = 1$ and $d = 0.3$ of system (11) and (12) are fixed. The parameter b of system (11) and (12) is replaced by the same x, where x is the state variable of system (13). All synchronizations for $q_1 = q_2 = 0.1$–0.9 are successfully obtained. For saving space, only results for $q_1 = q_2 = 0.1$ and 0.9 are shown in Figs. 1–4.

Case 2: The parameters $a = 0.05$, $c = 1$ and $d = 0.3$ of system (11) and (12) are fixed. The parameter b of system (11) and (12) is replaced by the same y, where y is the state variable of system (13). All synchronizations for

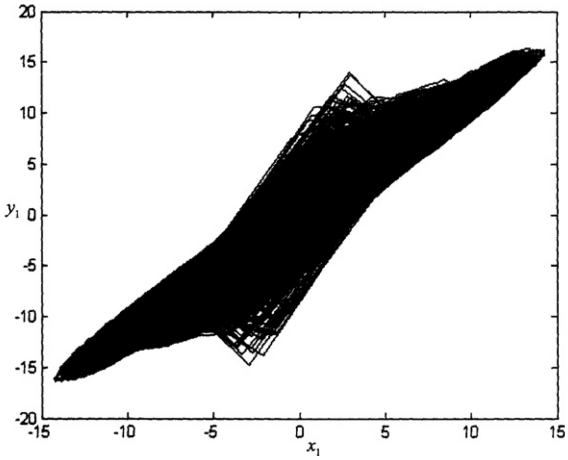

Fig. 15. The phase portrait and Poincaré map of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.1$ for *Case 4*.

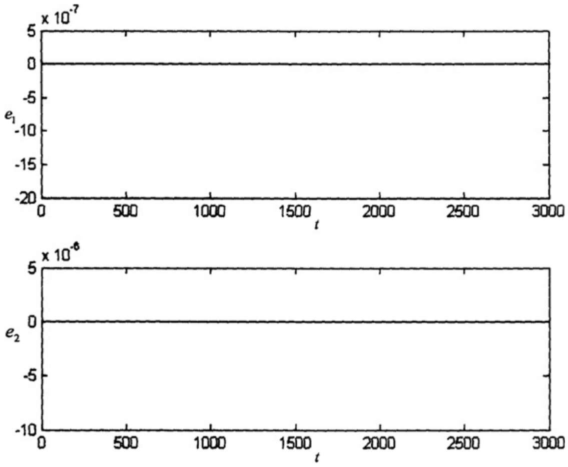

Fig. 16. The time histories of the errors of the states of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.1$ for *Case 4*.

$q_1 = q_2 = 0.1$–0.9 are successfully obtained. For saving space, only results for $q_1 = q_2 = 0.1$ and 0.9 are shown in Figs. 5–8.

Case 3: The parameters $a = 0.05$, $c = 1$ and $d = 0.3$ of system (11) and (12) are fixed. The parameter b of system (11) and (12) is replaced by the same x^2, where x is the state variable of system (13). All synchronizations for $q_1 = q_2 = 0.1$–0.9 are successfully obtained. For saving space, only results for $q_1 = q_2 = 0.1$ and 0.9 are shown in Figs. 9–12.

Case 4:The parameters $a = 0.05$, $c = 1$ and $d = 0.3$ of system (11) and (12) are fixed. The parameter b of system (11) and (12) is replaced by the same y^2, where y is the state variable of system (13). All synchronizations for $q_1 = q_2 = 0.1$–0.9 are successfully obtained. For saving space, only results for $q_1 = q_2 = 0.1$ and 0.9 are shown in Figs. 13–16.

Case 5:The parameters $a = 0.05$, $c = 1$ and $d = 0.3$ of system (11) and (12) are fixed. The parameter b of system (11) and (12) is replaced by the same xy, where x and y are the state variables of system (13). All synchronizations for $q_1 = q_2 = 0.1$–0.9 are successfully obtained. For saving space, only results for $q_1 = q_2 = 0.1$ and 0.9 are shown in Figs. 17–20.

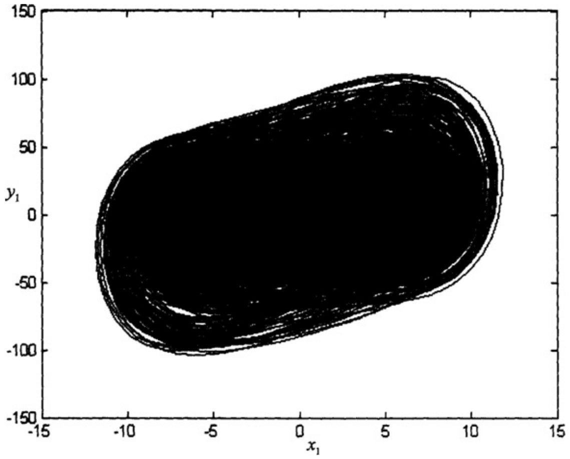

Fig. 17. The phase portrait and Poincaré map of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.9$ for *Case 5*.

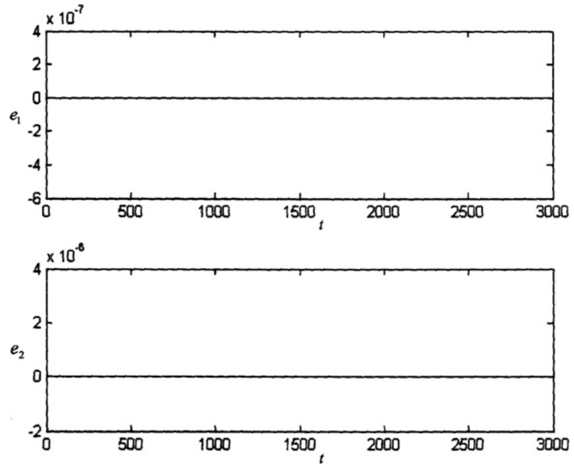

Fig. 18. The time histories of the errors of the states of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.9$ for *Case 5*.

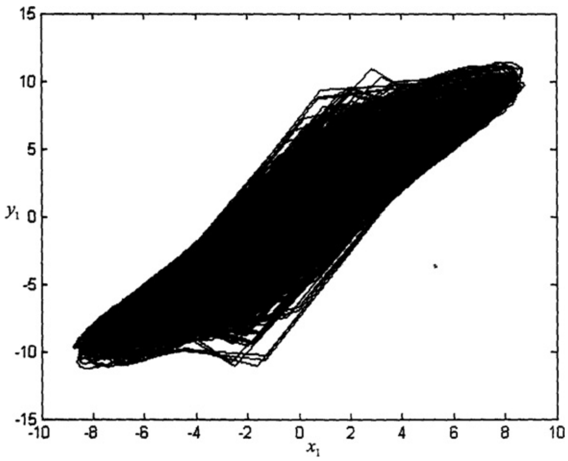

Fig. 19. The phase portrait and Poincaré map of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.1$ for *Case 5*.

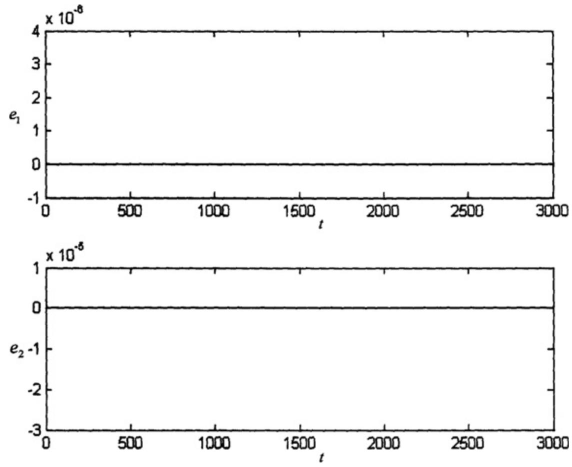

Fig. 20. The time histories of the errors of the states of the synchronized fractional order modified Duffing systems (11) and (12) with order $q_1 = q_2 = 0.1$ for *Case 5*.

5. Conclusions

In this paper, parameter excited chaos synchronizations of uncoupled integral and fractional order modified Duffing systems are studied by means of phase portrait, Poincaré map and the state error plots. It is found that this approach is very effective even for very low total fractional order 0.2.

Acknowledgement

This research was supported by the National Science Council, Republic of China, under grant number NSC 94-2212-E-009-013.

References

[1] Pecora LM, Carroll TL. Synchronization in chaotic systems. Phys Rev Lett 1990;64:821–4.
[2] Carroll TL, Heagy JF, Pecora LM. Transforming signals with chaotic synchronization. Phys Rev E 1996;54:4676–80.

[3] Kocarev L, Parlitz U. Generalized synchronization, predictability, and equivalence of unidirectionally coupled dynamical systems. Phys Rev Lett 1996;76:1816–9.
[4] Rosenblum MG, Pikovsky AS, Kurths J. Phase synchronization of chaotic oscillators. Phys Rev Lett 1996;76:1804–7.
[5] Yang SS, Duan CK. Generalized synchronization in chaotic systems. Chaos, Solitons & Fractals 1998;9:1703–7.
[6] Chen G, Liu ST. On generalized synchronization of spatial chaos. Chaos, Solitons & Fractals 2003;15:311–8.
[7] Kim CM, Rim S, Kye WH, Ryu JW, Park YJ. Anti-synchronization of chaotic oscillators. Phys Lett A 2003;320:39–46.
[8] Yang SP, Niu HY, Tian G, et al. Synchronizing chaos by driving parameter. Acta Phys Sin 2001;50:619–23.
[9] Dai D, Ma XK. Chaos synchronization by using intermittent parametric adaptive control method. Phys Lett A 2001;288:23–8.
[10] Chen HK. Synchronization of two different chaotic systems: A new system and each of the dynamical systems Lorenz, Chen and Lü. Chaos, Solitons & Fractals 2005;25:1049–56.
[11] Chen HK, Lin TN. Synchronization of chaotic symmetric gyros by one-way coupling conditions. ImechE Part C: J Mech Eng Sci 2003;217:331–40.
[12] Chen HK. Chaos and chaos synchronization of a symmetric gyro with linear-plus-cubic damping. J Sound Vib 2002;255:719–40.
[13] Ge ZM, Yu TC, Chen YS. Chaos synchronization of a horizontal platform system. J Sound Vib 2003:731–49.
[14] Ge ZM, Lin TN. Chaos, chaos control and synchronization of electro-mechanical gyrostat system. J Sound Vib 2003;259(3).
[15] Ge ZM, Chen YS. Synchronization of unidirectional coupled chaotic systems via partial stability. Chaos, Solitons & Fractals 2004;21:101–11.
[16] Ge ZM, Chen CC. Phase synchronization of coupled chaotic multiple time scales systems. Chaos, Solitons & Fractals 2004;20:639–47.
[17] Ge ZM, Lin CC, Chen YS. Chaos, chaos control and synchronization of vibrometer system. J Mech Eng Sci 2004;218:1001–20.
[18] Chen HK, Lin TN, Chen JH. The stability of chaos synchronization of the Japanese attractors and its application. Jpn J Appl Phys 2003;42(12):7603–10.
[19] Ge ZM, Shiue JS. Non-linear dynamics and control of chaos for Tachometer. J Sound Vib 2002;253(4).
[20] Ge ZM, Lee CI. Non-linear dynamics and control of chaos for a rotational machine with a hexagonal centrifugal governor with a spring. J Sound Vib 2003;262:845–64.
[21] Ge ZM, Hsiao CM, Chen YS. Non-linear dynamics and chaos control for a time delay Duffing system. Int J Nonlinear Sci Numer 2005;6(2):187–99.
[22] Ge ZM, Tzen PC, Lee SC. Parametric analysis and fractal-like basins of attraction by modified interpolates cell mapping. J Sound Vib 2002;253(3).
[23] Ge ZM, Lee SC. Parameter used and accuracies obtain in MICM global analyses. J Sound Vib 2004;272:1079–85.
[24] Ge ZM, Leu WY. Chaos synchronization and parameter identification for loudspeaker system. Chaos, Solitons & Fractals 2004;21:1231–47.
[25] Ge ZM, Chang CM. Chaos synchronization and parameter identification for single time scale brushless DC motor. Chaos, Solitons & Fractals 2004;20:889–903.
[26] Ge ZM, Lee JK. Chaos synchronization and parameter identification for gyroscope system. Appl Math Comput 2004;63:667–82.
[27] Ge ZM, Cheng JW. Chaos synchronization and parameter identification of three time scales brushless DC motor. Chaos, Solitons & Fractals 2005;24:597–616.
[28] Ge ZM, Chen YS. Adaptive synchronization of unidirectional and mutual coupled chaotic systems. Chaos, Solitons & Fractals 2005;26:881–8.
[29] Chen HK. Global chaos synchronization of new chaotic systems via nonlinear control. Chaos, Solitons & Fractals 2005;4:1245–51.
[30] Chen HK, Lee CI. Anti-control of chaos in rigid body motion. Chaos, Solitons & Fractals 2004;21:957–65.
[31] Ge ZM, Wu HW. Chaos synchronization and chaos anticontrol of a suspended track with moving loads. J Sound Vib 2004;270:685–712.
[32] Ge ZM, Yu CY, Chen YS. Chaos synchronization and chaos anticontrol of a rotational supported simple pendulum. JSME Int J, Ser C 2004;47(1):233–41.
[33] Ge ZM, Leu WY. Anti-control of chaos of two-degree-of-freedom louder speaker system and chaos system of different order system. Chaos, Solitons & Fractals 2004;20:503–21.
[34] Ge ZM, Cheng JW, Chen YS. Chaos anticontrol and synchronization of three time scales brushless DC motor system. Chaos, Solitons & Fractals 2004;22:1165–82.
[35] Ge ZM, Lee CI. Anticontrol and synchronization of chaos for an autonomous rotational machine system with a hexagonal centrifugal governor. Chaos, Solitons & Fractals 2005;282:635–48.
[36] Ge ZM, Lee CI. Control, anticontrol and synchronization of chaos for an autonomous rotational machine system with time-delay. Chaos, Solitons & Fractals 2005;23:1855–64.
[37] He GL, Zhou SP. What is the exact condition for fractional integrals and derivatives of Besicovitch functions to have exact box dimension? Chaos, Solitons & Fractals 2005;26:867–79.
[38] Yao K, Su WY, Zhou SP. On the connection between the order of fractional calculus and the dimensions of a fractal function. Chaos, Solitons & Fractals 2005;23:621–9.
[39] Guy J. Fractional master equation: Non-standard analysis and Liouville–Riemann derivative. Chaos, Solitons & Fractals 2001;12:2577–87.
[40] Elwakil SA, Zahran MA. Fractional integral representation of master equation. Chaos, Solitons & Fractals 1999;10:1545–8.

[41] Sun HH, Abdelwahad AA, Onaral B. IEEE Trans Autom Control 1984;29:441.
[42] Ichise M, Nagayanagi Y, Kojima T. J Electroanal Chem 1971;33:253.
[43] Heaviside O. Electromagnetic Theory. New York: Chelsea; 1971.
[44] Oustaloup A, Levron F, Nanot F, Mathieu B. Frequency band complex non integer differentiator: Characterization and synthesis. IEEE Trans CAS-I 2000;47:25–40.
[45] Chen YQ, Moore K. Discretization schemes for fractional-order differentiators and integrators. IEEE Trans CAS-I 2002;49:363–7.
[46] Hartley TT, Lorenzo CF. Dynamics and control of initialized fractional-order systems. Nonlinear Dyn 2002;29:2002–33.
[47] Hwang C, Leu JF, Tsay SY. A note on time-domain simulation of feedback fractional-order systems. IEEE Trans Auto Control 2002;47:625–31.
[48] Podlubny I, Petras I, Vinagre BM, OLeary P, Dorcak L. Analogue realizations of fractional-order controllers. Nonlinear Dyn 2002;29:281–96.
[49] Hartley TT, Lorenzo CF, Qammer HK. Chaos in a fractional order Chua's system. IEEE Trans CAS-I 1995;42:485–90.
[50] Arena P, Caponetto R, Fortuna L, Porto D. Chaos in a fractional order Duffing system. In: Proc ECCTD, Budapest, 1997, pp. 1259–62.
[51] Ahmad WM, Sprott JC. Chaos in fractional-order autonomous nonlinear systems. Chaos, Solitons & Fractals 2003;16:339–51.
[52] Ahmad WM, Harb WM. On nonlinear control design for autonomous chaotic systems of integer and fractional orders. Chaos, Solitons & Fractals 2003;18:693–701.
[53] Ahmad W, El-Khazali R, El-Wakil A. Fractional-order Wien-bridge oscillator. Electron Lett 2001;37:1110–2.
[54] Grigorenko L, Grigorenko E. Chaotic dynamics of the fractional Lorenz system. Phys Rev Lett 2003;91:034101.
[55] Arena P, Caponetto R, Fortuna L, Porto D. Bifurcation and chaos in noninteger order cellular neural networks. Int J Bifur Chaos 1998;7:1527–39.
[56] Arena P, Fortuna L, Porto D. Chaotic behavior in noninteger-order cellular neural networks. Phys Rev E 2000;61:776–81.
[57] Li CG, Chen G. Chaos and hyperchaos in fractional order Rössler equations. Physica A 2004;341:55–61.
[58] Li CG, Chen G. Chaos in the fractional order Chen system and its control. Chaos, Solitons & Fractals 2004;22:549–54.
[59] Li CP, Peng GJ. Chaos in Chen's system with a fractional order. Chaos, Solitons & Fractals 2004;22:443–50.
[60] Zaslavsky GM. Chaos, fractional kinetics, and anomalous transport. Phys Rep 2002;371:461–580.
[61] Lu JG. Chaotic dynamics and synchronization of fractional-order Arneodo's systems. Chaos, Solitons & Fractals 2005;26: 1125–33.
[62] Charef A, Sun HH, Tsao YY, Onaral B. Fractal system as represented by singularity function. IEEE Trans Auto Control 1992;37:1465–70.