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Stream depletion rate and volume from groundwater pumping in wedge-shape aquifers

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Summary   In river basins with several tributaries, wells in alluvial valleys are often located near the wedge-shaped confluence of two tributaries. Therefore, stream depletion characteristics, namely stream depletion rate (SDR) and stream depletion volume (SDV), induced by the pumping wells are distributed between two tributaries upstream from the confluence. Existing methods for the evaluation of stream depletion characteristics are applicable only for a wedge-shaped aquifer with special or uncommon wedge angles or for steady-state conditions. In water resources management, temporal stream depletion characteristics are important for the adjudication of water rights. A new and practically important method for analytical evaluation of transient SDR and SDV is developed for a wedge-shaped aquifer with arbitrary wedge angles. Results are obtained for each of two tributaries or the tributary segment lengths. A simple expression for the SDR fraction that originates from each tributary is presented for long period of pumping after which 100% of the pumping rate is supplied by stream depletion. The sensitivity analyses for different parameters are performed to examine their affects on the influence period of the total SDR and the results indicated that the SDR is sensitive to the angles of wedge-shaped aquifer and the pumping well. The results and sensitivity analysis presented in this study may provide useful information in the assessment of water balance or exploitation of river basins.
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Introduction

The hydrogeological concept of stream depletion is widely used for water resources management since the stream water is one of the main water sources. In river basins with several tributaries, wells in alluvial valleys are often located near the wedge-shaped confluence of two tributaries

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(Lambs, 2004). Therefore, stream depletion characteristics, namely stream depletion rate (SDR) and stream depletion volume (SDV), induced by the pumping well are distributed between two tributaries upstream from the confluence. The issues of depletion evaluation from the nearby streams have been addressed in the literature. Theis (1941) proposed a transient method to evaluate the effect of ground water pumping on a nearby stream. Glover and Balmer (1954) later generalized Theis' approach based on a series of idealistic assumptions and developed a formula expressed in terms of the distance of the well from the stream, the properties of the aquifer, and pumping time. Hantush (1965) extended this approach by considering an imperfect hydraulic connection between the aquifer and stream, but the extension has seen relatively little use in practice. Jenkins (1968) implemented the Theis/Glover-Balmer solutions to provide the standard tools and examples for the use in water management design. Using more realistic representation of streambed properties (see Zlotnik and Huang, 1999), Zlotnik et al. (1999) developed a semi-analytical solution for stream depletion due to pumping from a well near a stream. This solution considered the finite width of a stream with shallow penetration, streambed properties, and variations in aquifer transmissivity. In a special case of an aquifer of uniform saturated thickness, a fully analytical solution for stream depletion was developed. The solution is general enough to contain the solutions of Theis (1941), Glover and Balmer (1954), and Hantush (1965). Independently, Hunt (1999) derived compact analytical solutions for both stream depletion and aquifer drawdown with the assumptions that streambed penetration of the aquifer and dimensions of the streambed cross section are small. Using a sensitivity analysis, Christensen (2000) assessed potential stream depletion by using aquifer drawdown data and results of Hunt (1999). Butler et al. (2001) extended the results of Zlotnik et al. (1999) and Hunt (1999) to a heterogeneous aquifer of finite width. Hunt et al. (2001) and Kollet and Zlotnik (2003, 2007) applied these solutions to various field studies in New Zealand and the USA. Later, Fox et al. (2002) explored modifications of Hunt's solution (1999) to develop an analytical solution for drawdown in an aquifer with leakage from a finite-width stream. Hunt (2003a) generalized this solution to a case of delayed-yield aquifer. Moreover, he compared the new solution (Hunt, 2003a) with data obtained from a stream depletion field test (Hunt, 2003b). Lough and Hunt (2006) used data obtained from pumping tests for aquifer and streambed parameter evaluation based on the Hunt (2003) solution. A number of analytical models have been developed for leaky aquifers (e.g., Zlotnik, 2004; Butler et al., 2007; Zlotnik and Tartakovsky, 2008). And lately, Miller et al. (2007) described a modified Jenkins method to assess the effect of lateral impermeable aquifer boundaries.

However, in commonly occurring wedge-shaped aquifers, methods for evaluation of stream depletion characteristics are limited. Hantush (1967) considered the flow from a stream having a specific, right-angle bend to a nearby pumping well and developed an analytical solution to evaluate the stream depletion in such an aquifer system. Holzbecher (2005) gave a steady-state solution for an idealized case of two isopotentials, intersecting at an arbitrary angle, to determine the groundwater flow using a base flow component and pumping wells. Assumptions used for derivation of these solutions constrain their practical applications.

In water resources management, temporal stream depletion characteristics are important for the adjudication of water rights. This paper presents a new and practically important method for the analytical evaluation of transient SDR and SDV in a wedge-shaped aquifer with arbitrary wedge angles. Results are obtained for each of two tributaries or the tributary segment lengths. These solutions contain improper integrals that involve Bessel functions. The oscillatory nature of the integrands results in a slow integral convergence; therefore, a numerical approach that includes the roots search scheme, Gaussian quadrature, and Shanks' method, is proposed. This approach can evaluate the solutions accurately and quickly. A simple expression for SDR fraction that originates from each tributary is presented for very long times, when 100% of the pumping rate is supplied by stream depletion after a long period of pumping. Transient results for SDR are verified by comparing with a special case of Hantush (1967) solution. In addition, sensitivity analyses for different parameters are performed to study their affects on the influence period of the total SDR. Our results are obtained by assuming constant head along both tributaries. However, the slope of water level in the stream may be significant at large scale in some cases. Under these circumstances, the solutions given in Yeh and Chang (2006) can be used to generalize the estimations for SDR and SDV.

Mathematical model

Governing equation, boundary and initial conditions, and head solution

Consider radial system of coordinates (r, θ). A wedge-shaped aquifer with an angle of ϕ and constant head boundaries at $\theta = 0$ and $\theta = \phi$ is shown in Fig. 1. For a pumping well located at a point (r_0, θ_0) with a pumping rate Q, the partial differential equation governing the transient hydraulic head $h(r, \theta, t)$ at any point may be expressed as

$$T\left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2}\right) - S\frac{\partial h}{\partial t} = -\frac{Q}{r}\delta(r - r_0)\delta(\theta - \theta_0)$$
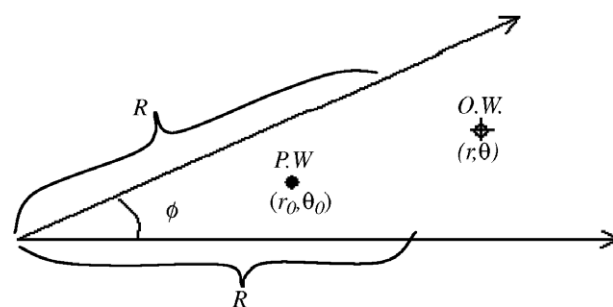
(1)

Figure 1 Schematic diagram of well location near the wedge-shaped confluence of two tributaries.

where T is the transmissivity (L^2/T); S is the storage coefficient; h is the hydraulic head (L); t is time from the start of the pumping test (T); r is the radial distance from the origin (L) and θ is the angle (radians) from the lower boundary. The initial and boundary conditions are as follows:

$$h(r, \theta, 0) = 0, \quad 0 \leqslant \theta \leqslant \phi, \ 0 \leqslant r \leqslant \infty \tag{2}$$

$$h(r, 0, t) = 0, \quad 0 \leqslant r \leqslant \infty \tag{3}$$

$$h(r, \phi, t) = 0, \quad 0 \leqslant r \leqslant \infty \tag{4}$$

The solution for hydraulic head in a wedge-shaped aquifer to Eqs. (1)–(4) can be expressed as (Chan et al., 1978; Yeh and Chang, 2006):

$$h(r, \theta, t) = \frac{2}{\phi} \frac{Q}{T} \sum_{n=1}^{\infty} \sin(\mu_n \theta) \sin(\mu_n \theta_0)(\Omega_1 + \Omega_2) \tag{5}$$

with

$$\Omega_1 = \int_0^{\infty} \left[-\exp\left(\frac{-u^2 T}{S} t\right) \right] J_{\mu_n}(ur_0) J_{\mu_n}(ur) \frac{du}{u} \tag{6}$$

and

$$\Omega_2 = \int_0^{\infty} J_{\mu_n}(ur_0) J_{\mu_n}(ur) \frac{du}{u} \tag{7}$$

and

$$\mu_n = \frac{n\pi}{\phi} \tag{8}$$

where u is the dummy variable, and $J_{\mu_n}(\cdot)$ is the Bessel function of the first kind with order μ_n.

Transient stream depletion rate (SDR) for a tributary segment length in wedge-shaped aquifer

The SDR for a tributary segment length that starts at the origin of coordinates and ends at radial distance R measured from the origin, can be written for each of tributaries. We denote the stream depletion characteristics related to the tributary at angle $\theta = 0$ by index "0" and stream depletion characteristics related to the tributary at angle $\theta = \phi$ by index "ϕ". SDR for each tributary is estimated as follows

$$\frac{q_0}{Q} = \int_0^R T \frac{1}{r} \frac{\partial h(r, \theta, t)}{\partial \theta} \, dr \bigg|_{\theta=0} \tag{9}$$

and

$$\frac{q_\phi}{Q} = -\int_0^R T \frac{1}{r} \frac{\partial h(r, \theta, t)}{\partial \theta} \, dr \bigg|_{\theta=\phi} \tag{10}$$

where h is given by Eq. (5).

The solutions for the SDR by Eqs. (9) and (10) from Eqs. (1)–(8) can be written as:

$$\frac{q_0}{Q} = D\left(\frac{t}{t_a}, \frac{R}{r_0}, 0\right) \tag{11}$$

and

$$\frac{q_\phi}{Q} = D\left(\frac{t}{t_a}, \frac{R}{r_0}, 1\right) \tag{12}$$

where t_a is a characteristic time scale introduced by Jenkins (1968) and defined as:

$$t_a = \frac{Sr_0^2}{T} \tag{13}$$

The function of $D(u,v,w)$ has two forms for different ranges of R/r_0:

$$
\begin{aligned}
D(u, v, w) = {}& -\frac{2}{\phi} \sum_{n=1}^{\infty} \left[(-1)^w\right]^n \mu_n \sin(\mu_n \theta_0) \\
& \times \int_0^{\infty} \frac{1}{\xi} \exp(-\xi^2 u) J_{\mu_n}(\xi) \int_0^v \frac{1}{\sigma} J_{\mu_n}(\xi\sigma) \, d\sigma \, d\xi \\
& + (-1)^w \frac{1}{\pi} \tan^{-1}\left[\frac{(-1)^w (R/r_0)^{\pi/\phi} \sin(\pi\theta_0/\phi)}{1 - (-1)^w (R/r_0)^{\pi/\phi} \cos(\pi\theta_0/\phi)} \right] \\
& \text{for } R/r_0 < 1
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
D(u, v, w) = {}& -\frac{2}{\phi} \sum_{n=1}^{\infty} \left[(-1)^w\right]^n \mu_n \sin(\mu_n \theta_0) \\
& \times \int_0^{\infty} \frac{1}{\xi} \exp(-\xi^2 u) J_{\mu_n}(\xi) \int_0^v \frac{1}{\sigma} J_{\mu_n}(\xi\sigma) \, d\sigma \, d\xi \\
& + (-1)^{w-1} \frac{1}{\pi} \tan^{-1}\left[\frac{(-1)^w (r_0/R)^{\pi/\phi} \sin(\pi\theta_0/\phi)}{1 - (-1)^w (r_0/R)^{\pi/\phi} \cos(\pi\theta_0/\phi)} \right] \\
& + (1-w) + (-1)^{w-1} \frac{\theta_0}{\phi} \quad \text{for } R/r_0 > 1
\end{aligned}
\tag{15}
$$

where σ and ξ are the dummy variables.

Transient stream depletion rate (SDR) for tributaries

The SDRs for both tributaries can be obtained by setting $R/r_0 = \infty$ in Eqs. (11) and (12) as:

$$\frac{q_0}{Q} = -\frac{2}{\phi} \sum_{n=1}^{\infty} \sin(\mu_n \theta_0) \int_0^{\infty} \frac{1}{\xi} \exp\left(-\xi^2 \frac{t}{t_a}\right) J_{\mu_n}(\xi) \, d\xi + 1 - \frac{\theta_0}{\phi} \tag{16}$$

and

$$\frac{q_\phi}{Q} = \frac{2}{\phi} \sum_{n=1}^{\infty} (-1)^n \sin(\mu_n \theta_0) \int_0^{\infty} \frac{1}{\xi} \exp\left(-\xi^2 \frac{t}{t_a}\right) J_{\mu_n}(\xi) \, d\xi + \frac{\theta_0}{\phi} \tag{17}$$

Steady stream depletion rate for tributaries

When the flow system reaches steady state, 100% of the pumping rate is supplied by stream depletion. Then a simple expression for SDR fraction for each tributary is as follows:

$$\frac{q_0}{Q} = 1 - \frac{\theta_0}{\phi} \tag{18}$$

and

$$\frac{q_\phi}{Q} = \frac{\theta_0}{\phi} \tag{19}$$

These simple relationships play an important role in constraining the SDR. To our knowledge, no such explicit relationships have been published previously.

Transient stream depletion volume (SDV) for a tributary segment length

Define the SDV as

$$\frac{V}{Qt} = \frac{1}{t} \int_0^t \frac{q}{Q} \, dt' \tag{20}$$

Then the SDV for the each tributary can, respectively, be written as

$$\frac{v_0}{Qt} = E\left(\frac{t}{t_a}, \frac{R}{r_0}, 0\right) \tag{21}$$

and

$$\frac{v_\phi}{Qt} = E\left(\frac{t}{t_a}, \frac{R}{r_0}, 1\right) \tag{22}$$

Accordingly, the functions of $E(u,v,w)$ for different ranges of R/r_0 are:

$$E(u,v,w) = \frac{2}{\phi}\frac{t_a}{t}\sum_{n=1}^{\infty}\left[(-1)^w\right]^n \mu_n \sin(\mu_n\theta_0)$$
$$\times \int_0^\infty \frac{1}{\xi^3}\left[\exp(-\xi^2 u) - 1\right]J_{\mu_n}(\xi)\int_0^v \frac{1}{\sigma}J_{\mu_n}(\xi\sigma)\,d\sigma\,d\xi$$
$$+ (-1)^w \frac{1}{\pi}\tan^{-1}\left[\frac{(-1)^w(R/r_0)^{\pi/\phi}\sin(\pi\theta_0/\phi)}{1-(-1)^w(R/r_0)^{\pi/\phi}\cos(\pi\theta_0/\phi)}\right]$$
$$\text{for } R/r_0 < 1 \tag{23}$$

and

$$E(u,v,w) = \frac{2}{\phi}\frac{t_a}{t}\sum_{n=1}^{\infty}\left[(-1)^w\right]^n \mu_n \sin(\mu_n\theta_0)$$
$$\times \int_0^\infty \frac{1}{\xi}\exp(-\xi^2 u)J_{\mu_n}(\xi)\int_0^v \frac{1}{\sigma}J_{\mu_n}(\xi\sigma)\,d\sigma\,d\xi$$
$$+ (-1)^{w-1}\frac{1}{\pi}\tan^{-1}\left[\frac{(-1)^w(r_0/R)^{\pi/\phi}\sin(\pi\theta_0/\phi)}{1-(-1)^w(r_0/R)^{\pi/\phi}\cos(\pi\theta_0/\phi)}\right]$$
$$+ (1-w) + (-1)^{w-1}\frac{\theta_0}{\phi} \quad \text{for } R/r_0 > 1 \tag{24}$$

Transient stream depletion volume for tributaries

The SDVs for each tributary can be obtained by setting $R/r_0 = \infty$ in Eqs. (21) and (22) as:

$$\frac{v_0}{Qt} = \frac{2}{\phi}\frac{t_a}{t}\sum_{n=1}^{\infty}\sin(\mu_n\theta_0)$$
$$\times \int_0^\infty \frac{1}{\xi^3}\left[\exp\left(-\xi^2\frac{t}{t_a}\right) - 1\right]J_{\mu_n}(\xi)\,d\xi + \left(1-\frac{\theta_0}{\phi}\right) \tag{25}$$

and

$$\frac{v_\phi}{Qt} = -\frac{2}{\phi}\frac{t_a}{t}\sum_{n=1}^{\infty}(-1)^n\sin(\mu_n\theta_0)$$
$$\times \int_0^\infty \frac{1}{\xi^3}\left[\exp\left(-\xi^2\frac{t}{t_a}\right) - 1\right]J_{\mu_n}(\xi)\,d\xi + \frac{\theta_0}{\phi} \tag{26}$$

Numerical evaluations

The improper integrals of the solutions of Eqs. (14)–(16), (17), (24)–(26) converge slowly because of oscillating Bessel functions. These integrals can be transformed as a sum of infinite series and each term of the series is obtained by integrating the area between the integrand and the horizontal axis from two consecutive roots. Therefore, Newton's method with suggested increments given below is employed to find the consecutive roots of the integrand

along the u-axis. For each area between the integrand and the horizontal axis, Gaussian quadrature (e.g., Gerald and Wheatley, 1989) is chosen to perform numerical integrations. Finally, Shanks' method is applied to accelerate the convergence when evaluating the Bessel functions, trigonometric functions and the infinite series in these solutions (e.g., Yang and Yeh, 2002; Peng et al., 2002; Yeh et al., 2003).

The integrands of the solutions are oscillatory due to the nature of Bessel function $J_v(u)$. One may find the roots of the integrand from the roots of $J_v(u)$, which may be found by a conjunctive use of the following suggested increments and Newton's method. An asymptotic expansion of the large positive ith root of $J_v(u)$ for a given v is (Abramowitz and Stegun, 1964)

$$j_{v,i} = \beta - (4v^2 - 1)/8\beta - [4(4v^2 - 1)(28v^2 - 31)]/3(8\beta)^3 \tag{27}$$

with $\beta = (\pi/4)(2v + 4i - 1)$. The increments Δ_1 from the origin to the first root can be approximated by $j_{v,1}$. The remaining increments Δ_i are chosen as $j_{v,i-1} - j_{v,i-2}$, and the remaining roots are approximately equal to $j_{v,i} = j_{v,i-1} + \Delta_i$, where $i = 2,3,\ldots$. The roots of $J_v(u)$ determined by Newton's method and the suggested increments are of accuracy to the seventh digit.

Sensitivity analysis of the parameters

Sensitivity analysis provides a way of examining the response of a model to changes in its parameters. It also helps in assessing how well parameters are likely to be estimated from the data available for model calibration. The parametric sensitivity is a measure of the effect of change in one factor on another factor. It may be mathematically expressed as (McCuen, 1985)
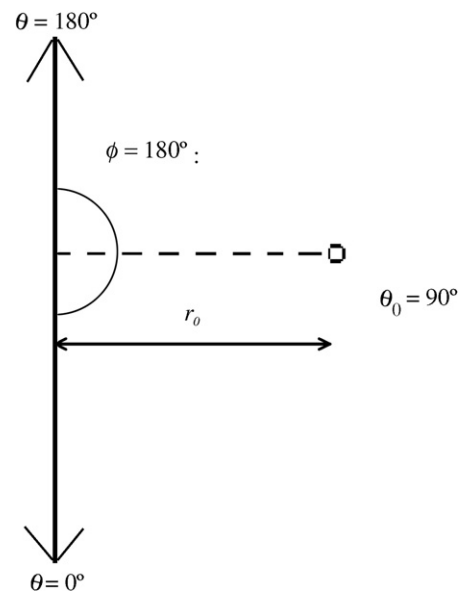

Figure 2 The configuration of a well pumping near an infinite stream.

Table 1 Values of q/Q for the present solution and Hantush's solution corresponding to the values of dimensionless time t/t_a

t/t_a	q/Q (Hantush's solution)	q/Q (Present solution)
5.00×10^{-3}	5.303×10^{-7}	5.960×10^{-7}
6.00×10^{-3}	4.991×10^{-6}	5.006×10^{-6}
7.00×10^{-3}	2.387×10^{-5}	2.384×10^{-5}
8.00×10^{-3}	7.722×10^{-5}	7.724×10^{-5}
9.00×10^{-3}	1.9397×10^{-4}	1.9395×10^{-4}
1.00×10^{-2}	4.069×10^{-4}	4.069×10^{-4}
2.00×10^{-2}	1.243405×10^{-2}	1.243405×10^{-2}
3.00×10^{-2}	4.161701×10^{-2}	4.161700×10^{-2}
4.00×10^{-2}	7.912989×10^{-2}	7.912991×10^{-2}
5.00×10^{-2}	0.11931377	0.11931379
6.00×10^{-2}	0.15948459	0.15948461
7.00×10^{-2}	0.19834209	0.19834213
8.00×10^{-2}	0.23526249	0.23526247
9.00×10^{-2}	0.26998323	0.26998324
0.1	0.30244242	0.30244244
0.15	0.4340228	0.43402286
0.2	0.52674783	0.52674781
0.25	0.59435953	0.59435953
0.3	0.64547785	0.64547786
0.35	0.68534840	0.68534841
0.4	0.71725890	0.71725888
0.45	0.74334990	0.74334991
0.5	0.76506645	0.76506645
0.55	0.78341593	0.78341594
0.6	0.79912062	0.79912063
0.65	0.81271115	0.81271116
0.7	0.8245858	0.82458581
0.75	0.8350491	0.83504911
0.8	0.84433780	0.84433778
0.85	0.85263864	0.85263864
0.9	0.86010089	0.86010090
0.95	0.86684525	0.86684525
1	0.87297037	0.87297037
1.1	0.88367482	0.8836748
1.2	0.89271750	0.89271750
1.3	0.90045709	0.90045709
1.4	0.90715608	0.90715608
1.5	0.91301094	0.91301093
1.6	0.91817166	0.91817165
1.7	0.92275469	0.92275470
1.8	0.92685185	0.92685186
1.9	0.93053648	0.93053647
2	0.93386785	0.93386785
2.2	0.93965621	0.93965621
2.4	0.94451314	0.94451313
2.6	0.94864665	0.94864664
2.8	0.95220713	0.95220713
3	0.95530599	0.95530600
3.5	0.96154049	0.96154049
4	0.96624874	0.96624874
4.5	0.96993003	0.96993003
5	0.97288730	0.97288730
5.5	0.97531502	0.97531502
6	0.97734373	0.97734372
7	0.98054204	0.98054201

Table 1 (*continued*)

t/t_a	q/Q (Hantush's solution)	q/Q (Present solution)
8	0.98294906	0.98294904
9	0.98482613	0.98482613
10	0.98633090	0.98633092
15	0.99086201	0.99086202
20	0.99313700	0.99313701
25	0.99450503	0.99450504
30	0.99541836	0.99541832
35	0.99607128	0.99607129
40	0.99656133	0.99656136
45	0.99694271	0.99694272
50	0.99724790	0.99724794
100	0.99862283	0.99862282
200	0.99931111	0.99931112
300	0.99954068	0.99954068
400	0.99965548	0.99965549
500	0.99972437	0.99972438
600	0.99977030	0.99977031
700	0.99980310	0.99980310
800	0.99982771	0.99982772
900	0.99984685	0.99984686
1000	0.99986216	0.99986217
2000	0.99993109	0.99993108
3000	0.99995406	0.99995405
4000	0.99996553	0.99996554
5000	0.99997244	0.99997243

$$S_{pi} = \frac{\partial O}{\partial P_i} = \frac{f(P_i + \Delta P_i; P_{j|j \neq i}) - f(P_1, P_2, \ldots P_n)}{\Delta P_i} \quad (28)$$

where O is the output function of the system and P_i is the ith input parameter of the system. The sensitivity of Eq. (28) can be normalized by the parameter value so that the sensitivity coefficient with respect to any given parameter is in the same unit. The normalized sensitivity is defined as

$$S_{i,t} = \frac{\partial O}{\partial P_i / P_i} = P_i \frac{\partial O}{\partial P_i} \quad (29)$$

where $S_{i,t}$ is the sensitivity coefficient of ith input parameter at time t. The partial derivative of Eq. (29) can be approximated by a finite difference formula expressed as:

$$\frac{\partial O}{\partial P_i} = \frac{O(P_i + \Delta P_i) - O(P_i)}{\Delta P_i} \quad (30)$$

The increment in the denominator may be chosen as the parameter value times a factor of 10^{-3}, i.e., $\Delta P_i = 10^{-3} P_i$ (Yeh, 1987). Eq. (29) measures the influence that the fractional change in the parameter, or its relative error, exerts on the output (Huang and Yeh, 2007).

Results and discussion

Transient stream depletion rate/volume for different angle configurations of wedge confluence and pumping well

Several cases are considered in this paper to illustrate the SDR and SDV for wedge-shaped aquifers of different angles.
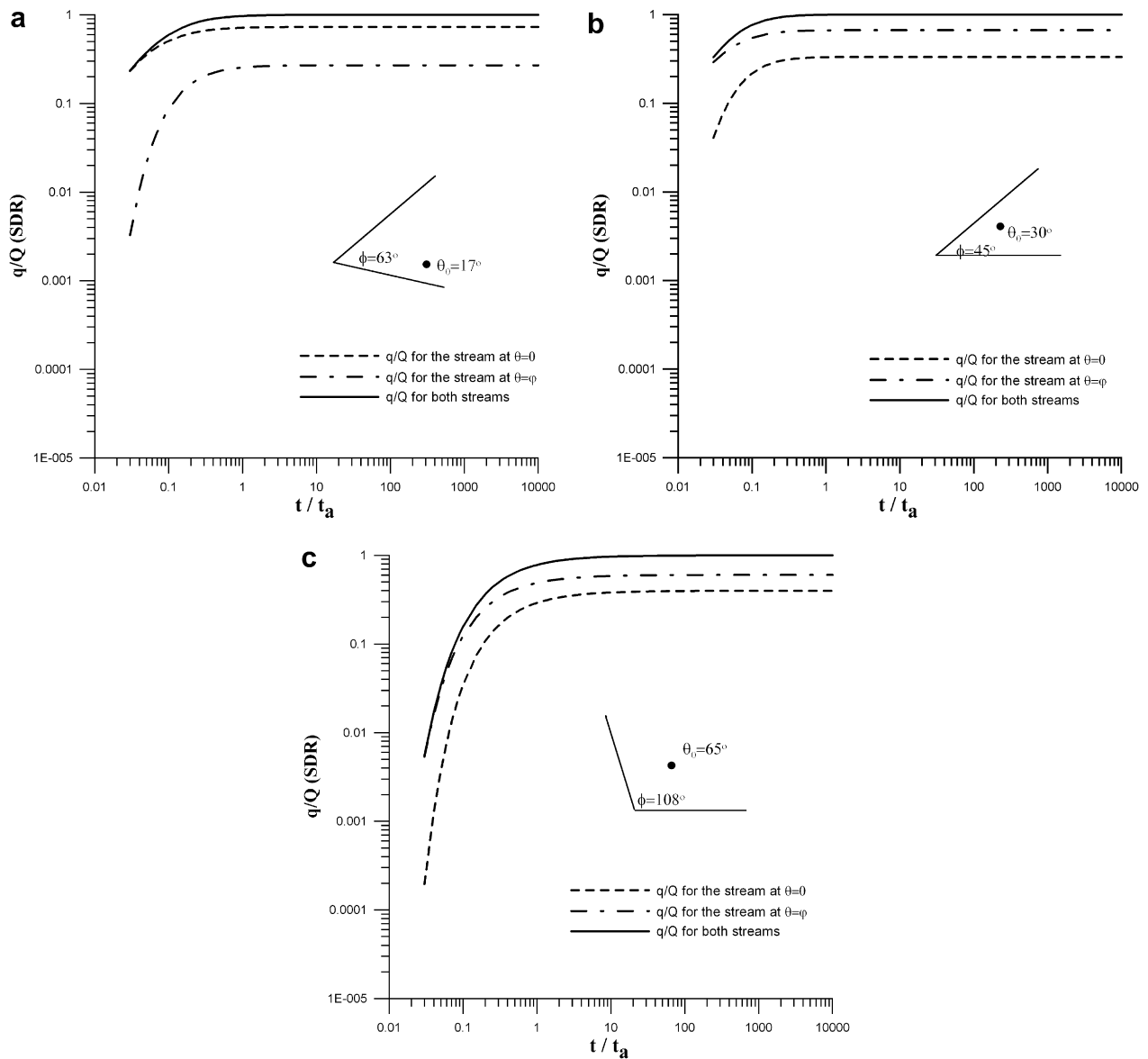
Figure 3   The dimensionless transient SDR against dimensionless time t/t_a for infinite tributaries at $\theta = 0$ and $\theta = \phi$ with different angle configurations of wedge confluence and pumping well (a) $\phi = 63°$ and $\theta_0 = 17°$; (b) $\phi = 45°$ and $\theta_0 = 30°$ and (c) $\phi = 108°$ and $\theta_0 = 65°$.

For groundwater pumping near an infinite stream as shown in Fig. 2 (i.e., $\phi = \pi$), the solution should reduce to that of Glover and Balmer (1941):

$$\frac{q}{Q} = \frac{q_0}{Q} + \frac{q_\pi}{Q} = 1 - \text{erf}\left(\frac{1}{2}\sqrt{\frac{t_a}{t}}\right) \tag{31}$$

This is demonstrated in Appendix.

For the depletion of flow in right-angle stream bends (i.e., $\phi = \pi/2$), the values of stream depletion rate evaluated from the presented solution should be identical to Hantush (1967) solution:

$$\frac{q}{Q} = \frac{q_0}{Q} + \frac{q_{\pi/2}}{Q} = 1 - \text{erf}\left(\frac{\sin\theta_0}{2}\sqrt{\frac{t_a}{t}}\right)\text{erf}\left(\frac{\cos\theta_0}{2}\sqrt{\frac{t_a}{t}}\right) \tag{32}$$

However, identity of our solution to Eq. (32) could not be proven using analytical techniques. Therefore, we compared values of q/Q for both Hantush's solution and the present solution with $\phi = \pi/2$ using numerical techniques. As shown in Table 1, there are only small differences in the seventh decimal place between Hantush's solution and the present solution. The solution obtained by Hantush contains the error function which was evaluated by the function DERF of IMSL (1997). The error function calculated by the DERF is of accuracy less than the seventh digit if compared with the error function value given in Abramowitz and Stegun (1964). Thus, the differences may arise from the numerical evaluation in DERF.

Figs. 3a–c and 4a–c show the transient SDR and SDV, respectively, assuming $R/r_0 = \infty$ at $\theta = 0$ and $\theta = \phi$ with different combinations of angle for the wedge confluence and
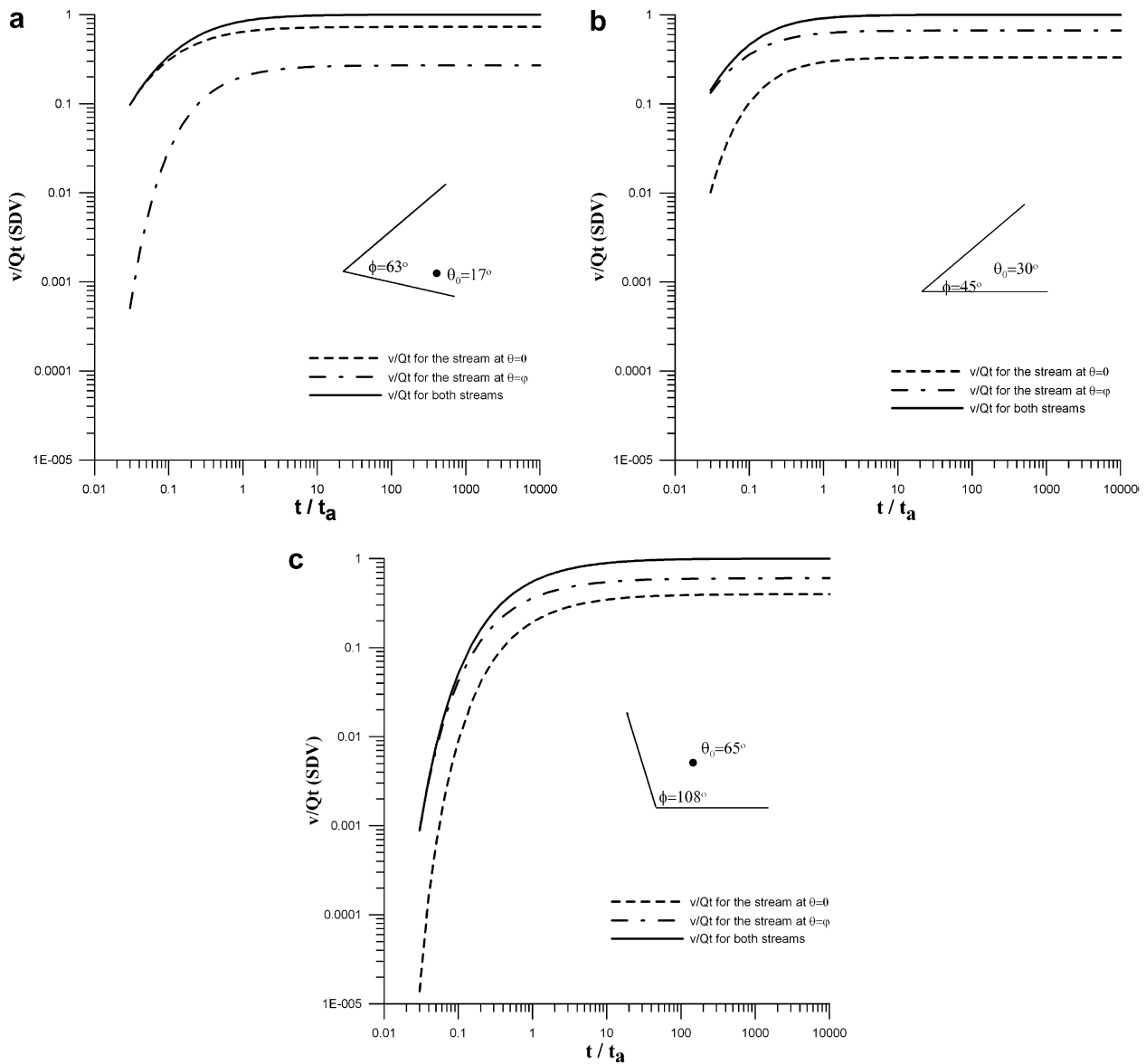
Figure 4 The dimensionless transient SDV against dimensionless time t/t_a for infinite tributaries at $\theta = 0$ and $\theta = \phi$ with different angle configurations of wedge confluence and pumping well (a) $\phi = 63°$ and $\theta_0 = 17°$; (b) $\phi = 45°$ and $\theta_0 = 30°$ and (c) $\phi = 108°$ and $\theta_0 = 65°$.

pumping well. The solid lines denote total SDR/SDV for both streams, dash lines represent SDR/SDV from the tributary at $\theta = 0$ and the dot-dash lines represent SDR/SDV from the tributary at $\theta = \phi$. As indicated in these figures, the stream near the pumping well contributes more recharge to the pumping well. These figures illustrate how SDR and SDV depend on the wedge angle and the angles of the pumping well from the tributaries.

Transient stream depletion rate/volume with different tributary segment lengths at $\theta = 0$

Cases with different tributary segment lengths are considered to assess the influence of tributary segment length on the quantities of SDR and SDV at $\theta = 0$ in a wedge-shaped aquifer with $\phi = 63°$ and $\theta_0 = 17°$. Fig. 5a and b illustrate the

values of SDR and SDV, respectively, with different tributary segment lengths $R/r_0 = 0.25, 0.5, 0.75, 1, 5$ and ∞ at $\theta = 0$. Those two figures show that the values of SDR and SDV increase with the tributary segment length and these two values for the case of $R/r_0 = 5$ are almost identical with those of $R/r_0 = \infty$. It indicates that the SDR and SDV induced by the pumping well are mostly contributed from tributary segment lengths within $R/r_0 = 5$ at $\theta = 0$.

Sensitivity analysis of parameters

Consider that the distance r_0 from the confluence to the pumping well is $r_0 = 400$ m, the transmissivity T is 1 m^2/min (6.944×10^{-4} m^2/day), storage coefficient S is 0.2, the angle of wedge-shaped aquifer ϕ is $80°$, and the angle of pumping well θ_0 is $40°$. The total transient SDR and the
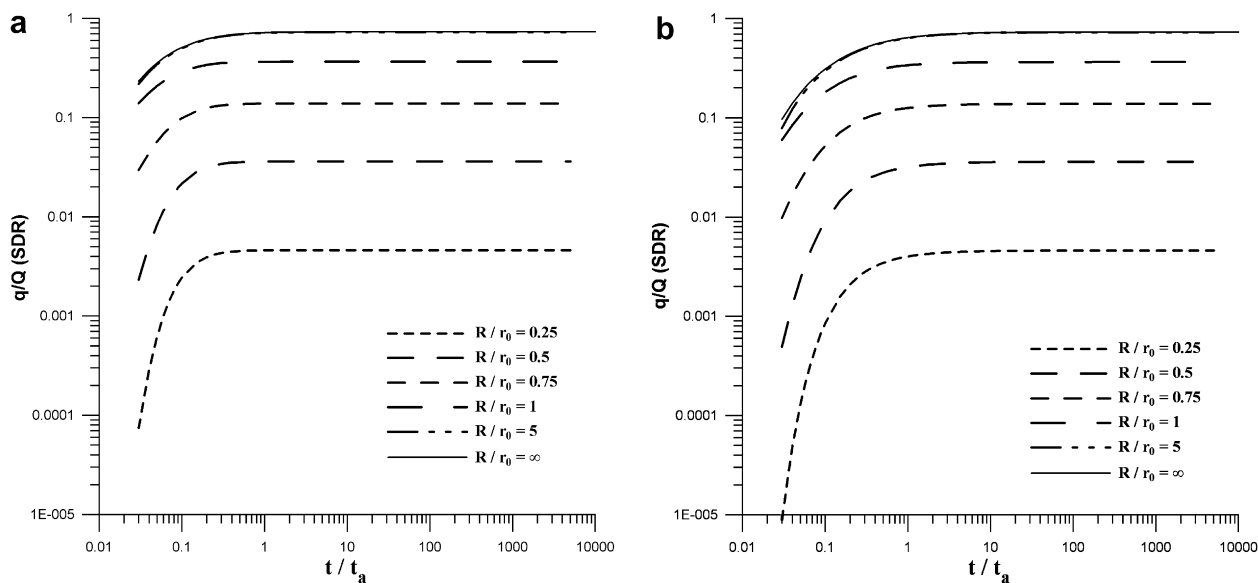
Figure 5 (a) The dimensionless transient SDR against dimensionless time t/t_a for different tributary segment lengths $R/r_0 = 0.25$, 0.5, 0.75, 1, 5 and ∞ and (b) The dimensionless transient SDV against dimensionless time t/t_a for different tributary segment lengths $R/r_0 = 0.25$, 0.5, 0.75, 1, 5 and ∞.

sensitivity coefficients of T, S and r_0 are plotted in Fig. 6. This figure indicates that the temporal distribution of each sensitivity coefficient of the parameters reflects the temporal change of the SDR in response to the relative change of each parameter. The non-zero periods in the sensitivity coefficient curves imply that the parameter T has a positive influence and the parameters S and r_0 have negative influences on the total SDR. In addition, those sensitivity coefficient curves show that T, S and r_0 influence SDR beginning at $t = 500$ min (0.347 days), maximizing at $t = 4000$ min (2.777 days), and then terminating at $t = 10^6$ min (694.44 days). In fact, those sensitivity coefficient curves indicate that
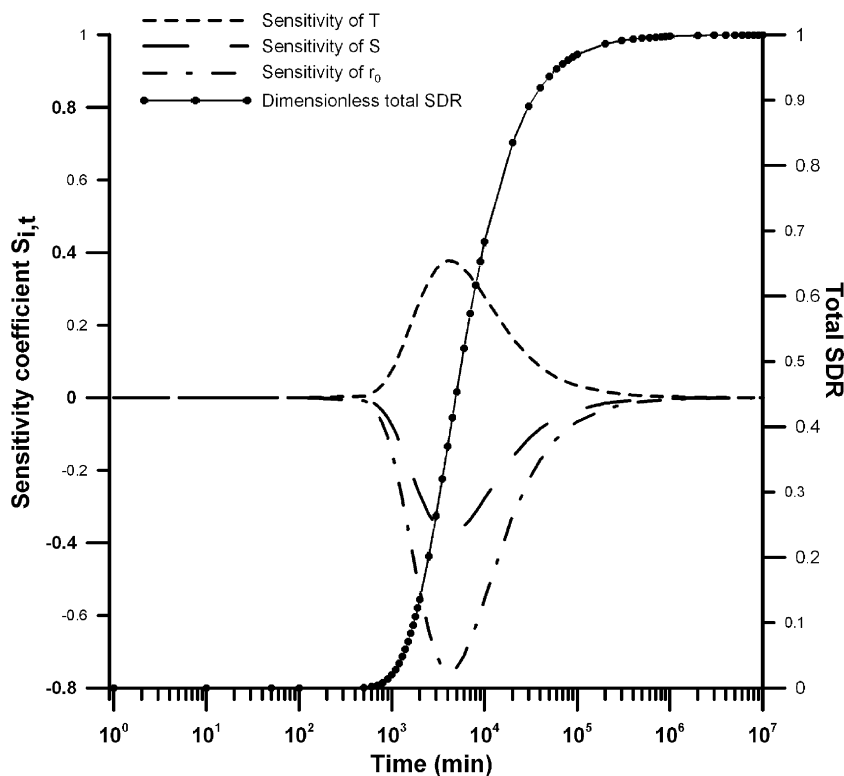

Figure 6 The dimensionless transient SDR and the sensitivity coefficients of the wedge-shaped aquifer parameters T, S and r_0 ($r_0 = 400$ m, $T = 1$ m^2/min, $S = 0.2$, $\theta_0 = 40°$ and $\phi = 80°$).
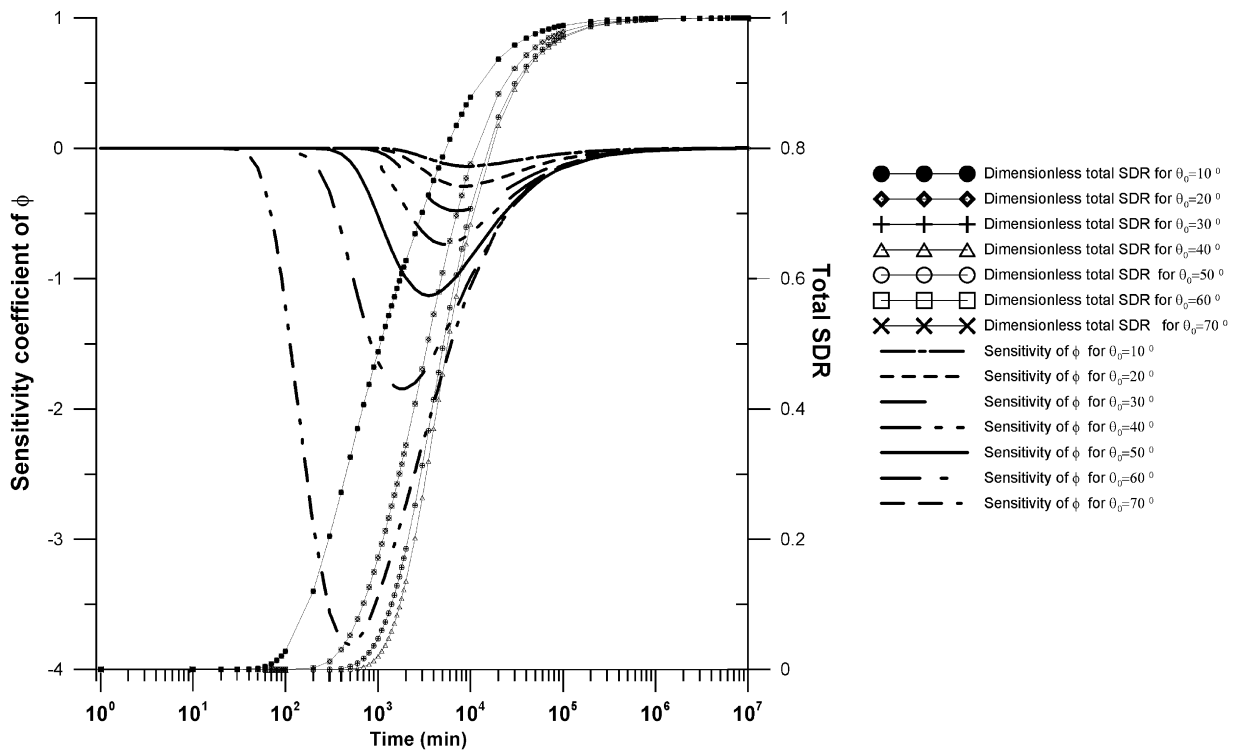
Figure 7 The dimensionless transient SDR and the sensitivity coefficients of ϕ for different angles of pumping well θ_0 ranging from 10° to 70° with an increment of 10°. ($r_0 = 400$ m, $T = 1$ m^2/min, $S = 0.2$, and $\phi = 80$°).

the total SDR starts to react to the pumping at 500 min (0.347 days), has an inflection point at 4000 min (2.777 days), and reaches steady state at 10^6 min. Fig. 7 shows the total transient SDR and the sensitivity of ϕ for θ_0 from 10° to 70° with an increment of 10°. As illustrated in the figure, the time that the total transient SDR starts to response increases with θ_0 and the curves of total SDR for θ_0 equal 10° and 70°, 20° and 60°, and 30° and 50° are the same since they are symmetrical to the line at $\theta_0 = 40$°. This figure also indicates that the parameter ϕ has a negative influence on the total SDR, and the maximum sensitivity of ϕ increases with θ_0. Such a phenomenon can be related to the configuration of the systems. The total SDR is mainly contributed from the stream at $\theta = 0$ when $\theta_0 = 10$° and the total SDR is mainly contributed from the stream at $\theta = \phi$ when $\theta_0 = 70$°. Thus, the total SDR induced by a pumping well which is closer to the stream at $\theta = \phi$ is more sensitive to the parameter ϕ. Fig. 8 shows the sensitivity of parameter θ_0 for θ_0 from 10° to 70° with an increment of 10°. The maximum sensitivity coefficient of θ_0 decreases when θ_0 increases and approaches zero at $\theta_0 = 40$°. Notice that the maximum sensitivity of θ_0 begins to be positive when $\theta_0 = 50$° and the maximum sensitivity of θ_0 increases with θ_0. These results are attributed to the location of the pumping well. If the pumping well is closer to the stream, the total SDR will be more sensitive to the θ_0.
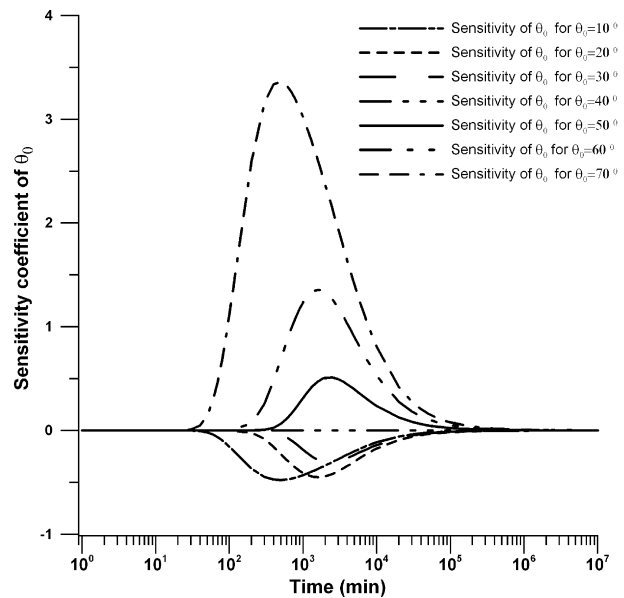
Concluding remarks

New analytical solutions of transient and steady-state stream depletion rate (SDR) and stream depletion volume (SDV) for a wedge-shaped aquifer are developed for arbi-

Figure 8 The sensitivity coefficients of θ_0 for different angles of pumping well θ_0 ranging from 10° to 70° with an increment of 10°. ($r_0 = 400$ m, $T = 1$ m^2/min, $S = 0.2$, and $\phi = 80$°).

trary angles of the wedge confluence. In the steady-state case, these solutions have a very simple structure that does not involve complicated computations. In transient cases, the solutions consist of infinite series with an improper integral involving Bessel functions. Therefore, a numerical

approach, including the use of Newton's method, Gaussian quadrature, and Shanks' method, is used for efficiently evaluating the solutions.

The solution for evaluating SDR in an infinite stream has been shown analytically to be identical to that of Glover and Balmer (1954). For a right-angle stream, it was shown numerically that the values of the proposed solution are almost identical to these of Hantush's solution (1967). These newly-derived solutions can be used to evaluate the transient SDR and SDV for a wedge-shaped aquifer with arbitrary wedge angles and tributary segment lengths.

From the results of the sensitivity analysis of parameters, it was found that the total SDR has different sensitivities for different angles of wedge-shaped aquifer or pumping well, and the differences in the angles of the wedge-shaped aquifer and the pumping well can be an important factor. The sensitivity analyses for different parameters in the SDR and SDV solutions in investigating their influence period of SDR and SDV provide useful information to examine the response of the wedge-shape aquifers to the change of the related parameters.

The results of this paper are useful in computerized assessment of water balance of river basins in the wedge-shaped aquifers. For rapid stream depletion assessment and practical applications, the present graphs can be utilized with reasonable accuracy.

Acknowledgements

Appendix A

For groundwater pumping near an infinite stream, the total SDR can be obtained by Eq. (A1) and substituting $\phi = \pi$ and $\theta_0 = \frac{\pi}{2}$ into Eqs. (16) and (17). Then,

$$\frac{q}{Q} = \frac{q_0}{Q} + \frac{q_\pi}{Q}, \tag{A1}$$

and

$$\frac{q}{Q} = 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} [1 - (-1)^n] \sin\left(n\frac{\pi}{2}\right)$$
$$\times \int_0^\infty \exp\left(-\xi^2 \frac{t}{t_a}\right) J_n(\xi) \frac{d\xi}{\xi} \tag{A2}$$

Rearranging Eq. (A2), the total SDR can be expressed as

$$\frac{q}{Q} = 1 - \frac{4}{\pi} \int_0^\infty \exp\left(-\xi^2 \frac{t}{t_a}\right) \sum_{n=0}^{\infty} J_{2n+1}(\xi)$$
$$\times \sin\left[(2n+1)\frac{\pi}{2}\right] \frac{d\xi}{\xi} \tag{A3}$$

The summation of Eq. (A3) can be obtained based on the formula (8.514−6) of Jeffrey et al. (1980) as

$$\sum_{n=0}^{\infty} J_{2n+1}(\xi) \sin\left[(2n+1)\frac{\pi}{2}\right] = \frac{\sin(\xi)}{2} \tag{A4}$$

Therefore, Eq. (A3) can be simplified as

$$\frac{q}{Q} = 1 - \frac{2}{\pi} \int_0^\infty \exp\left(-\xi^2 \frac{t}{t_a}\right) \sin(\xi) \frac{d\xi}{\xi} \tag{A5}$$

In addition, the integrand of Eq. (A5) can be expressed as the summation form based on the formula (3.956−6) of Jeffrey et al. (1980) as

$$\frac{2}{\pi} \int_0^\infty \exp\left(-\xi^2 \frac{t}{t_a}\right) \sin(\xi) \frac{d\xi}{\xi}$$
$$= 1 - \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\sqrt{\frac{t_a}{4t}}\right)^{2k+1}}{k!(2k+1)} = 1 - \mathrm{erf}\left(\frac{1}{2}\sqrt{\frac{t_a}{t}}\right) \tag{A6}$$

The total SDR for the impact of groundwater pumping near an infinite stream can then be written as

$$\frac{q}{Q} = 1 - \mathrm{erf}\left(\frac{1}{2}\sqrt{\frac{t_a}{t}}\right) \tag{A7}$$

Thus, the present solution is identical to the solution in Glover and Balmer (1954), and their solution can be considered as a special case of our transient SDR solution for infinite tributaries.

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