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Piecewise linear maps, Liapunov exponents and entropy

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Abstract

Let $\mathcal{L}_{\mathbf{A}} = \{f_{\mathbf{A},\mathbf{x}}\colon\ \mathbf{x}$ is a partition of $[0, 1]\}$ be a class of piecewise linear maps associated with a transition matrix $\mathbf{A}$. In this paper, we prove that if $f_{\mathbf{A},\mathbf{x}} \in \mathcal{L}_{\mathbf{A}}$, then the Liapunov exponent $\lambda(\mathbf{x})$ of $f_{\mathbf{A},\mathbf{x}}$ is equal to a measure theoretic entropy $h_{m_{\mathbf{A},\mathbf{x}}}$ of $f_{\mathbf{A},\mathbf{x}}$, where $m_{\mathbf{A},\mathbf{x}}$ is a Markov measure associated with $\mathbf{A}$ and $\mathbf{x}$. The Liapunov exponent and the entropy are computable by solving an eigenvalue problem and can be explicitly calculated when the transition matrix $\mathbf{A}$ is symmetric. Moreover, we also show that $\max_{\mathbf{x}} \lambda(\mathbf{x}) = \max_{\mathbf{x}} h_{m_{\mathbf{A},\mathbf{x}}} = \log(\lambda_1)$, where λ_1 is the maximal eigenvalue of $\mathbf{A}$.

Keywords: Piecewise linear map; Liapunov exponents; Entropy; Ergodic theory

1. Introduction

Our main concern here is to relate entropy to Liapunov exponents. Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ transition matrix, i.e., $a_{ij} = 0$ or 1 for all i, j. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, where $x_i > 0$ for all i and $\sum_{i=1}^n x_i = 1$. We shall call such $\mathbf{x}$ a partition of the interval [0, 1]. Let $I_j = [\sum_{i=1}^{j-1} x_i, \sum_{i=1}^{j} x_i]$, where $x_0 = 0$, and $\alpha_i = \{1 \leqslant j \leqslant n\colon\ a_{ij} = 1\}$. We then define $\mathcal{L}_{\mathbf{A}}$ as

$$\mathcal{L}_{\mathbf{A}} = \bigl\{f_{\mathbf{A},\mathbf{x}}\colon\ \mathbf{x} \text{ is a partition of } [0, 1]\bigr\}. \tag{1.1}$$

Here $f_{\mathbf{A},\mathbf{x}}$ is a piecewise linear map satisfying

$$f(I_i) = \bigcup_{j \in \alpha_i} I_j \tag{1.2a}$$

and

$$f'(t) = \frac{(\mathbf{A}\mathbf{x})_i}{x_i} =: s_i \quad \text{for } t \in \mathring{I}_i - D_i, \tag{1.2b}$$

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where $\mathring{I}_i$ is the interior of I_i and $D_i \subset I_i$ is a finite set. If, for instance, $\mathbf{A} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathbf{x} = (\frac{1}{2}, \frac{1}{2})^T$, then

$$f_{\mathbf{A},\mathbf{x}}(t) = \begin{cases} 2t, & 0 \leqslant t \leqslant \frac{1}{2}, \\ t - \frac{1}{2}, & \frac{1}{2} \leqslant t \leqslant t. \end{cases}$$

We prove, in this paper, that the Liapunov exponent $\lambda(\mathbf{x})$ of $f_{\mathbf{A},\mathbf{x}}$ is equal to a measure theoretic entropy $h_{m_{\mathbf{A},\mathbf{x}}}$ of $f_{\mathbf{A},\mathbf{x}}$, where $m_{\mathbf{A},\mathbf{x}}$ is a Markov measure associated with $\mathbf{A}$ and $\mathbf{x}$. Here, the Liapunov exponent and the entropy are computable by solving an eigenvalue problem and can be explicitly calculated when the transition matrix $\mathbf{A}$ is symmetric. Moreover, we also show that $\max_{\mathbf{x}} \lambda(\mathbf{x}) = \max_{\mathbf{x}} h_{m_{\mathbf{A},\mathbf{x}}} = \log(\lambda_1)$, where λ_1 is the maximal eigenvalue of $\mathbf{A}$.

The relationship between entropy and Liapunov exponents have been studied by many authors (see, e.g., [3,4,6,8] and the works cited therein). In [8], Ruelle proved that

$$h_\mu(f) = \int_M \chi \, d\mu, \tag{1.3}$$

where μ is any invariant measure for f and $\chi(x) = \sum_{\lambda_j(x) \geqslant 0} m_j(x)$ in which $\lambda_j(x)$ denotes the jth Liapunov exponent of f at x and $m_j(x)$ is it's multiplicity. It was shown by Pesin [6] that the equality in (1.3) holds provided that f is Hölder C^1 and μ is absolutely continuous with respect to Lebesgue measure of M. In [3], Ledrappier studied the relationship between dimension, entropy and Liapunov exponent for piecewise differential interval maps. In [4], Liu studied Pesin's formula for C^2 noninvertible maps.

2. Preliminaries

For ease of references, we shall recall some definitions and known results. Let $(X, \mathcal{B}, m)$ be a measure space. Here $\mathcal{B}$ denotes the σ-algebra of all measurable sets in X and m denotes the measure on X. Let $f : X \to X$ be a measurable function, f is said to be *measure preserving* with respect to the measure m if $m(S) = m(f^{-1}(S))$ for all $S \in \mathcal{B}$. Here m is called an *invariant measure* for f.

Definition 2.1. Let f be measure preserving on $(X, \mathcal{B}, m)$. A set $S \in \mathcal{B}$ is called *f-invariant* if $f^{-1}(S) = S$. f is said to be *ergodic* if every f-invariant set has measure 0 or full measure.

Definition 2.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function. For each point $t_0 \in \mathbb{R}$, define the Liapunov exponent at t_0 as follows:

$$\lambda(t_0) = \limsup_{n\to\infty} \frac{1}{n} \log\left|\left(f^{(n)}\right)'(x)\right| = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log\left|f'(x_j)\right|, \tag{2.1}$$

where $x_j = f^j(x)$.

Proposition 2.1. *(See, e.g., [7, p. 86].) Let $f : [0,1] \to [0,1]$ be differentiable. If f is ergodic with respect to the measure m, then the Liapunov exponent for f is constant a.e. and is given by*

$$\lambda(x) = \int_0^1 \log\left|f'(t)\right| dm \quad \text{a.e.} \tag{2.2}$$

Definition 2.3. Let $(X, \mathcal{B}, m)$ be a measure space and $\mathcal{P}$ be a partition of X, the *entropy of partition* $\mathcal{P}$ is defined to be

$$H(\mathcal{P}) = -\sum_{P \in \mathcal{P}} m(P) \log m(P).$$

Let $f : X \to X$ be measure preserving. The *entropy of f with respect to $\mathcal{P}$* is defined by

$$h(f, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} f^{-j}(\mathcal{P})\right). \tag{2.3}$$

Here the notation $\bigvee_{j=0}^{n-1} f^{-j}(\mathcal{P})$ denotes the partition whose elements are of the form $A_0 \cap \cdots \cap A_{n-1}$ for $A_i \in f^{-j}(\mathcal{P})$, $i = 0, \ldots, n-1$, satisfying $m(A_0 \cap \cdots \cap A_{n-1}) \neq 0$. The *measure theoretic entropy* of f is given by

$$h_m(f) = \sup_{\mathcal{P}\text{: partition}} h(f, \mathcal{P}).$$

Proposition 2.2. *(See [5, Proposition IV.3.2].) The limit in* (2.3) *is well defined and exists.*

Let $\mathbf{A}$ be an $n \times n$ transition matrix. $\mathbf{P} = (p_{ij}) \in M_{n\times n}(\mathbb{R})$ is said to be a *stochastic matrix* associated with $\mathbf{A}$ if

(1) $p_{ij} = 0$ if and only if $a_{ij} = 0$ for $1 \leqslant i, j \leqslant n$.
(2) $0 \leqslant p_{ij} \leqslant 1$ for all $1 \leqslant i, j \leqslant n$.
(3) $\sum_j p_{ij} = 1$.

Clearly, there exists a left eigenvector $\mathbf{q}^T = (q_1, q_2, \ldots, q_n)$ satisfying the following:

$$\mathbf{q}^T \mathbf{P} = \mathbf{q}^T \tag{2.4a}$$

and

$$\sum_{i=1}^{n} q_i = 1. \tag{2.4b}$$

We then define a Markov measure $\mu = \mu_{\mathbf{P},\mathbf{q}}$ associated with $(\mathbf{P}, \mathbf{q})$ by

$$\mu\big(C(i_0, i_1, \ldots, i_k)\big) = q_{i_0} p_{i_0,i_1} \cdots p_{i_{k-1},i_k}, \tag{2.5}$$

where $C(i_0, i_1, \ldots, i_k) = \{(j_0, j_1, \ldots) \in \Sigma_{\mathbf{A}}\colon\ j_0 = i_0, \ldots, j_k = i_k\}$ is called a *cylinder*.

Proposition 2.3. *(See, e.g., [5, Theorem I-10.1].)* $\mu = \mu_{\mathbf{P},\mathbf{q}}$ *is an invariant measure of the Markov shift* $\sigma_{\mathbf{A}}$.

Theorem 2.1. *(See, e.g., [5, p. 221].) Let* $\mathbf{A}$ *be an* $n \times n$ *transition matrix and* $\mu_{\mathbf{P},\mathbf{q}} = \mu$ *be the invariant Markov measure defined by* $(\mathbf{P}, \mathbf{q})$ *associated with* $\mathbf{A}$. *Then*

$$h_\mu(\sigma_{\mathbf{A}}) = -\sum_{ij} q_i p_{ij} \log p_{ij}.$$

Definition 2.4. Let $(X_i, \mathcal{B}_i, m_i)$, $i = 1, 2$, be measure spaces and $f_i : X_i \to X_i$ be measure preserving. We say that f_1 is equivalent to f_2 if there exist $F : X_1 \to X_2$ and $G : X_2 \to X_1$ satisfying the following properties:

(a) For any $A_2 \in \mathcal{B}_2$, $F^{-1}(A_2) \in \mathcal{B}_1$ and $m_1(F^{-1}(A_2)) = m_2(A_2)$.
(b) For any $A_1 \in \mathcal{B}_1$, $G^{-1}(A_1) \in \mathcal{B}_2$ and $m_2(G^{-1}(A_1)) = m_1(A_1)$.
(c) $G \circ F = \mathrm{id}_{X_1}$ a.e. and $F \circ G = \mathrm{id}_{X_2}$ a.e.
(d) $f_2 \circ F = F \circ f_1$ a.e.

This is obviously an equivalence relation. Equivalence maps have the following important properties that can be easily derived.

Proposition 2.4. *(See [5, Proposition IV.4.1].) Equivalent maps have the same measure theoretic entropy and ergodicity.*

3. Main results

By adopting the standard techniques for solving the Perron–Frobenius equation, we are able to obtain an invariant measure of $f_{\mathbf{A},\mathbf{x}}$ on $[0, 1]$.

Theorem 3.1. *Let* $\mathbf{y}^T = (y_1, \ldots, y_n)$ *be the left eigenvector of* $\operatorname{diag}(s_1^{-1}, \ldots, s_n^{-1})\mathbf{A}$ *corresponding to eigenvalue* 1 *with* $\mathbf{y}^T\mathbf{x} = 1$*, i.e.,*

$$\mathbf{y}^T \operatorname{diag}(s_1^{-1}, \ldots, s_n^{-1})\mathbf{A} = \mathbf{y}^T \tag{3.1}$$

and

$$\mathbf{y}^T\mathbf{x} = 1. \tag{3.2}$$

Let $\rho(t) = y_i$ *for* $t \in I_i$*. Then the measure* m *on* $[0, 1]$ *defined as* $m(S) = m_{\mathbf{A},\mathbf{x}}(S) = \int_S \rho(t)\,dt$ *is an invariant measure of* $f_{\mathbf{A},\mathbf{x}}$ *on* $[0, 1]$.

Proof. Since $\operatorname{diag}(s_1^{-1}, \ldots, s_n^{-1})\mathbf{A}\mathbf{x} = \mathbf{x}$, the existence of the left eigenvector $\mathbf{y}$ satisfying (3.1) and (3.2) is guaranteed. To see m is an invariant measure of $f_{\mathbf{A},\mathbf{x}}$, it suffices to show that for $J \subset I_i$, $1 \leqslant i \leqslant n$, $m(J) = m(f_{\mathbf{A},\mathbf{x}}^{-1}(J))$. To this end, let $\alpha_i = \{j\colon \alpha_{ji} = 1\}$. We see that

$$f_{\mathbf{A},\mathbf{x}}^{-1}(J) = \bigcup_{j\in\alpha_i} J_j,$$

where $J_j \subset I_j$ and $\ell(J) = s_j \ell(J_j)$. Here $\ell(J)$ is the Lebesgue measure of J. Now,

$$m\big(f_{\mathbf{A},\mathbf{x}}^{-1}(J)\big) = m\Bigg(\bigcup_{j\in\alpha_i} J_j\Bigg) = \sum_{j\in\alpha_i} y_j \ell(J_j) = \sum_{j\in\alpha_i} \frac{y_j}{s_j}\ell(J) = y_i \ell(J) = m(J).$$

We have used (3.1) to justify the second equality above. □

Theorem 3.1 is indeed a special case of [1, Proposition 3.3.1]. In the following, we show that the eigenvalue problem (3.1) and (3.2) can be explicitly solved when $\mathbf{A}$ is symmetric.

Corollary 3.1. *If* $\mathbf{A}$ *is a symmetric transition matrix, then the row vector* $\mathbf{y}^T$ *as given in Theorem* 3.1*, is*

$$\mathbf{y}^T = \frac{\mathbf{x}^T\mathbf{A}}{\mathbf{x}^T\mathbf{A}\mathbf{x}}.$$

Proof. Since $\mathbf{A}$ is symmetric, $\mathbf{x}^T\mathbf{A} = ((\mathbf{A}\mathbf{x})_1, \ldots, (\mathbf{A}\mathbf{x})_n)$. Hence

$$\begin{aligned}\mathbf{x}^T\mathbf{A}\operatorname{diag}(s_1^{-1}, \ldots, s_n^{-1}) &= \big((\mathbf{A}\mathbf{x})_1, \ldots, (\mathbf{A}\mathbf{x})_n\big)\operatorname{diag}\bigg(\frac{x_1}{(\mathbf{A}\mathbf{x})_1}, \ldots, \frac{x_1}{(\mathbf{A}\mathbf{x})_n}\bigg)\\ &= (x_1, \ldots, x_n) = \mathbf{x}^T.\end{aligned}$$

Thus $\mathbf{y}^T$ is as asserted. □

From here on to the end of this section, we set $\mathbf{P}$ and $\mathbf{q}$ as follows:

$$\mathbf{P} = (\operatorname{diag}\mathbf{A}\mathbf{x})^{-1}\mathbf{A}\operatorname{diag}\mathbf{x}, \tag{3.3}$$

and

$$\mathbf{q}^T = \mathbf{y}^T\operatorname{diag}\mathbf{x} \quad \bigg(= \frac{\mathbf{x}^T\mathbf{A}(\operatorname{diag}\mathbf{x})}{\mathbf{x}^T\mathbf{A}\mathbf{x}} \text{ if } \mathbf{A} \text{ is symmetric}\bigg). \tag{3.4}$$

Here $\mathbf{A}$ is a transition matrix and $\mathbf{x}$ is defined in (3.1). Note that $\mathbf{P}$ is a stochastic matrix associated with $\mathbf{A}$ and $\mathbf{q}^T$ is a left eigenvector corresponding to eigenvalue 1. Thus $\mu_{\mathbf{P},\mathbf{q}}$ is an invariant measure for $\sigma_{\mathbf{A}}$.

Theorem 3.2. *Consider* $f_{\mathbf{A},\mathbf{x}}$ *and* $\sigma_{\mathbf{A}}$ *on* $([0, 1], m_{\mathbf{A},\mathbf{x}})$ *and* $(\Sigma_{\mathbf{A}}, \mu_{\mathbf{P},\mathbf{q}})$*, respectively. Here* $\mathbf{P}$ *and* $\mathbf{q}$ *are described as above. Then* $f_{\mathbf{A},\mathbf{x}}$ *is equivalent to* $\sigma_{\mathbf{A}}$.

Proof. Given $1 < i_0, \dots, i_k < n$, let

$$I_{i_0,i_1,\dots,i_k} = \left\{t \in [0,1]:\ f^j(t) \in I_{i_j},\ 0 \leqslant j \leqslant k\right\}.$$

It is well known that $f_{\mathbf{A},\mathbf{x}}$ is topological conjugate to $\sigma_{\mathbf{A}}$ (i.e., satisfying condition (c) and (d) in Definition 2.4, see, e.g., [2, Theorem 3.18]). Therefore, it suffices to show that

$$m_{\mathbf{A},\mathbf{x}}(I_{i_0,\dots,i_k}) = \mu_{\mathbf{P},\mathbf{q}}(C_{i_0,\dots,i_k}).$$

Writing $s_i^{-1} = \frac{x_i}{(\mathbf{Ax})_i}$, we get that

$$\begin{aligned} m_{\mathbf{A},\mathbf{x}}(I_{i_0,i_1,\dots,i_k}) &= y_{i_0} s_{i_0}^{-1} a_{i_0,i_1} s_{i_1}^{-1} \cdots a_{i_{k-1},i_k} x_{i_k} \\ &= (y_{i_0} x_{i_0}) \left(\frac{1}{(\mathbf{Ax})_{i_0}} a_{i_0,i_1} x_{i_1}\right) \cdots \left(\frac{1}{(\mathbf{Ax})_{i_{k-1}}} a_{i_{k-1},i_k} x_{i_k}\right) \\ &= (q_{i_0})(p_{i_0,i_1})(p_{i_1,i_2}) \cdots (p_{i_{k-1},i_k}) \\ &= \mu_{\mathbf{P},\mathbf{q}}(C_{i_0,\dots,i_k}). \end{aligned}$$

We thus complete the proof of Theorem 3.2. □

Theorem 3.3. *Let* $\mathbf{A}$ *be an* $n \times n$ *transition matrix which is irreducible, then the Liapunov exponent* $\lambda(\mathbf{x})$ *of* $f_{\mathbf{A},\mathbf{x}}$ *is*

$$\lambda(\mathbf{x}) = \mathbf{y}^T \log \operatorname{diag}(s_1, \dots, s_n)\mathbf{x}.$$

Proof. According to the equivalence of $f_{\mathbf{A},\mathbf{x}}$ and $\sigma_{\mathbf{A}}$, we see that $f_{\mathbf{A},\mathbf{x}}$ is ergodic if and only if $\mathbf{A}$ is irreducible. Hence, it follows from (2.2) and the construction of $f_{\mathbf{A},\mathbf{x}}$, that the Liapunov exponent λ of $f_{\mathbf{A},\mathbf{x}}$ is

$$\lambda = \int_0^1 \log\left|f'_{\mathbf{A},\mathbf{x}}(t)\right| dm_{\mathbf{A},\mathbf{x}} = \mathbf{y}^T \log \operatorname{diag}(s_1, \dots, s_n)\mathbf{x}.$$

We just proved the theorem. □

We are now ready to state the main result of the paper.

Theorem 3.4. *Let* $\mathbf{A}$ *be an* $n \times n$ *transition matrix which is irreducible. Let* $\lambda = \lambda(\mathbf{x})$ *be the Liapunov exponent of* $f_{\mathbf{A},\mathbf{x}}$*, then*

(1) $\lambda(\mathbf{x}) = h_{m_{\mathbf{A},\mathbf{x}}}(f_{\mathbf{A},\mathbf{x}})$ *where* $m_{\mathbf{A},\mathbf{x}}$ *is the invariant measure for* $f_{\mathbf{A},\mathbf{x}}$ *given as in Theorem* 3.1.
(2) $\sup_{\mathbf{x}:\ partition} \lambda(\mathbf{x}) = h_{\text{top}}(f_{\mathbf{A},\mathbf{x}}) = \log \lambda_1$*, where* λ_1 *is the maximal eigenvalue of* $\mathbf{A}$*. The sup attains while* $\mathbf{x}$ *is chosen to be the eigenvector of* $\mathbf{A}$ *corresponding to eigenvalue* λ_1 *with* $\sum x_i = 1$.

Proof. Let $\mathbf{P} = (p_{ij})$, and so $(p_{ij}) = (\frac{x_j}{(\mathbf{Ax})_i} a_{ij})$. Set $\widetilde{\mathbf{P}} = (p_{ij} \log p_{ij})$, and $\mathbf{e} = (1, \dots, 1)^T$, it follows from Theorem 3.2, (3.3), (3.4) and Theorem 2.1(1) that

$$h_{m_{\mathbf{A},\mathbf{x}}}(f_{\mathbf{A},\mathbf{x}}) = h_{\mu_{\mathbf{P},\mathbf{q}}}(\sigma_{\mathbf{A}}) = -\mathbf{q}^T \widetilde{\mathbf{P}}\mathbf{e} = -\mathbf{y}^T (\operatorname{diag} \mathbf{x})\widetilde{\mathbf{P}}\mathbf{e}. \tag{3.5}$$

Now,

$$\begin{aligned} \widetilde{\mathbf{P}}\mathbf{e} &= \left(\frac{x_j}{(\mathbf{Ax})_i} a_{ij} \log\left(\frac{x_j}{(\mathbf{Ax})_i} a_{ij}\right)\right)_{n \times n} \mathbf{e} \\ &= (\operatorname{diag} \mathbf{Ax})^{-1} \left(a_{ij} \log\left(\frac{x_j}{(\mathbf{Ax})_i} a_{ij}\right)\right)_{n \times n} \operatorname{diag}(x_1, \dots, x_n)\mathbf{e} \\ &= (\operatorname{diag} \mathbf{Ax})^{-1} \left(a_{ij} \log\left(\frac{x_j}{(\mathbf{Ax})_i} a_{ij}\right)\right)_{n \times n} \mathbf{x}. \end{aligned} \tag{3.6}$$

Moreover, we have that

$$-\left(a_{ij}\left(\log\frac{x_j}{(\mathbf{Ax})_i}a_{ij}\right)\right)_{n\times n}=-(a_{ij}\log a_{ij})_{n\times n}+\left(a_{ij}\log(\mathbf{Ax})_i\right)_{n\times n}-(a_{ij}\log x_j)_{n\times n}. \tag{3.7}$$

Since either $a_{ij}=0$ or $a_{ij}=1$, we see that $a_{ij}\log a_{ij}=0$. We also note that

$$\left(a_{ij}\log(\mathbf{Ax})_i\right)_{n\times n}=\log(\operatorname{diag}\mathbf{Ax})\mathbf{A}$$

and

$$(a_{ij}\log x_j)_{n\times n}=\mathbf{A}\log\operatorname{diag}\mathbf{x}.$$

Substituting (3.7) into (3.6), we get that

$$-\widetilde{\mathbf{P}}\mathbf{e}=(\operatorname{diag}\mathbf{Ax})^{-1}\log(\operatorname{diag}\mathbf{Ax})\mathbf{Ax}-(\operatorname{diag}\mathbf{Ax})^{-1}\mathbf{A}(\log\operatorname{diag}\mathbf{x})\mathbf{x}. \tag{3.8}$$

Here $\log\mathbf{A}=(\log a_{ij})$ and

$$\operatorname{diag}\mathbf{x}=\begin{pmatrix} x_1 & & & 0\\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_n\end{pmatrix}.$$

To further simplify (3.5), we note that

$$\mathbf{y}^T\operatorname{diag}\mathbf{x}(\operatorname{diag}\mathbf{Ax})^{-1}\mathbf{A}=\mathbf{y}^T \tag{3.9}$$

and

$$\begin{aligned}\mathbf{y}^T\operatorname{diag}\mathbf{x}(\operatorname{diag}\mathbf{Ax})^{-1}\log(\operatorname{diag}\mathbf{Ax})\mathbf{Ax}&=\mathbf{y}^T\log(\operatorname{diag}\mathbf{Ax})(\operatorname{diag}\mathbf{x})(\operatorname{diag}\mathbf{Ax})^{-1}\mathbf{Ax}\\&=\mathbf{y}^T\log(\operatorname{diag}\mathbf{Ax})(\operatorname{diag}\mathbf{x})\mathbf{e}\\&=\mathbf{y}^T\log(\operatorname{diag}\mathbf{Ax})\mathbf{x}.\end{aligned} \tag{3.10}$$

It then follows from (3.8)–(3.10) and Theorem 3.2, (3.5) becomes

$$\begin{aligned}h_{\mu_{\mathbf{P},\mathbf{q}}}(\sigma_{\mathbf{A}})&=\mathbf{y}^T\log(\operatorname{diag}\mathbf{Ax})\mathbf{x}-\mathbf{y}^T(\log\operatorname{diag}\mathbf{x})\mathbf{x}\\&=\mathbf{y}^T\log\bigl(\operatorname{diag}(s_1,\dots,s_n)\bigr)\mathbf{x}\\&=\lambda(\mathbf{x}).\end{aligned}$$

We thus complete the proof of the theorem. □

Remark 3.1. It is shown in [3, Proposition 4] that for any piecewise differentiable map f with $\lambda(f,\mu)>0$, the equality

$$HD(\mu)\lambda(f,\mu)=h_\mu(f)$$

holds. Here $HD(\mu)$ denotes the fractal dimension of the ergodic invariant measure μ on a compact space. In the case of this paper, it can be shown that $HD(\mu)=1$, and hence, Theorem 3.4(1) can be obtained when the Liapunov exponent of $f_{\mathbf{A},\mathbf{x}}$ is positive. In this work, we mainly provide an alternative proof without using the dimension. On the other hand, combining Theorems 3.3 and 3.4(1), it reveals that the Liapunov exponent and the measure theoretic entropy of the maps $f_{\mathbf{A},\mathbf{x}}$ are computable by solving the eigenvalue problem (3.1) (e.g., using the power method). In particular, while the corresponding transition matrix $\mathbf{A}$ is symmetric, it follows from Corollary 3.1 that the Liapunov exponent (as well as the entropy) can be explicitly solved by

$$\lambda(\mathbf{x})=h_{m_{\mathbf{A},\mathbf{x}}}(f_{\mathbf{A},\mathbf{x}})=\sum_{ij}\frac{1}{\mathbf{x}^T\mathbf{z}}a_{ij}x_ix_j\log\frac{a_{ij}x_j}{z_i} \tag{3.11}$$

where $\mathbf{z}=\mathbf{Ax}$. From the viewpoint of an eigenvalue problem, using Theorem 3.4(2), (3.11) also gives a nontrivial lower bound for the maximal eigenvalue of a symmetric transition matrix.

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