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EXISTENCE FOR A MATRIX EQUATION ARISING IN MICROELECTRONICS

JONQ JUANG

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ABSTRACT. In this paper we rigorously show the existence of solutions of a matrix equation which arises in the design of micro electronical circuits. This equation was studied by Szidarovszky and Palusinsk [Appl. Math. Comput. **64**, 115-119(1994)], who also presented an iterative algorithm for its solution. We show, via an example, that this algorithm could converge extremely slow in certian cases. The solution can then be used to minimize the reflection coefficients of the active signals.

1. INTRODUCTION

Consider the following matrix equation of the form

$$R = (M + X)^{-1}(M - X). \tag{1}$$

Here M is an M-matrix [3], which is given. A matrix A is called an *M-matrix* if A is invertible, A^{-1} is nonnegative (in the componentwise sense), and $a_{ij} \leq 0$ for all $i, j = 1, \cdots, n,\ i \neq j$. The unknown matrices R and X have the following constraints:

(C1) The matrix $X =$ diag $(x_1, x_2, \cdots, x_n),\ x_i > 0$ for all $i = 1, 2, \cdots, n$.
(C2) The diagonal of matrix R contains only zero elements.

Equation (1) arises in microelectronics. The matrix M is the characteristic admittance matrix, which represents various signal propagation properties of the interconnections of high speed electronic circuits and systems. The diagonal admittance matrix X gives the load of the resistive terminating network. The reflection matrix R describes the ratios of the amplitudes of the incident and reflected waves. Physically, matrix M has non-positive off-diagonal elements, a positive diagonal and a nonnegative inverse with positive diagonal. In practice, we wish to select the load of the resistive terminating elements so that the reflection coefficients of the active signals are equal to zero. That is, given an M-matrix M, find a diagonal matrix X with positive diagonal such that $R = (M + X)^{-1}(M - X)$ has zero diagonal.

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Some numerical procedures for solving equation (1) were proposed in [2]. Note that the boundedness of the sequence in [2] has not been established. For more physical details see, for example, [1, 2].

The purpose of this paper is two-fold. First, a priori upper and lower bounds for X are obtained. The existence of solutions to (1) will be established via degree theory. Second, an example is given to illustrate that the algorithm in [2] could converge extremely slow in certain cases.

2. Main results

Define the mapping $\mathbf{F} : R^{n\times n} \to R^{n\times n}$ such that for any $n \times n$ matrix $A = (a_{ij})$, $\mathbf{F}(A) =$ diag $(a_{11}, a_{22}, \cdots, a_{nn})$. We note, as in [2], that, from (1),

$$R = (M + X)^{-1}(M + X - 2X) = I - 2(M + X)^{-1}X.$$

Therefore, all diagonal elements of $(M + X)^{-1}X$ must be equal to $\frac{1}{2}$. Equation (1) can thus be decoupled as

$$\mathbf{F}((M + X)^{-1}) = \text{ diag } (\frac{1}{2x_1}, \frac{1}{2x_2}. \cdots, \frac{1}{2x_n}),$$

which is equivalent to the following fixed point problem:

$$X = \frac{1}{2}\mathbf{F}^{-1}((M + X)^{-1}) := \mathbf{G}(X). \tag{2}$$

Notation. 1. Let $\mathbf{A}$ be the set of all $n \times n$ matrices with only nonnegative elements.

2. Let $A, B, \in R^{n\times n}$; we write $A \geq B$ if $A - B \in \mathbf{A}$.

To be complete, we recall the following well-known results, see e.g., 2.4.10 of [3] and Theorem 13.2.11 of [4] respectively.

Theorem 1. *Let two $n \times n$ matrices $A_i, i = 1, 2$, be, respectively, decomposed as $A_i = D_i - B_i$, where $D_i, i = 1, 2$, are diagonal parts of $A_i, i = 1, 2$. Suppose A_1 is an M-matrix, $D_1 \leq D_2$ and $B_1 \geq B_2$. Then A_2 is an M-matrix and $A_2^{-1} \leq A_1^{-1}$.*

Theorem 2. *Let $D \subset R^n$ be an open bounded set. Suppose that $\phi : \bar{D} \to R^n$ is continuous. Assume that no solution of $\phi(x) = p$ lies on ∂D. Then the following hold:*

(i) *Homotopy Invariance. Let $\mathbf{H}_t$ be a homotopy, and suppose that $\mathbf{H}_t(x) \neq p$ for any $x \in \partial D$ and $t \in [0, 1]$. Then $d(\mathbf{H}_t, p, D)$ is independent of t.*

(ii) *$d(I, p, D) = 1$ if $p \in D$, $d(I, p, D) = 0$ if $p \notin D$.*

(iii) *If $d(\phi, p, D) \neq 0$, the equation $\phi(x) = p$ has at least one solution in D.*

Consider the following one-parameter family of equations:

$$\begin{aligned} X &= \tfrac{1}{2}\mathbf{F}^{-1}((M + tX)^{-1}) \\ &:= \mathbf{G}_t(X),\ 0 \leq t \leq 1. \end{aligned} \tag{3}$$

We next establish a priori bounds for X and all t.

Lemma 1. *Let $X \in \mathbf{A}$ be a solution of (3). Then $\frac{1}{2}F^{-1}(M^{-1}) \leq X \leq F(M)$ for all $0 \leq t \leq 1$.*

Proof. Let X be as assumed. Using Theorem 1, we see that

$$\begin{aligned} X &= \frac{1}{2}F^{-1}((M + tX)^{-1}) \leq \frac{1}{2}F^{-1}((M + X)^{-1}) \\ &\leq \frac{1}{2}F^{-1}((F(M) + X)^{-1}) = \frac{1}{2}(F(M) + X). \end{aligned}$$

Consequently, the a priori upper bound as asserted follows. The a priori lower bound can be obtained by dropping the term tX.

Theorem 3. *There exist R and X satisfying the constraints (C1) and (C2) and equation* (1). *Moreover, for any solution $X \in \mathbf{A}$ of equation* (2), *the corresponding solution R of* (1) *has the property that $-R \in \mathbf{A}$.*

Proof. Let $D = \{X \in R^{n\times n} : \frac{1}{4}F^{-1}(M^{-1}) < X < 2F(M)\}$. D is evidently a non-empty bounded open subset of $R^{n\times n}$, and $\mathbf{G}_t : \bar{D} \to R^{n\times n}$ is continuous. It is also clear, via Lemma 1, that if $X - \mathbf{G}_t X = 0$ for $X \in \bar{D}$, then $X \in D$. The preparations for the use of degree theory are now complete. Consider the homotopy $\mathbf{H}_t = I - \mathbf{G}_t$. Hence by homotopy invariance

$$d(I - \mathbf{G}_0, 0, D) = d(I - \mathbf{G}_1, 0, D).$$

But

$$d(I - \mathbf{G}_0, 0, D) = d(I, \frac{1}{2}\mathbf{F}^{-1}(M^{-1}), D) = 1$$

by Theorem 2-ii, as $\frac{1}{2}\mathbf{F}^{-1}(M^{-1}) \in D$. The first assertion of the theorem now follows from Theorem 2-iii. The last assertion of the theorem follows from the constraints (C2) and the fact that $(M + X)^{-1} \geq 0$.

Our second result deals with the asymptotic convergence rate of the algorithm in [2]. Consider now the iteration procedure:

$$X^{(k+1)} = \mathbf{G}(X^{(k)}), \tag{4a}$$

$$X^{(0)} = 0. \tag{4b}$$

Applying the theorem in [2], we obtain that $\{X^{(k)}\}$ is a bounded, increasing sequence. Hence, it converges upward to a limit, say X^*. Note that in this case X^* is the smallest positive solution of (2). In the following, we shall illustrate, via an example, that such an algorithm can be extremely slow to converge. Let

$$M = \begin{pmatrix} 1 & -(1-\varepsilon) \\ -(1-\varepsilon) & 1 \end{pmatrix},$$

where ε is a small positive parameter. Clearly, M is an M-matrix. Writing equation (2) in component form, we obtain that

$$x_1 = \frac{(1+x_1)(1+x_2) - (1-\varepsilon)^2}{2(1+x_2)} := f_1(x_1, x_2; \varepsilon), \tag{5a}$$

$$x_2 = \frac{(1+x_1)(1+x_2) - (1-\varepsilon)^2}{2(1+x_2)} := f_2(x_1, x_2; \varepsilon). \tag{5b}$$

A simple calculation gives that the unique positive solution $x^* = (x_1^*, x_2^*)$ to (5) is $x_1^* = x_2^* = \sqrt{1-(1-\varepsilon)^2}$. Define : $f : R^2 \to R^2$ by

$$f(x; \varepsilon) = (f_1(x; \varepsilon), f_2(x; \varepsilon)), \tag{5c}$$

where $x = (x_1, x_2)$.

Then the iteration procedure (4) is equivalent to

$$x^{(k+1)} = f(x^{(k)}; \varepsilon), \tag{6a}$$

$$x^{(0)} = 0. \tag{6b}$$

Note that the asymptotic convergence rate of the algorithm is determined by the spectral radius $\sigma(f'(x^*;\varepsilon))$ of the Jacobian matrix $f'(x^*;\varepsilon)$. A direct calculation yields that

$$\sigma(f'(x^*;\varepsilon)) = \frac{1}{2}[1 + \frac{(1-\varepsilon)^2}{(1+\sqrt{1-(1-\varepsilon)^2})}] = 1 - h(\varepsilon),$$

where $h(\varepsilon) > 0$ and $h(\varepsilon) = o(\varepsilon^{\frac{1}{2}})$.

We also note that if M is a diagonal matrix, the convergence rate of (4) is $\frac{1}{2}$. It is observed that as M becomes less diagonally dominate, the convergence rate of the algorithm deteriorates.

In conclusion, we note that if the initial sequence $X^{(0)}$ is chosen to be $\mathbf{F}(M)$, then $\{X^{(k)}\}$ is decreasing and bounded below. These observations are a direct consequence of Lemma 1. The sequence then converges to the largest positive solution X^{**} of (2). Applying Lemma 1, we obtain the following corollary.

Corollary. *Let $R^* = (r^*_{ij})$ and $r^{**} = (r^{**}_{ij})$, respectively, be the corresponding solutions of equation* (1) *with respect to the solutions X^* and X^{**} of equation* (2). *We further assume that $R = (r_{ij})$ is the corresponding solution of* (1) *associated with any positive solution X of* (2). *Then $r^{**}_{ij} \leq r_{ij} \leq r^*_{ij} \leq 0$ for all i, j.*

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Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan, Republic of China

E-mail address: jjuang@math.nctu.edu.tw