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Decision Support

A fuzzy ranking method with range reduction techniques

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Abstract

For ranking alternatives based on pairwise comparisons, current analytic hierarchy process (AHP) methods are difficult to use to generate useful information to assist decision makers in specifying their preferences. This study proposes a novel method incorporating fuzzy preferences and range reduction techniques. Modified from the concept of data envelopment analysis (DEA), the proposed approach is not only capable of treating incomplete preference matrices but also provides reasonable ranges to help decision makers to rank decision alternatives confidently.
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1. Introduction

This paper addresses the range computation for pairwise comparison preference rating. The motivation, purpose and advantages of the proposed approach are introduced first. Then, the concept and insufficiencies of conventional fuzzy AHP models are described. Next, the range reduction and fuzzy ranking models are proposed and constructed. Finally, a numerical example is used to illustrate the solving process.

The analytic hierarchy process (AHP), developed by Saaty (1977), is a popular approach to rank alternatives. Through the ratio-scaled assessment of pairwise preferences between alternatives, the ranks of alternatives are found by computing the eigenvalues of the preference matrix. Conventional AHP, however, cannot treat incomplete preference matrices. In addition, AHP has been proven to be a mathematically flawed system in deriving weights and synthesizing scores of attributes by several authors (Barzilai, 1997, 2001, 2005; Brugha, 2000, 2004; etc.).

Since fuzziness and vagueness commonly exist in many decision-making problems (Levary and Wan, 1998; Ribeiro, 1996), numerous ranking methods (Graan, 1980; Laarhoven and Pedrycz, 1983; Boender et al., 1989;

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Chang, 1996; Ruoning and Xiaoyan, 1996; Leung and Cao, 2000; Yu, 2002) have been developed to solve fuzzy decision problems with pairwise comparison matrices. A major disadvantage of conventional fuzzy AHP methods is that no range information is provided to help a decision maker to specify preferences conveniently. Conventionally, Saaty (1980) ratio scale of [1/9, 9] is used as default upper and lower bounds, yet the ranges are usually too big for a decision maker to use as a useful reference. In addition, most of current AHP methods require a decision maker to specify a complete pairwise comparison matrix.

Data envelopment analysis (DEA) is another commonly used technique in ranking decision alternatives. The DEA technique is intended to evaluate the efficiency of each alternative using CCR models (Charnes et al., 1978) or BCC models (Banker et al., 1984) based on the concept of maximizing the ratio of outputs to inputs. However, there are some insufficiencies of conventional DEA models in ranking alternatives. First, current DEA models may generate too many efficient alternatives with the same rank. The lack of discrimination among alternatives prohibits its applications in real cases (Angulo-Meza and Lins, 2002). In addition, most DEA methods do not incorporate the preferences specified by the decision maker.

This study proposes a novel ranking method with pair-wise preference comparisons. The proposed model first adopts a modified DEA model to generate reasonable upper and lower bounds of preference ratios. By referring to these ranges, a decision maker then specifies his/her fuzzy preferences partially. A goal-programming model with minimal approximation errors and maximal fulfillment of a decision maker's preferences is proposed to solve the fuzzy decision problem.

The major advantages of the proposed approach are listed as follows:

(i) Reasonable upper and lower bounds are provided to help a decision maker to articulate related fuzzy preferences.
(ii) Incomplete preference matrix can be handled.
(iii) Various fuzzy preferences with convex, concave or mixed convex–concave features are treated to obtain a crisp optimal solution efficiently.

2. Conventional fuzzy AHP models

Consider a set of n alternatives $A = \{A_i | i = 1, \dots, n\}$ for solving a decision problem. From the basis of AHP (Saaty, 1980), the pairwise comparison of A_i over A_j, denoted as $h_{i,j}$, is the preference specified by a decision maker as the ratio of the weights of A_i to A_j. Let $h_{i,j} = \frac{w_i}{w_j}$ measure the relative dominance of A_i over A_j in terms of priority weighs $w_1 > 0, \dots, w_n > 0$. Following Saaty, $h_{i,j}$ are specified as 1–9 numerical rates. Denote $\mathbf{H} = (h_{i,j})$, where $h_{j,i} = \frac{1}{h_{i,j}}$ is assumed. A fuzzy AHP problem can be expressed as follows:

$$\begin{aligned} &\text{Min} && \sum_{i=1}^{n} \sum_{j>i}^{n} \left| \frac{w_i}{w_j} - \tilde{h}_{i,j} \right| \\ &\text{Max} && \sum_{i=1}^{n} \sum_{j>i}^{n} \mu(\tilde{h}_{i,j}) \\ &\text{Subject to} && \sum_{i=1}^{n} w_i = 1, \\ & && w_i, \quad \tilde{h}_{i,j} \geqslant 0, \end{aligned} \tag{2.1}$$

where $\tilde{h}_{i,j}$ is a fuzzy number representing how many times is A_i preferred over A_j, which is specified by the decision maker. $\mu(\tilde{h}_{i,j})$ is the membership function of $\tilde{h}_{i,j}$. The first objective is to minimize the sum of deviations resulted from approximation, and the second objective is to maximize the sum of membership functions of $\tilde{h}_{i,j}$. Model (2.1) is in the form of goal-programming (Cooper, 2005). This model can be solved by weights method (Taha, 2003) to optimize both objectives jointly.

A most commonly used membership function is a triangle type as shown in Fig. 1, where $h_{i,j,1}$ and $h_{i,j,3}$ are, respectively, the lower and upper bounds of $h_{i,j}$, and $h_{i,j,2}$ is the $h_{i,j}$ value which is the most likely to occur.

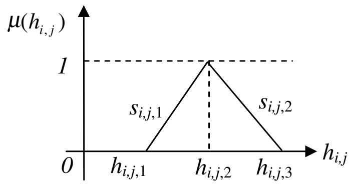

Fig. 1. A triangular membership function of $h_{i,j}$.

Many methods have been developed to solve fuzzy AHP problems. For examples, Graan (1980) generated a fuzzy priority vector by assigning fuzzy weights. Laarhoven and Pedrycz (1983) and Boender et al. (1989) proposed logarithmic least squares methods to generate a priority vector under fuzzy environment. However, most conventional fuzzy AHP methods use repetitive extension principal processes or tedious arithmetic calculations to solve problems. Besides, the obtained fuzzy priority vector needs extra defuzzification techniques to generate a crisp solution.

Yu (2002) proposed a goal-programming (GP) AHP model for solving group decision-making fuzzy AHP problems based on the work of Li and Yu (1999). If there are e decision-makers in the group, the GP-AHP model is formulated as follows:

$$\begin{aligned}
\text{Min} \quad & \sum_{i=1}^{n}\sum_{j>i}^{n}\sum_{e=1}^{E}\left|(\ln w_i - \ln w_j) - \ln h^e_{i,j}\right| \\
\text{Max} \quad & \sum_{i=1}^{n}\sum_{j>i}^{n}\sum_{e=1}^{E}\mu(\ln h^e_{i,j}) \\
\text{Subject to} \quad & \ln h^e_{i,j} = \left\{\mu\left(\ln h^e_{i,j}\right) + \left(s^e_{i,j,2} - s^e_{i,j,1}\right)\ln h^e_{i,j,2} - \left(s^e_{i,j,2} - s^e_{i,j,1}\right)d^e_{i,j} + s^e_{i,j,1}\ln h^e_{i,j,1}\right\}/s^e_{i,j,2}, \\
& \ln h^e_{i,j} - \ln h^e_{i,j,2} + d^e_{i,j} \geqslant 0, \\
& d^e_{i,j}, h^e_{i,j} \geqslant 0 \quad \forall i,j, \\
& w_i \geqslant 0 \quad \forall i,
\end{aligned} \tag{2.2}$$

where $h^e_{i,j}$ indicates the eth decision-maker's fuzzy preference of A_i over A_j. The deviation variable, $d^e_{i,j}$, is used to treat the absolute term. The triangular membership, $\mu\left(\ln\left(h^e_{i,j}\right)\right)$, is a function of $\ln\left(h^e_{i,j}\right)$, where $\ln\left(h^e_{i,j,1}\right)$, $\ln\left(h^e_{i,j,2}\right)$ and $\ln\left(h^e_{i,j,3}\right)$ are lower, middle and upper values of $\ln\left(\tilde{h}^e_{i,j}\right)$. The slopes of the two line segments in the triangular membership function, $s^e_{i,j,1}$ and $s^e_{i,j,2}$, are given by $s^e_{i,j,1} = \frac{\mu\left(\ln\left(h^e_{i,j,2}\right)\right) - \mu\left(\ln\left(h^e_{i,j,1}\right)\right)}{\ln\left(h^e_{i,j,2}\right) - \ln\left(h^e_{i,j,1}\right)}$ and $s^e_{i,j,2} = \frac{\mu\left(\ln\left(h^e_{i,j,3}\right)\right) - \mu\left(\ln\left(h^e_{i,j,2}\right)\right)}{\ln\left(h^e_{i,j,3}\right) - \ln\left(h^e_{i,j,2}\right)}$.

Yu applied a linearization technique to solve fuzzy AHP problems involving triangular, convex and mixed concave-convex fuzzy estimates under a group decision-making environment. Instead of tedious computations, a GP-AHP approach can obtain a crisp solution efficiently.

A major disadvantage for current fuzzy AHP methods (Yu, 2002; Boender et al., 1989; Laarhoven and Pedrycz, 1983; Graan, 1980) is that there is no bound information about the piecewise preferences. A core issue of a fuzzy pairwise comparison model is how to specify the membership function of a preference. For instance, how to specify $h_{i,j,1}$ and $h_{i,j,3}$ in Fig. 1. All current methods assume that a decision maker can tell these values. In fact, without additional information, it is quite difficult for a decision maker to guess these values. If the range is too wide (as $h_{i,j,1} = 1/9$ and $h_{i,j,3} = 9$), it is meaningless to specify the preferences. If the range is too narrow (as $h_{i,j,1} = 6$ and $h_{i,j,3} = 7$), then some good solutions may be eliminated from the solution space.

This study proposes a novel ranking method, which can provide reasonable range information to help a decision maker specifying their fuzzy preferences with an incomplete pairwise comparison matrix.

3. Proposed fuzzy ranking models

Given a set of n alternatives, $A = (A_1, A_2, \ldots, A_n)$, for solving a decision problem, where each alternative contains m criteria $A_i = A_i(c_{i,1}, c_{i,2}, \ldots, c_{i,m})$. Denote w_k as the weight of criterion k. Since the experience has shown multi-criteria syntheses are difficult, all weights are assumed to be positive to avoid making them any more complicated. All criteria values are transformed to the same positive format by subtracting from upper bound, and normalized to a scale from 1 to 9 in advance.

Denote $c_{i,k}$ as the transformed kth criterion value of alternative A_i. Based on the concept of Brugha (2000, 2004), relative measured weights and scores should be synthesized using a power function. Instead of an arithmetic synthesis of score function by AHP, the score function of A_i is assumed to be in a non-linear Cobb–Douglas (Cobb and Douglas, 1928) form with constant return to scale, expressed below

$$S_i(\mathbf{w}) = c_{i,1}^{w_1} c_{i,2}^{w_2}, \ldots, c_{i,m}^{w_m}, \tag{3.1}$$

where $w_1, \ldots, w_m \geqslant 0$, $\sum_{k=1}^{m} w_k = 1$ and $1 \leqslant S_i \leqslant 9$.

Define a *relative dominance matrix* $\mathbf{R} = (r_{i,j})$ as a $n \times n$ matrix, where element $r_{i,j} = \frac{\text{Score}_i}{\text{Score}_j}$ expresses the ratio of scores of A_i over A_j. $r_{j,i} = \frac{1}{r_{i,j}}$ is assumed. Section 3.1 illustrates how to articulate the reduced ranges of $r_{i,j}$ to help a decision maker to specify related preferences. Section 3.2 describes the proposed fuzzy ranking model.

3.1. Range reduction techniques

This study proposes a range reduction technique, a modified DEA ranking method with rank minimization, which is modified from the concept of multiplicative DEA models (Charnes et al., 1982, 1983, 1996). Denote Score_j^p as the score of A_j and w_k^p as the weight of criteria k while A_p is chosen as the target alternative (i.e. the score or rank of A_p is optimized). Denote Rank_p as the rank of A_p. $1 \leqslant \text{Rank}_p \leqslant n$. Let $\text{Rank}_p = 1$ if A_p is the best choice. A_p is superior to A_j (denoted as $A_p \succ A_j$) if and only if $\text{Rank}_p < \text{Rank}_j$.

Remark 1. $\text{Rank}_j < \text{Rank}_p$ if and only if $\text{Score}_j^p > \text{Score}_p^p$.

Since Score_p^p is the maximum score that A_p can have, $\text{Score}_j^p > \text{Score}_p^p$ implies that $\text{Score}_j^p > \text{Score}_p^p > \varepsilon$ no matter how we specify w_k^p. A_j therefore is clearly superior to A_p. Denote $\text{Sup}(p)$ as *a superior set* of A_p. $\text{Sup}(p)$ is a collection of A_j which are superior to A_p, expressed as

$$\text{Sup}(p) = \{A_j | \text{Score}_j^p > \text{Score}_p^p \text{ for } j = 1, 2, \ldots, n\}. \tag{3.2}$$

Rank_p can then be computed as

$$\text{Rank}_p = 1 + \|\text{Sup}(p)\|, \tag{3.3}$$

where $\|\text{Sup}(p)\|$ is the number of elements in $\text{Sup}(p)$.

For a target alternative A_p, the proposed DEA ranking model with rank minimization is formulated below.

Model 1 (a modified DEA model)

$$\text{Min} \quad \sum_{j=1, j\neq p}^{n} t_{p,j} \tag{3.4}$$

$$\text{Subject to} \quad \text{Score}_p^p + M \times t_{p,j} \geqslant \text{Score}_j^p \quad \forall j = 1, 2, \ldots, n, \tag{3.5}$$

$$t_{p,j} \in \{0, 1\}, \quad M \text{ is a large value}, \tag{3.6}$$

$$1 \leqslant \text{Score}_j^p \leqslant 9 \quad \forall j, \tag{3.7}$$

$$\sum_{k=1}^{m} w_k^p = 1, \tag{3.8}$$

$$w_1, \ldots, w_m \geqslant 0. \tag{3.9}$$

The objective is to minimize the rank of A_p. If $t_{p,j} = 0$ for all j then Score_p^p has the maximal value. Expression (3.5) means that if $\text{Score}_p^p \geqslant \text{Score}_j^p$ then $t_{p,j} = 0$, and otherwise $t_{p,j} = 1$. A superior set Sup(p) of A_p can be obtained by checking all $t_{p,j}$. If $t_{p,j} = 1$, then A_j is in the superior set of A_p.

Model 1 can be converted directly into following linear 0–1 programs:

$$\text{Min} \quad \sum_{j=1, j\neq p}^{n} t_{p,j}$$

$$\text{Subject to} \quad \sum_{k=1}^{m} w_k^p \ln(c_{p,k}) + M \times t_{p,j} \geqslant \sum_{k=1}^{m} w_k^p \ln(c_{j,k}) \quad \forall j = 1, \ldots, n, \tag{3.10}$$

$$\ln(1) \leqslant \sum_{k=1}^{m} w_k^p \ln(c_{j,k}) \leqslant \ln(9) \quad \forall j = 1, \ldots, n, \tag{3.11}$$

$$t_{p,j} \in \{0, 1\}, \quad M \text{ is a large value},$$

$$\sum_{k=1}^{m} w_k^p = 1,$$

$$w_1, \ldots, w_m \geqslant 0.$$

Let $\overline{r_{i,j}}$ and $\underline{r_{i,j}}$ be, respectively, the upper and lower bound of $r_{i,j}$ with $\underline{r_{i,j}} \leqslant r_{i,j} \leqslant \overline{r_{i,j}}$. Here $\overline{r_{i,j}}$ is obtained by maximizing $r_{i,j}$ under the constraint that no other alternative getting a score greater than 1. Similarly, $\underline{r_{i,j}}$ is found by minimizing $r_{i,j}$ subjected to the same constraints, as described in Model 2.

Model 2 (range reduction model)

$$\text{Max(Min)} \quad r_{i,j} = \frac{\text{Score}_i}{\text{Score}_j} \tag{3.12}$$

$$\text{Subject to} \quad \text{Score}_p < \text{Score}_q \quad \forall A_q \in \text{Sup}(p) \;\; \forall p = 1, \ldots, n, \tag{3.13}$$

$$1 \leqslant \text{Score}_i \leqslant 9 \quad \forall i,$$

$$\sum_{k=1}^{m} w_k = 1,$$

$$w_1, \ldots, w_m \geqslant 0.$$

The restrictions "$\text{Score}_p < \text{Score}_q$" (3.13) are imbedded into the constraint set for all $A_q \in \text{Sup}(p)$. By incorporating the superior sets obtained from Model 1, Model 2 can substantially tighten the ranges of $r_{i,j}$. It is important to note that both $\overline{r_{i,j}}$ and $\underline{r_{i,j}}$ are suggested bounds to assist the decision maker to articulate their preferences. The decision make can still revise both bounds directly. Model 2 can also be converted into a linear 0–1 program as Model 1.

3.2. Proposed fuzzy ranking model

By incorporating the reduced ranges of $r_{i,j}$, a proposed fuzzy ranking model can then be formulated as follows:

Model 3 (a fuzzy ranking model)

$$\text{Min} \quad \text{Obj1} = \sum_{\tilde{r}_{i,j}} \left| \frac{\text{Score}_i}{\text{Score}_j} - \tilde{r}_{i,j} \right| \tag{3.14}$$

$$\text{Max} \quad \text{Obj2} = \sum_{\tilde{r}_{i,j}} \mu(\tilde{r}_{i,j}) \tag{3.15}$$

$$\text{Subject to}\quad \underline{r_{i,j}} \leqslant \frac{\text{Score}_i}{\text{Score}_j} \leqslant \overline{r_{i,j}}, \tag{3.16}$$
$$1 \leqslant \text{Score}_i \leqslant 9 \quad \forall i,$$
$$\sum_{k=1}^{m} w_k = 1,$$
$$w_1, \ldots, w_m \geqslant 0,$$

where $\tilde{r}_{i,j}$ is a fuzzy number representing how many times alternative i is preferred over j, specified by the decision maker. The first objective is to minimize the sum of deviations resulting from the approximation. The second objective tries to maximize the sum of membership functions, which indicate the fulfillment of the decision maker's preferences. Expression (3.16) sets the reduced ranges of $r_{i,j}$.

A piecewise linear function with triangular membership function is illustrated here. Given a triangular fuzzy preference $\tilde{r}_{i,j} = (r_{i,j,1}, r_{i,j,2}, r_{i,j,3})$, a piecewise linear function of $\ln(\tilde{r}_{i,j})$ can be expressed below

$$\mu(\ln(r_{i,j})) = s_{i,j,1} \times (\ln(r_{i,j}) - \ln(r_{i,j,1})) + \frac{(s_{i,j,2} - s_{i,j,1})}{2} \times \left(|\ln(r_{i,j}) - \ln(r_{i,j,2})| + \ln(r_{i,j}) - \ln(r_{i,j,2})\right), \tag{3.17}$$

where $s_{i,j,1} = \frac{\mu(\ln(r_{i,j,2})) - \mu(\ln(r_{i,j,1}))}{\ln(r_{i,j,2}) - \ln(r_{i,j,1})}$ and $s_{i,j,2} = \frac{\mu(\ln(r_{i,j,3})) - \mu(\ln(r_{i,j,2}))}{\ln(r_{i,j,3}) - \ln(r_{i,j,2})}$. $|o|$ is the absolute value of o.

After taking logarithms, Model 3 can then be transferred into a linear program as follows:

$$\text{Min}\quad \text{Obj1} = \sum_{\tilde{r}_{i,j}} (\ln(\text{Score}_i) - \ln(\text{Score}_j) - \ln(r_{i,j}) + 2z_{i,j})$$
$$\text{Max}\quad \text{Obj2} = \sum_{\tilde{r}_{i,j}} \mu(\ln(r_{i,j}))$$
$$\text{Subject to}\quad \mu(\ln(r_{i,j})) = s_{i,j,1} \times (\ln(r_{i,j}) - \ln(r_{i,j,1})) + (s_{i,j,2} - s_{i,j,1}) \times (\ln(r_{i,j}) - \ln(r_{i,j,2}) + d_{i,j}) \quad \forall \tilde{r}_{i,j}, \tag{3.18}$$
$$\ln(r_{i,j}) - \ln(r_{i,j,2}) + d_{i,j} \geqslant 0 \quad \forall \tilde{r}_{i,j}, \tag{3.19}$$
$$d_{i,j} \geqslant 0 \quad \forall \tilde{r}_{i,j}, \tag{3.20}$$
$$\ln(\underline{r_{i,j}}) \leqslant \ln(\text{Score}_i) - \ln(\text{Score}_j) \leqslant \ln(\overline{r_{i,j}}) \quad \forall i, j > i, \tag{3.21}$$
$$\sum_{\tilde{r}_{i,j}} (\ln(\text{Score}_i) - \ln(\text{Score}_j) - \ln(r_{i,j}) + z_{i,j}) \geqslant 0 \quad \forall \tilde{r}_{i,j}, \tag{3.22}$$
$$z_{i,j} \geqslant 0 \quad \forall \tilde{r}_{i,j}, \tag{3.23}$$
$$\ln(\text{Score}_i) = \sum_{k=1}^{m} w_k \ln(c_{i,k}) \quad \forall i, \tag{3.24}$$
$$\ln(1) \leqslant \ln(\text{Score}_i) \leqslant \ln(9) \quad \forall i,$$
$$\sum_{k=1}^{m} w_k = 1,$$
$$w_1, \ldots, w_m \geqslant 0.$$

Expressions (3.18)–(3.20) are based on Yu (2002). Expression (3.21) is from (3.16). In order to linearize the absolute term in Obj1, constraints (3.22) and (3.23) are added into the model based on the work of Li (1996). Expression (3.24) is from (3.1). Model 3 is a multi-objective linear optimization problem, which can be solved by many techniques to get a global optimum. One of commonly used methods is formulated below:

$$\text{Min}\quad \text{Obj1} - \text{Obj2}$$
$$\text{Subject to}\quad \text{All other constraints are in Model 3}$$

4. A numerical example

Considering the implications of a tendency of multicriteria decision-making, Brugha (2004) used screening, ordering and choosing phases to find a preference. The solving process of the proposed approach is illustrated by these three phases as listed below:

(i) The screening phase: the DM specifies upper and lower bounds of attributes to screen out of poor alternatives.
(ii) The ordering phase: the DM tries to put a preference order on the remaining alternatives.
  (a) All criteria values are transformed to the same positive format by subtracting from upper bound, and then normalized to a scale from 1 to 9.
  (b) Use the proposed DEA ranking model (Model 1) to get the superior set of each alternatives.
  (c) Apply range reduction model (Model 2) to provide reasonable upper and lower bounds of $r_{i,j}$, where $r_{i,j}$ represents a pairwise comparison of A_i over A_j.
  (d) Decision makers specify fuzzy preferences based on the support of suggested ranges. Apply the fuzzy ranking model (Model 3) to get the weights of each criterion.
  (e) Calculate the scores of each alternative and get a preference order.
(iii) The choosing phase: the DM makes a choice between two or three close alternatives.

The following example, modified from Harvard Business Review (Hammond et al., 1998), is applied to illustrate above concepts. The example describes a business problem for renting an office. A decision maker defines four major objectives to fulfill in selecting his/her office: (i) a short commute time from home to office, (ii) good access to his clients, (iii) sufficient space, and (iv) low costs. The commuting time is the average time in minutes needed to travel to work during rush hour. The percentage of his clients within an hour's drive of the office is used to measure the access to clients. Office size is measured in square feet, and cost is measured by monthly rent. The DM hopes to keep monthly cost and commuting time as small as possible and remaining criteria larger. There are thirty available alternatives.

(i) The screening phase
Suppose the DM sets the upper bounds of monthly cost and commute time to be 2200 and 60, respectively, and the lower bounds of office size and customer access to be 500 and 50%, respectively. Twenty alternatives are screened out. The remaining 10 alternatives are listed in Table 1.
(ii) The ordering phase
  (a) Monthly cost and commute time are transformed to the positive format by subtracting from upper bound: inexpensiveness instead of costs, convenience instead of commute time. Then, all criteria values are normalized to a scale from 1 to 9, as listed in Table 2.

Table 1
Original criteria values for renting an office

Alternative	Minimization		Maximization	
	Monthly cost ($)	Commute time (minutes)	Office size (square feet)	Customer access (%)
A_1	1850	45	800	50
A_2	1700	25	700	80
A_3	1500	20	500	70
A_4	1900	25	950	85
A_5	1750	30	700	75
A_6	1950	40	950	65
A_7	1800	60	850	60
A_8	1600	45	1000	50
A_9	2200	50	900	75
A_{10}	2000	45	1050	85

Table 2
Transformed criteria values with positive format

Alternative	Maximization			
	Inexpensive	Convenience	Office size (square feet)	Customer access (%)
A_1	5.00	4.00	5.36	1.00
A_2	6.71	8.00	3.91	7.86
A_3	9.00	9.00	1.00	5.57
A_4	4.43	8.00	7.55	9.00
A_5	6.14	7.00	3.91	6.71
A_6	3.86	5.00	7.55	4.43
A_7	5.57	1.00	6.09	3.29
A_8	7.86	4.00	8.27	1.00
A_9	1.00	3.00	6.82	6.71
A_{10}	3.29	4.00	9.00	9.00

(b) Let $M = 1000$ and $\varepsilon = 0.1$, solving the office-renting example by Model 1 yields the best rank and the corresponding score of each alternative in the last three columns of Table 3. Taking A_1 for instance, let $p = 1$, solving Model 1 yields $\text{Score}_1^1 = 5.04$. By the proposed model, there are four alternatives better than A_1, that is $\text{Sup}(1) = \{A_2, A_4, A_5, A_8\}$. The best rank of A_1 is 5.
In order to compare the proposed model with conventional DEA models, the optimal score and the corresponding rank of each alternative by a conventional DEA model (multiplicative CCR model) are listed in the second and third column of Table 3. Taking A_1 for instance, the rank of A_1 is 7. The proposed model can obtain a better rank ($\text{Rank}_1 = 5$) for alternative A_1 than that of conventional DEA model ($\text{Rank}_1 = 7$).

(c) Next, in order to provide reasonable ranges of $r_{i,j}$, a range reduction model is applied to the example. Applying Model 2 to the example yields the upper $(\overline{r_{i,j}})$ and lower $(\underline{r_{i,j}})$ bound of $r_{i,j}$, as listed in Table 4. For simplicity, only the upper-right parts of the matrix are shown. Each element is divided into two parts, where upper and lower values indicate the upper and lower bounds, respectively. The ranges of $r_{i,j}$ are significantly reduced by adding the superior set constraints. Taking $r_{1,2}$ for instance, the original range of $r_{1,2}$ is $1/9 \leqslant r_{1,2} \leqslant 9$ because $1 \leqslant \text{Score}_i \leqslant 9$, $\forall i$. After taking Model 2, the range of $r_{1,2}$ is reduced to $0.24 \leqslant r_{1,2} \leqslant 0.98$.
In order to help the decision makers specify preferences conveniently, the values of $r_{i,j}$ are transferred to a discrete numerical rating $r'_{i,j}$ based on the pairwise comparison scale (Saaty, 1980) listed in Table 5. The reduced upper and lower bounds of $r_{i,j}$ in Table 4 can then be transferred to a corresponding matrix in Saaty's scale in Table 6. These reduced ranges provide reasonable upper and lower bounds to help the decision maker specify their preferences.

Table 3
Results and comparisons of the office-renting example

	Conventional DEA model		Proposed DEA model		
	Maximal score	Rank	Best rank	Score	Superior set Sup(k)
A_1	5.36	7	5.04	5	$\{A_2, A_4, A_5, A_8\}$
A_2	8.00	2	7.17	1	
A_3	9.00	1	7.23	1	
A_4	9.00	1	8.33	1	
A_5	7.00	4	5.69	2	$\{A_2\}$
A_6	7.55	3	6.02	2	$\{A_4\}$
A_7	6.09	6	5.58	2	$\{A_8\}$
A_8	8.27	2	5.92	1	
A_9	6.82	5	6.74	3	$\{A_4, A_{10}\}$
A_{10}	9.00	1	7.40	1	

Table 4
The reduced ranges $[\underline{r_{i,j}}, \overline{r_{i,j}}]$ of $r_{i,j}$

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	1	0.98	3.31	1.00	1.00	1.21	4.00	1.00	4.16	1.43
		0.24	0.27	0.22	0.28	0.40	0.70	0.64	0.41	0.31
A_2		1	3.38	1.47	1.16	1.75	8.00	4.18	6.13	2.04
			0.77	0.56	1.02	0.62	0.77	0.68	0.63	0.48
A_3			1	1.91	1.44	2.26	9.00	3.67	7.94	2.66
				0.17	0.30	0.18	0.23	0.21	0.19	0.14
A_4				1	1.83	1.82	8.00	4.49	4.16	2.00
					0.74	1.09	0.85	0.64	1.14	0.86
A_5					1	1.59	7.00	3.61	5.59	1.85
						0.60	0.76	0.64	0.62	0.47
A_6						1	5.00	2.47	3.52	1.25
							0.72	0.55	1.00	0.75
A_7							1	1.00	4.91	1.56
								0.25	0.33	0.25
A_8								1	6.38	2.10
									0.41	0.31
A_9									1	0.76
										0.32
A_{10}										1

Table 5
The mapping of $r_{i,j}$ and the pairwise comparison scale for AHP preferences (Saaty, 1980)

Numerical rating	Verbal judgments of preferences	Value of $r_{i,j}$
9	Extremely preferred	$r_{i,j} \geqslant 8.5$
8	Very strongly to extremely	$7.5 \leqslant r_{i,j} < 8.5$
7	Very strongly preferred	$6.5 \leqslant r_{i,j} < 7.5$
6	Strongly to very strongly	$5.5 \leqslant r_{i,j} < 6.5$
5	Strongly preferred	$4.5 \leqslant r_{i,j} < 5.5$
4	Moderately to strongly	$3.5 \leqslant r_{i,j} < 4.5$
3	Moderately preferred	$2.5 \leqslant r_{i,j} < 3.5$
2	Equally to moderately	$1.5 \leqslant r_{i,j} < 2.5$
1	Equally preferred	$1 \leqslant r_{i,j} < 1.5$

Table 6
The reduced ranges of $r_{i,j}$ in Saaty's scale

Max	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	1	1	3	1	1	1	4	1	4	1
		1/4	1/4	1/4	1/4	1/2	1	1/2	1/2	1/3
A_2		1	3	1	1	2	8	4	6	2
			1	1/2	1	1/2	1	1	1/2	1/2
A_3			1	2	1	2	9	4	8	3
				1/6	1/3	1/5	1/4	1/5	1/5	1/7
A_4				1	2	2	8	4	4	2
					1	1	1	1/2	1	1
A_5					1	2	7	4	6	2
						1/2	1	1/2	1/2	1/2
A_6						1	5	2	4	1
							1	1/2	1	1
A_7							1	1	5	2
								1/4	1/3	1/4
A_8								1	6	2
									1/2	1/3
A_9									1	1
										1/3
A_{10}										1

(d) Suppose the decision maker specifies the membership functions as $\tilde{r}_{1,4} = (\frac{1}{4}, \frac{1}{3}, \frac{1}{2})$, $\tilde{r}_{4,7} = (2, 3, 5)$ and $\tilde{r}_{2,7} = (3, 5, 6)$, Model 3 can be formulated as follows:

$$\text{Min} \quad \text{Obj1} - \text{Obj2}$$

$$\text{Obj1} = \left(\sum_{k-1}^{4} w_k \times \ln(c_{1,k}) - \sum_{k-1}^{4} w_k \times \ln(c_{4,k}) - \ln(r_{1,4}) + 2 \times z_{1,4}\right) + \left(\sum_{k-1}^{4} w_k \times \ln(c_{4,k}) - \sum_{k-1}^{4} w_k \times \ln(c_{7,k}) - \ln(r_{4,7}) + 2 \times z_{4,7}\right) + \left(\sum_{k-1}^{4} w_k \times \ln(c_{2,k}) - \sum_{k-1}^{4} w_k \times \ln(c_{7,k}) - \ln(r_{2,7}) + 2 \times z_{2,7}\right)$$

$$\text{Obj2} = \mu(\ln(r_{1,4})) + \mu(\ln(r_{4,7})) + \mu(\ln(r_{2,7}))$$

Subject to

$$\mu(\ln(r_{1,4})) = 3.48 \times \left(\ln(r_{1,4}) - \ln\left(\frac{1}{4}\right)\right) + (-2.47 - 3.48) \times \left(\ln(r_{1,4}) - \ln\left(\frac{1}{3}\right) + d_{1,4}\right),$$

$$\mu(\ln(r_{4,7})) = 2.47 \times (\ln(r_{4,7}) - \ln(2)) + (-1.96 - 2.47) \times (\ln(r_{4,7}) - \ln(3) + d_{4,7}),$$

$$\mu(\ln(r_{2,7})) = 1.96 \times (\ln(r_{2,7}) - \ln(3)) + (-5.48 - 1.96) \times (\ln(r_{2.7}) - \ln(5) + d_{2,7}),$$

$$\ln(r_{1,4}) - \ln\left(\frac{1}{3}\right) + d_{1,4} \geqslant 0,$$

$$\ln(r_{4,7}) - \ln(3) + d_{4,7} \geqslant 0,$$

$$\ln(r_{2,7}) - \ln(5) + d_{2,7} \geqslant 0$$

$$d_{1,4} \geqslant 0, \quad d_{4,7} \geqslant 0, \quad d_{2,7} \geqslant 0,$$

$$\left(\sum_{k-1}^{4} w_k \times \ln(c_{1,k}) - \sum_{k-1}^{4} w_k \times \ln(c_{4,k}) - \ln(r_{1,4}) + z_{1,4}\right) \geqslant 0,$$

$$\left(\sum_{k-1}^{4} w_k \times \ln(c_{4,k}) - \sum_{k-1}^{4} w_k \times \ln(c_{7,k}) - \ln(r_{4,7}) + z_{4,7}\right) \geqslant 0,$$

$$\left(\sum_{k-1}^{4} w_k \times \ln(c_{2,k}) - \sum_{k-1}^{4} w_k \times \ln(c_{7,k}) - \ln(r_{2,7}) + z_{2,7}\right) \geqslant 0,$$

$$z_{1,4} \geqslant 0, \quad z_{4,7} \geqslant 0, \quad z_{2,7} \geqslant 0,$$

$$\ln(\underline{r_{i,j}}) \leqslant \sum_{k=1}^{4} w_k \times \ln(c_{i,k}) - \sum_{k=1}^{4} w_k \times \ln(c_{j,k}) \leqslant \ln(\overline{r_{i,j}}) \quad \forall i = 1, \ldots, n-1, \quad j = i+1, \ldots, n,$$

$$\ln(1) \leqslant \sum_{k=1}^{4} w_k \times \ln(c_{i,k}) \leqslant \ln(9) \quad \forall i,$$

$$\sum_{k=1}^{m} w_k = 1,$$

$$w_1, \ldots, w_m \geqslant 0.$$

Solving the above program by Lingo software yields a global optimal solution with obj1 = 0.465, obj2 = 0, $w_1 = 0.24$, $w_2 = 0.36$, $w_3 = 0$, $w_4 = 0.4$, $\mu(\ln(r_{1,4})) = 1$, $\mu(\ln(r_{4,7})) = 1$, $\mu(\ln(r_{2,7})) = 1$, $\ln(r_{1,4}) = -1.099$, $\ln(r_{4,7}) = 1.099$, $\ln(r_{2,7}) = 1.609$ and $d_{1,4} = d_{4,7} = d_{2,7} = 0$.
Represented in Saaty's ratio scale, $r_{1,4} = \frac{1}{3}$, $r_{4,7} = 3$ and $r_{2,7} = 5$. The approximation error, obj1, is equal to 0.463. Obj2 is equal to 3, which implies high fulfillment of the decision maker's preferences.

Table 7
The final score and rank of each alternative

	Score	Rank
A_1	2.43	9
A_2	7.62	1
A_3	7.43	2
A_4	7.28	3
A_5	6.67	4
A_6	4.48	6
A_7	2.43	9
A_8	2.70	8
A_9	3.18	7
A_{10}	5.27	5

(e) Substituting the values of w_1, w_2, w_3, and w_4 into Expression (3.1) yields the score and rank of each alternatives, as listed in Table 7. A_2 is the best choice, following by A_3, A_4, A_5, A_{10}, A_6, A_9 and A_8. A_1 and A_7 are at the same score and ranked the worst.

(iii) The choosing phase
Since the scores of the top three alternatives A_2, A_3 and A_4 are close to each other, the DM may make a final choice among these three alternatives.
This office-renting example demonstrates how proposed approach provides reasonable upper and lower bounds information of preferences based on the concept of DEA. By referring to these ranges, a decision maker can specify his/her fuzzy preferences partially, and obtain the optimized ranks of alternatives.

5. Concluding remarks

This study proposes a novel ranking method which incorporates fuzzy preferences specified by a decision maker. Based on a modified DEA model, reasonable upper and lower bounds are provided to assist a decision maker in articulating related preferences. A goal-programming model with minimal approximation errors and maximal fulfillment of a decision maker's preferences is proposed to solve the fuzzy preference problem directly and efficiently.

A comparison with other ranking methods, such as AHP methods (Saaty, 1977, 1980, etc.) and Fuzzy AHP methods (Graan, 1980; Chang, 1996; Leung and Cao, 2000; Yu, 2002, etc.), indicates the following advantages of the proposed method:

(i) The proposed method provides reasonable upper and lower bounds information about specifying preferences, which are not provided by other methods.
(ii) The proposed method can treat incomplete pairwise comparison matrices; while most of the other methods cannot deal with them.
(iii) The proposed fuzzy ranking method results in a crisp solution directly; however, most of fuzzy AHP methods require extra defuzzification techniques to obtain such a solution.

Two issues could be studied in the future research. First, to enhance the fuzzy rating of the proposed method, the fuzzy set gradual membership grid technique (Badiru and Cheung, 2002) can be incorporated in the proposed fuzzy ranking method. Second, in order to improve some restrictions resulting from linear programming methods whose solutions are found at corners of combinations of constraints, non-linear or fuzzy constraints can be applied in the range limits.

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