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K-INVARIANT KAEHLER STRUCTURES ON $K_{\mathbf{C}}/N$ AND THE ASSOCIATED LINE BUNDLES

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Abstract. Let K be a compact semi-simple Lie group, and let N be a maximal unipotent subgroup of the complexified group $K_{\mathbf{C}}$. In this paper, we classify all the K-invariant Kaehler structures on $K_{\mathbf{C}}/N$. For each Kaehler structure ω, let $\mathbf{L}$ be the line bundle with connection whose curvature is ω. We then study the holomorphic sections of $\mathbf{L}$, which constitute a K-representation space.

1. Introduction

Let K be a compact semi-simple Lie group, let $G = K_{\mathbf{C}}$ be its complexification, and let KAN be an Iwasawa decomposition of G. Since G and N are complex, the space $X = G/N$ is a complex homogeneous space, with left K action. We denote by T the centralizer of A in K; T is a Cartan subgroup of K here. Since T normalizes N, it acts on X on the right.

Given a suitable K-invariant symplectic structure ω on X, the process of geometric quantization [5] converts it into a K-representation space V. A desired property of V is that every irreducible K-representation occurs with multiplicity one (termed a *model* in [3], if V is in addition unitary). Several years ago, A.S. Schwarz suggested the space $X = G/N$ as a candidate for this process [6], and this was worked out in [1] for $K \times T$-invariant Kaehler structures on X.

In this paper, we classify all the K-invariant Kaehler structures on X. For each K-invariant Kaehler structure, we study its associated line bundle whose holomorphic sections constitute the K-representation space V.

Let $H = TA = T_{\mathbf{C}}$ (which is a Cartan subgroup of G), with Lie algebra $\mathfrak{h}$. Let n be the rank of G, and denote by $\lambda_1, ..., \lambda_n \in \mathfrak{h}^*$ the positive simple roots. For each positive simple root λ_j, let $\chi_j : H \longrightarrow \mathbf{C}^*$ be the character that satisfies $\chi_j(\exp v) = \exp \lambda_j(v)$. We say that a differential form β transforms by χ_j if $R_t^*\beta = \chi_j(t)\beta$, where $t \in T$ and R_t is the right T action. We shall prove

Theorem I. *Every K-invariant Kaehler structure on X can be uniquely written as*

$$\omega = \sqrt{-1}\partial\bar{\partial}F + \sum_1^n d\beta_j,$$

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where $\sqrt{-1}\partial\bar{\partial}F$ *is a* $K \times T$*-invariant Kaehler structure; each* β_j *is* K*-invariant and transforms by* χ_j *under the right* T *action.*

Since the $K \times T$-invariant component $\sqrt{-1}\partial\bar{\partial}F$ has been described carefully in [1], this theorem completely classifies all the K-invariant Kaehler structures on X. Observe that ω is $K \times T$-invariant if and only if the component $\sum d\beta_j$ vanishes. We shall see in the next theorem that this is in fact the desired property to perform geometric quantization.

Let ω be a K-invariant Kaehler structure on X. By Theorem I, ω is exact, hence is in particular integral. Therefore, we can consider the complex line bundle $\mathbf{L}$ on X whose Chern class is $[\omega] = 0$. It is equipped with a connection ∇ whose curvature is ω. Let $\mathfrak{k}$ be the Lie algebra of K. For $\xi \in \mathfrak{k}$, we denote by $\xi^\sharp$ the vector field on X induced by the K action. There is a canonical representation of $\mathfrak{k}$ on the smooth sections of $\mathbf{L}$, given by the operators

$$\nabla_{\xi^\sharp} + \sqrt{-1}\phi^\xi, \;\; \xi \in \mathfrak{k},$$

where ϕ is the moment map associated to the K action on (X, ω) ([2], [5]). We shall assume that this representation is K-invariant; namely, it lifts to a holomorphic K action on $\mathbf{L}$. Thus $\mathcal{O}(\mathbf{L})$, the space of holomorphic sections of $\mathbf{L}$, becomes a K-representation space. The following theorem asserts that $\mathcal{O}(\mathbf{L})$ suits the purpose of geometric quantization best when ω is $K \times T$-invariant:

Theorem II. *The following are equivalent:*
(i) ω *is* $K \times T$*-invariant ;*
(ii) ω *has a potential function ;*
(iii) *every irreducible* K*-representation occurs in* $\mathcal{O}(\mathbf{L})$ *with multiplicity one.*

2. K-invariant Kaehler structures on $K_{\mathbf{C}}/N$

In this section, we prove Theorem I, which classifies all the K-invariant Kaehler structures on $X = K_{\mathbf{C}}/N$. Let $\partial, \bar{\partial}$ be the Dolbeault operators on X, and $Z_K^{0,1}(X,\mathbf{C})$ be the space of K-invariant $\bar{\partial}$-closed *(0,1)*-forms on X. We shall see that every K-invariant Kaehler structure ω on X can be written as

$$\omega = \partial\alpha + \overline{\partial\alpha},$$

where $\alpha \in Z_K^{0,1}(X, \mathbf{C})$. Therefore, we now develop some machineries to calculate $Z_K^{0,1}(X, \mathbf{C})$.

Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}, \mathfrak{t}$ be the Lie algebras of G, K, A, N, T respectively. Let $n = \text{rank}\, G = \dim_{\mathbf{C}} H$. Let $\lambda_{\pm 1}, ..., \lambda_{\pm m} \in \Delta$ be the root system of $\mathfrak{g}$, where $\lambda_1, ..., \lambda_n$ are positive simple roots, and $n \leq m$. Let

$$\{\xi_j, \xi_{-j}\} \subset \mathfrak{g}/\mathfrak{h} \;\; , \;\; \xi_{\pm j} \in \mathfrak{g}_{\pm\lambda_j} \tag{2.1}$$

be a Weyl basis ([4], p. 421) of $\mathfrak{g}/\mathfrak{h}$. Then

$$\zeta_j = \xi_j - \xi_{-j} \;, \; \gamma_j = \sqrt{-1}(\xi_j + \xi_{-j}) \;\in \mathfrak{k} \;\; ; \;\; j = 1, ..., m. \tag{2.2}$$

In fact, under the image of $\mathfrak{k} \longrightarrow \mathfrak{k}/\mathfrak{t}$, $\{\zeta_j, \gamma_j\}$ form a basis of $\mathfrak{k}/\mathfrak{t}$. By Iwasawa, $\mathfrak{g}/\mathfrak{n} \cong \mathfrak{k} + \mathfrak{a}$, which induces an almost complex structure J on $\mathfrak{k} + \mathfrak{a}$. Then

$$J\zeta_j = \gamma_j \; ; \; J\gamma_j = -\zeta_j. \tag{2.3}$$

The Killing form identifies these vectors with ζ_j^* , $\gamma_j^* \in \mathfrak{k}^*$. Consider

$$q_j = \zeta_j + \sqrt{-1}\gamma_j \in \wedge^{0,1}(\mathfrak{k} + \mathfrak{a}) \;, \; v_j = \zeta_j^* - \sqrt{-1}\gamma_j^* \in \wedge^{0,1}(\mathfrak{k} + \mathfrak{a})^* \;, \tag{2.4}$$

for $j = 1, ..., m$. By Iwasawa, $X = G/N = KA$. Therefore, we may identify $\wedge^{0,1}(\mathfrak{k}+\mathfrak{a})$ and $\wedge^{0,1}(\mathfrak{k}+\mathfrak{a})^*$ with the $K \times A$-invariant anti-holomorphic vector fields and complex *(0,1)*-forms on X.

Let $\xi \in \mathfrak{t}$, and $ad_\xi^* : \mathfrak{k}^* \longrightarrow \mathfrak{k}^*$. Then

$$\begin{aligned} ad_\xi^* \zeta_j^* &= ad_\xi^*(\xi_j^* - \xi_{-j}^*) \\ &= \lambda_j(\xi)\xi_j^* + \lambda_j(\xi)\xi_{-j}^* \\ &= -\sqrt{-1}\lambda_j(\xi)\gamma_j^*, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} ad_\xi^* \gamma_j^* &= ad_\xi^* \sqrt{-1}(\xi_j^* + \xi_{-j}^*) \\ &= \sqrt{-1}(\lambda_j(\xi)\xi_j^* - \lambda_j(\xi)\xi_{-j}^*) \\ &= \sqrt{-1}\lambda_j(\xi)\zeta_j^*. \end{aligned} \tag{2.6}$$

Note that, in (2.5) and (2.6), the root λ_j satisfies $\sqrt{-1}\lambda_j(\xi) \in \mathbf{R}$ for $\xi \in \mathfrak{t}$.

For $\xi \in \mathfrak{t}$, the action of ad_ξ^* on $\wedge^{0,1}(\mathfrak{g})^*$ preserves $\wedge^{0,1}(\mathfrak{n})^*$. Therefore ad_ξ^* acts on $\wedge^{0,1}(\mathfrak{g}/\mathfrak{n})^* = \wedge^{0,1}(\mathfrak{k}+\mathfrak{a})^*$. Let $v_j \in \wedge^{0,1}(\mathfrak{k}+\mathfrak{a})^*$ be the *(0,1)*-form given in (2.4). Then (2.5) and (2.6) give

$$ad_\xi^* v_j = \lambda_j(\xi) v_j. \tag{2.7}$$

We now go from Lie algebra representation to group representation; so consider

$$Ad_t^* : \wedge^{0,1}(\mathfrak{k}+\mathfrak{a})^* \longrightarrow \wedge^{0,1}(\mathfrak{k}+\mathfrak{a})^*,$$

for $t \in T$. Also, for each root λ_j, we define the character $\chi_j : T \to \mathbf{C}^*$ which satisfies

$$\chi_j(\exp \xi) = \exp(\lambda_j, \xi) \tag{2.8}$$

for all $\xi \in \mathfrak{t}$. Then (2.7) implies that

$$Ad_t^* v_j = \chi_j(t) v_j \tag{2.9}$$

for all $t \in T$.

Since T normalizes N, there is a right T action on $X = G/N$, which induces T representation on the $K \times A$-invariant *(0,1)*-forms $\wedge^{0,1}(\mathfrak{k}+\mathfrak{a})^*$. For $t \in T$, let L_t and R_t denote the left and right T actions on X respectively. Then, by (2.9),

$$\begin{aligned} R_t^* v_j &= R_t^* L_t^* v_j \\ &= Ad_t^* v_j \\ &= \chi_j(t) v_j. \end{aligned} \tag{2.10}$$

Let $\{\zeta_j, \gamma_j\}$ be the vectors in (2.2), and let

$$V = \oplus_1^m \mathbf{R}(\zeta_j, \gamma_j) \subset \mathfrak{k}. \tag{2.11}$$

Then (2.3) says that V is preserved by the almost complex structure on $\mathfrak{k}+\mathfrak{a} = \mathfrak{g}/\mathfrak{n}$. In fact,

$$\mathfrak{k}+\mathfrak{a} = V \oplus \mathfrak{h} \tag{2.12}$$

is a decomposition of $\mathfrak{k}+\mathfrak{a}$ into complex vector subspaces. This decomposition is orthogonal with respect to the Killing form on $\mathfrak{k}+\mathfrak{a} = \mathfrak{g}/\mathfrak{n}$. It induces the inclusions

$$\wedge^{0,k}(V)^*, \wedge^{0,k}(\mathfrak{h})^* \subset \wedge^{0,k}(\mathfrak{k}+\mathfrak{a})^*,$$

where $\wedge^{0,k}(V)^*$ annihilates $\wedge^{0,k}(\mathfrak{h})$ and $\wedge^{0,k}(\mathfrak{h})^*$ annihilates $\wedge^{0,k}(V)$. Note that the *(0,k)*-forms in $\wedge^{0,k}(\mathfrak{h})^*$ are $K \times T$-invariant: If $\xi \in \wedge^{0,k}(\mathfrak{h})^*$, then $ad_v^*\xi = 0$ for all $v \in \mathfrak{h}$. Hence $Ad_t^*\xi = \xi$ for all $t \in T$. It follows that

$$R_t^*\xi = R_t^* L_t^*\xi = Ad_t^*\xi = \xi \tag{2.13}$$

for all $t \in T$.

Let $v_1, ..., v_m$ be the $K \times A$-invariant *(0,1)*-forms in (2.4). We want to consider the $K \times A$-invariant *(0,2)*-forms $\{\bar{\partial} v_j\} \subset \wedge^{0,2}(\mathfrak{k} + \mathfrak{a})^*$. Fix $j \in \{1, ..., m\}$, and the general expression for $\bar{\partial} v_j$ is

$$\bar{\partial} v_j = w + \sum_k u_k \wedge v_k + \sum_{r<s} b_{rs} v_r \wedge v_s, \tag{2.14}$$

for some $w \in \wedge^{0,2}(\mathfrak{h})^*$, $u_k \in \wedge^{0,1}(\mathfrak{h})^*$, $b_{rs} \in \mathbf{C}$. The following lemma describes w, u_k and b_{rs}. Recall that $\lambda_1, ..., \lambda_n$ are simple, among the positive roots $\lambda_1, ..., \lambda_m$. Then,

Lemma 2.1. *In (2.14), $w = 0$; and $u_k = 0$ if and only if $k \neq j$. Finally, all b_{rs} vanish if and only if $j = 1, ..., n$.*

Proof. In view of (2.10),

$$R_t^* \bar{\partial} v_j = \bar{\partial} R_t^* v_j = \chi_j(t) \bar{\partial} v_j$$

for all $t \in T$. Therefore, we also need RHS of (2.14) to transform by χ_j under the right T action. But

$$\begin{aligned} R_t^*(u_k \wedge v_k) &= R_t^* u_k \wedge R_t^* v_k \\ &= L_{t^{-1}}^* u_k \wedge \chi_k(t) v_k \\ &= \chi_k(t) u_k \wedge v_k, \end{aligned} \tag{2.15}$$

and

$$R_t^*(v_r \wedge v_s) = R_t^* v_r \wedge R_t^* v_s = \chi_r(t)\chi_s(t) v_r \wedge v_s. \tag{2.16}$$

Since the non-zero elements of $\{w, u_k \wedge v_k, v_r \wedge v_s\} \subset \wedge^{0,2}(\mathfrak{k} + \mathfrak{a})^*$ are linearly independent, the vectors that do not transform by χ_j have to vanish. Therefore, (2.13) and (2.15) imply that

$$w = 0\ , \ \text{ and } u_k = 0 \text{ if } k \neq j.$$

However, $u_j \neq 0$ in (2.14): Let q_j be the vector in (2.4). By arguments similar to the ones in (2.5) and (2.6), we see that $[\xi, q_j] = \lambda_j(\xi) q_j$ for all $\xi \in \wedge^{0,1}(\mathfrak{h})$. Choose ξ such that $\lambda_j(\xi) \neq 0$. Then

$$\begin{aligned} 0 \neq \lambda_j(\xi)(v_j, q_j) &= (v_j, [\xi, q_j]) \\ &= (\bar{\partial} v_j, \xi \wedge q_j). \end{aligned}$$

Since $\wedge^{0,1}(V)^*$ annihilates $\wedge^{0,1}(\mathfrak{h})$, $(w, \xi \wedge q_j) = (b_{rs} v_r \wedge v_s, \xi \wedge q_j) = 0$. It follows that $(u_j \wedge v_j, \xi \wedge q_j) \neq 0$, i.e. $u_j \neq 0$.

We next compute the b_{rs}, and show that they all vanish if and only if $j = 1, ..., n$. If $j = 1, ..., n$, then λ_j is simple so $\chi_r \chi_s \neq \chi_j$ for all $r, s \in \{1, ..., m\}$. Hence by (2.16), all $b_{rs} = 0$.

On the other hand, consider $j = n+1, ..., m$, so that λ_j is not simple. There exist some roots λ_k, λ_l such that $\lambda_k + \lambda_l = \lambda_j$, and ξ_k, ξ_l, ξ_j be the eigenvectors in (2.1) such that

$$[\xi_k, \xi_l] = c\, \xi_j, \tag{2.17}$$

where $c \in \mathbf{C}$ is non-zero. Let p_k, p_l, v_j be the $K \times A$-invariant vector fields and differential form given in (2.4). With some computations following (2.4), we can conclude from (2.17) that

$$(v_j, [p_k, p_l]) \neq 0.$$

But $(\bar{\partial} v_j, p_k \wedge p_l) = (v_j, [p_k, p_l])$, which means that $b_{kl} \neq 0$ in (2.14). This completes the proof of the lemma. □

Let $\Omega_K^{0,1}(X, \mathbf{C})$ be the space of K-invariant *(0,1)*-forms on X. Since we identify $\wedge^{0,1}(\mathfrak{k} + \mathfrak{a})^*$ with the $K \times A$-invariant *(0,1)*-forms on X, it follows that

$$\Omega_K^{0,1}(X, \mathbf{C}) = C_K^\infty(X, \mathbf{C}) \otimes \wedge^{0,1}(\mathfrak{k} + \mathfrak{a})^*.$$

However, by Iwasawa $X = KA$, so a K-invariant function on X is simply a function on A. Therefore,

$$\Omega_K^{0,1}(X, \mathbf{C}) = C^\infty(A, \mathbf{C}) \otimes \wedge^{0,1}(\mathfrak{k} + \mathfrak{a})^*. \tag{2.18}$$

We are interested in

$$Z_K^{0,1}(X, \mathbf{C}) \ = \ \{\alpha \in \Omega_K^{0,1}(X, \mathbf{C}) \ ; \ \bar{\partial}\alpha = 0\}.$$

For all positive simple roots $\lambda_1, ..., \lambda_n$ with their characters χ_j defined in (2.8), let

$$Z_{K,\lambda_j}^{0,1}(X, \mathbf{C}) \ = \ \{\alpha \in Z_K^{0,1}(X, \mathbf{C}) \ ; \ R_t^* \alpha = \chi_j(t)\alpha \text{ for all } t \in T\}.$$

Similarly, let $Z_{KT}^{0,1}(X, \mathbf{C})$ denote the elements in $Z_K^{0,1}(X, \mathbf{C})$ that are invariant under the right T action. Then

Proposition 2.2. *(i) For every positive simple root λ_j, $Z_{K,\lambda_j}^{0,1}(X, \mathbf{C})$ is one dimensional ;*

(ii) $Z_K^{0,1}(X, \mathbf{C}) = Z_{KT}^{0,1}(X, \mathbf{C}) \oplus (\oplus_1^n Z_{K,\lambda_j}^{0,1}(X, \mathbf{C}))$.

Proof. Let $\alpha \in \Omega_K^{0,1}(X, \mathbf{C})$. By (2.18), we have

$$\alpha = w + \sum_1^m f_j v_j, \tag{2.19}$$

where $w \in C^\infty(A, \mathbf{C}) \otimes \wedge^{0,1}(\mathfrak{h})^*$, $f_j \in C^\infty(A, \mathbf{C})$, and $v_j \in \wedge^{0,1}(V)^*$ are the *(0,1)*-forms in (2.4).

Clearly $C^\infty(A, \mathbf{C})$ is $K \times T$-invariant. It follows from (2.13) that w is $K \times T$-invariant, and from (2.10) that each $f_j v_j$ transforms by χ_j under the right T action.

Since R_t^* commutes with $\bar{\partial}$,

$$\bar{\partial} w \in \Omega_{KT}^{0,2}(X, \mathbf{C}) \ , \ \bar{\partial}(f_j v_j) \in \Omega_{K,\lambda_j}^{0,2}(X, \mathbf{C})$$

for all $j = 1, ..., m$. Therefore, in (2.19), $\bar{\partial}\alpha = 0$ if and only if $\bar{\partial} w = \bar{\partial}(f_1 v_1) = ... = \bar{\partial}(f_m v_m) = 0$; so we can investigate these components separately. Clearly if $\bar{\partial}\alpha = 0$, then $w \in Z_{KT}^{0,1}(X, \mathbf{C})$.

Suppose that $\bar{\partial}(f_j v_j) = 0$, for $j = 1, ..., n$. By Lemma 2.1, $\bar{\partial} v_j = u_j \wedge v_j$, for some $u_j \in \wedge^{0,1}(\mathfrak{h})^*$. Then

$$\begin{aligned} 0 = \bar{\partial}(f_j v_j) \ &= (\bar{\partial} f_j) \wedge v_j + f_j \bar{\partial} v_j \\ &= (\bar{\partial} f_j + f_j u_j) \wedge v_j. \end{aligned} \tag{2.20}$$

If $0 \neq \bar{\partial} f_j + f_j u_j \in \wedge^{0,1}(\mathfrak{h})^*$, then $\bar{\partial} f_j + f_j u_j$ and v_j are linearly independent, which contradicts (2.20). Therefore,

$$\bar{\partial} f_j + f_j u_j = 0. \tag{2.21}$$

We claim that the solutions f_j of (2.21) form a one dimensional vector space: We make the identification

$$f_j \in C_K^\infty(X, \mathbf{C}) = C^\infty(A, \mathbf{C}) \ , \ u_j \in \wedge^{0,1}(\mathfrak{h})^* = Hom(\mathfrak{a}, \mathbf{C}) = \Omega_A^1(A, \mathbf{C}),$$

so that f_j and u_j are a complex function and an invariant form on A respectively. However, the Lie group A is isomorphic to its Lie algebra $\mathfrak{a}$ via the exponential map, and by a choice of Euclidean coordinates, $\mathfrak{a} = \mathbf{R}^n$. Let $dx_1, ..., dx_n$ be the standard 1-forms on $\mathbf{R}^n$. Then, under these identifications, u_j becomes a complex linear 1-form on $\mathbf{R}^n$. Namely, $u_j = \sum_k c_{jk} dx_k$ for some $c_{jk} \in \mathbf{C}$. Also, the operator $\bar\partial$ on $C_K^\infty(X, \mathbf{C})$ is identified with the operator d on $C^\infty(A, \mathbf{C})$. Therefore, (2.21) becomes

$$0 = df_j + f_j u_j = \sum_k \frac{\partial f_j}{\partial x_k} dx_k + c_{jk} f_j dx_k,$$

which means that

$$\frac{\partial f_j}{\partial x_k} = -c_{jk} f_j$$

for all $k = 1, ..., n$. This equation can be solved with

$$f_j(x) = a \exp(-\sum_k c_{jk} x_k),$$

and is unique up to the constant $a \in \mathbf{C}$. Hence the space of solutions of (2.21) is one dimensional, as claimed. This proves part (i) of the proposition.

In order to complete the proof, we need to show that $f_{n+1}, ..., f_m = 0$ in (2.19). Since $\bar\partial \alpha = 0$, $\bar\partial(f_j u_j) = 0$ for all j. Let $j \in \{n+1, ..., m\}$. Then

$$\begin{aligned} 0 = \bar\partial(f_j v_j) &= (\bar\partial f_j) \wedge v_j + f_j (\bar\partial v_j) \\ &= (\bar\partial f_j) \wedge v_j + f_j u_j \wedge v_j + f_j x \\ &= (\bar\partial f_j + f_j u_j) \wedge v_j + f_j x, \end{aligned} \tag{2.22}$$

where $u_j \in \wedge^{0,1}(\mathfrak{h})^*$, and $0 \neq x \in \wedge^{0,2}(V)^*$ by Lemma 2.1. But $(\bar\partial f_j + f_j u_j) \wedge v_j$ and $f_j x$ are linearly independent if they are both non-zero. So (2.22) implies $f_j x = 0$, and hence $f_j = 0$. This proves the proposition. ☐

We have shown that every $\alpha \in Z_K^{0,1}(X, \mathbf{C})$ can be uniquely written as

$$\alpha = \alpha_0 + \alpha_1 + ... + \alpha_n,$$

where α_0 is $K \times T$-invariant and $R_t^* \alpha_j = \chi_j(t) \alpha_j$ for all $j = 1, ..., n$. With this result, we now consider a K-invariant Kaehler structure ω on X. Since K is semi-simple,

$$H^2(X, \mathbf{R}) = H^2(KA, \mathbf{R}) = H^2(K, \mathbf{R}) = 0.$$

Therefore ω, being closed, can be written as

$$\omega = d\beta,$$

for some real 1-form β on X. Let

$$\beta = \alpha + \bar\alpha$$

be its Dolbeault decomposition, where α and $\bar\alpha$ are *(0,1)* and *(1,0)*-forms respectively. Averaging by K if necessary, we may assume that $\beta, \alpha, \bar\alpha$ are K-invariant. Since ω is of type *(1,1)*,

$$\omega = \partial\alpha + \overline{\partial\alpha} \tag{2.23}$$

and

$$\bar{\partial}\alpha = \partial\bar{\alpha} = 0.$$

Therefore, $\alpha \in Z_K^{0,1}(X, \mathbf{C})$. We apply Proposition 2.2 and write

$$\alpha = \sum_0^n \alpha_j, \tag{2.24}$$

where $\alpha_0 \in Z_{KT}^{0,1}(X, \mathbf{C})$ and $\alpha_j \in Z_{K,\lambda_j}^{0,1}(X, \mathbf{C})$ for $j = 1, ..., n$. We claim that α_0 is $\bar{\partial}$-exact:

Recall from (2.19) that α_0 can be written as an element of $C^\infty(A, \mathbf{C}) \otimes \wedge^{0,1}(\mathfrak{h})^*$. We make the natural identification

$$\begin{aligned} \alpha_0 \quad &\in C^\infty(A, \mathbf{C}) \otimes \wedge^{0,1}(\mathfrak{h})^* \\ &= C^\infty(A, \mathbf{C}) \otimes Hom(\mathfrak{a}, \mathbf{C}) \\ &= C^\infty(A, \mathbf{C}) \otimes \Omega^1_A(A, \mathbf{C}) \\ &= \Omega^1(A, \mathbf{C}), \end{aligned}$$

so that α_0 is identified with a complex 1-form on A. Then α_0, being a $\bar{\partial}$-closed *(0,1)*-form, is identified with a closed 1-form on A. Since

$$H^1(A, \mathbf{C}) = 0,$$

it means that α_0 is identified with an exact 1-form on A. Hence

$$\alpha_0 = \bar{\partial} f \in C^\infty(A, \mathbf{C}) \otimes \wedge^{0,1}(\mathfrak{h})^*, \tag{2.25}$$

for some $f \in C^\infty(A, \mathbf{C})$, as claimed.

Set $F = \sqrt{-1}(\bar{f} - f)$ and $\beta_j = \alpha_j + \bar{\alpha}_j$. Then (2.23), (2.24) and (2.25) imply that

$$\omega = \sqrt{-1}\partial\bar{\partial}F + \sum_1^n d\beta_j,$$

which satisfies the decomposition for ω described in Theorem I.

Let $\iota : H \hookrightarrow X$ be the natural holomorphic imbedding of the Cartan subgroup H into X. Then each $\iota^*\beta_j$ is a T-invariant form that transforms by χ_j under the right T action. Since H and T are abelian, $L_t = R_{t^{-1}}$. Therefore

$$\iota^* d\beta_j = d\iota^*\beta_j = 0, \tag{2.26}$$

which means that each $d\beta_j$ degenerates along H. Hence if $\omega = \sqrt{-1}\partial\bar{\partial}F + \sum d\beta_j$ is Kaehler, then $\sqrt{-1}\partial\bar{\partial}F$ cannot vanish. We shall show that more is true: $\sqrt{-1}\partial\bar{\partial}F$ has to be Kaehler.

Let $\mathfrak{k} + \mathfrak{a} = V \oplus \mathfrak{h}$ be the decomposition of $\mathfrak{k} + \mathfrak{a}$ into complex subspaces V and $\mathfrak{h}$, given in (2.12). Note that

$$\mathfrak{k} = V + \mathfrak{t}. \tag{2.27}$$

For each positive simple root λ_j, we let $\chi_j : H \longrightarrow \mathbf{C}^*$ be its corresponding character. We then say that a differential form β transforms by χ_j if $R_t^*\beta = \chi_j(t)\beta$. The following proposition completes the proof of Theorem I.

Proposition 2.3. *Let $\omega = \sqrt{-1}\partial\bar{\partial}F + \sum d\beta_j$ be a K-invariant Kaehler structure, where $\sqrt{-1}\partial\bar{\partial}F$ is $K \times T$-invariant, and each β_j transforms by χ_j under the right T action. Then $\sqrt{-1}\partial\bar{\partial}F$ is necessarily Kaehler.*

Proof. For simplicity, we write $\omega = \omega' + \omega''$, where $\omega' = \sqrt{-1}\partial\bar{\partial}F$ and $\omega'' = \sum d\beta_j$. Since X is diffeomorphic to KA, the points on X can be written as $ka, k \in K, a \in A$.

Suppose that ω' is not Kaehler. Since it is K-invariant, ω'_a is degenerate for some $a \in A$. Given $\xi \in \mathfrak{k}$, let $\xi^\sharp$ be the infinitesimal vector field on X generated by the K action. Let $V \subset \mathfrak{k}$ be the subspace given in (2.11), generated by the basis $\{\zeta_j, \gamma_j\}$ in (2.2). Then

$$(V^\sharp)_a \oplus (\mathfrak{t}^\sharp)_a \oplus J(\mathfrak{t}^\sharp)_a \;=\; T_aX.$$

Further, $(V^\sharp)_a$ and $(\mathfrak{t}^\sharp)_a \oplus J(\mathfrak{t}^\sharp)_a$ are complementary with respect to ω'_a (see [1]). Therefore, one of the following two cases is valid:

Case 1. ω'_a is degenerate on $(\mathfrak{t}^\sharp)_a \oplus J(\mathfrak{t}^\sharp)_a$. Then, together with (2.26), we see that ω_a is degenerate.

Case 2. ω'_a is degenerate on $V^\sharp$. There exists a non-zero vector

$$\eta = \sum_1^m a_j\zeta_j + b_j\gamma_j \;\in\; V$$

such that $\omega'(\eta^\sharp, J\eta^\sharp)_a \leq 0$. Let

$$\pi : \mathfrak{k} = V \oplus \mathfrak{t} \longrightarrow \mathfrak{t}$$

be the projection onto the second factor, and let

$$\Phi : X \longrightarrow \mathfrak{k}^*$$

be the moment map associated to the K action on (X, ω'). Then

$$\begin{aligned} 0 \quad &\geq \omega'(\eta^\sharp, J\eta^\sharp)_a \\ &= (\Phi(a), [\eta, J\eta]) \\ &= (\Phi(a), \pi[\eta, J\eta]) \qquad\qquad \text{as } \Phi(a) \in \mathfrak{t}^* \text{ [1]}, \\ &= \textstyle\sum_1^m (a_j^2 + b_j^2)(\Phi(a), \lambda_j). \end{aligned}$$

Since η is non-zero, there exists some positive root λ_j such that

$$(\Phi(a), \lambda_j) \leq 0. \tag{2.28}$$

For this λ_j, we see that

$$\omega(\zeta_j^\sharp, J\zeta_j^\sharp)_a = \omega(\zeta_j^\sharp, \gamma_j^\sharp)_a = (\Phi(a), \lambda_j) + (\sum_i \beta_i, [\zeta_j, \gamma_j]^\sharp)_a. \tag{2.29}$$

But in view of (2.26) and $[\zeta_j, \gamma_j] \in \mathfrak{t}$,

$$(\sum_i \beta_i, [\zeta_j, \gamma_j]^\sharp)_a = 0. \tag{2.30}$$

Combining equations (2.28), (2.29) and (2.30), we get

$$\omega(\zeta_j^\sharp, J\zeta_j^\sharp)_a \leq 0,$$

i.e. ω is not Kaehler. This solves the situation of Case 2, hence Proposition 2.3.

□

We have thus proved Theorem I. The $K \times T$-invariant component, $\sqrt{-1}\partial\bar{\partial}F$, has been studied carefully in [1], and we briefly state it here: By K-invariance and the exponential map, F becomes a function on $\mathfrak{a}$. Then $\sqrt{-1}\partial\bar{\partial}F$ is Kaehler if and only if the following conditions hold.

(i) $F : \mathfrak{a} \longrightarrow \mathbf{R}$ is strictly convex.

(ii) Let $\Phi : X \longrightarrow \mathfrak{k}^*$ be the moment map corresponding to the K action on $(X, \sqrt{-1}\partial\bar{\partial}F)$. Then the image of Φ intersects $\mathfrak{t}^*$ inside the positive Weyl chamber.

Hence this result, together with Theorem I, classifies all the K-invariant Kaehler structures on X.

3. Line bundles on $K_{\mathbf{C}}/N$

Let ω be a K-invariant Kaehler structure on $X = K_{\mathbf{C}}/N$. We write ω in the canonical form

$$\omega = \sqrt{-1}\partial\bar{\partial}F + \sum_1^n d\beta_j,$$

as expressed in Theorem I.

We claim that ω is $K \times T$-invariant if and only if it has a potential function:

Since each β_j transforms by the character χ_j, we know that ω is $K \times T$-invariant if and only if $\sum d\beta_j$ vanishes. This will imply that ω has a potential function F. Conversely, suppose that ω has a potential function F. Then, averaging by K if necessary, we may assume that F is K-invariant. But by Iwasawa, $X = KA$; so the K-invariant function F is just a function on A. Consequently F, and hence ω, are $K \times T$-invariant. We have shown that the first two properties of Theorem II, the $K \times T$-invariance and the existence of a potential function, are equivalent.

As we shall see, this is the desired property to perform geometric quantization. The above formula proves that ω is exact, and hence is in particular integral. Therefore, there exists a complex line bundle $\mathbf{L}$ whose Chern class is $[\omega] = 0$; it is equipped with a connection ∇ whose curvature is ω. There is a natural $\mathfrak{k}$ representation on the smooth sections of $\mathbf{L}$ given by the operators

$$\nabla_{\xi^\sharp} + \sqrt{-1}\phi^\xi,\ \xi \in \mathfrak{k},$$

where ϕ is the moment map corresponding to the K action on X ([2], [5]). We shall assume that this representation is induced from a holomorphic K action on $\mathbf{L}$. With nice topological conditions, this assumption is always valid. For instance, it is always possible to do this if K is simply-connected [5]. This way, we get a K representation on $\mathcal{O}(\mathbf{L})$, the space of holomorphic sections of $\mathbf{L}$. In [1], we see that if ω is $K \times T$-invariant, then $\mathcal{O}(\mathbf{L})$ contains every irreducible K representation with multiplicity one. We shall show that $\mathcal{O}(\mathbf{L})$ is not so nice if ω is not invariant under the right T action. Therefore, the most appropriate setting to perform geometric quantization is a $K \times T$-invariant Kaehler manifold.

Proposition 3.1. *Suppose ω is not invariant under the right T action. Then there is no non-vanishing holomorphic section on $\mathbf{L}$.*

Proof. As in (2.23), we write

$$\omega = \partial\alpha + \overline{\partial\alpha},$$

where α is a *(0,1)*-form and $\bar{\partial}\alpha = 0$. Since ω is not $K \times T$-invariant, it has no potential function; hence α is not $\bar{\partial}$-exact. Write

$$\beta = \alpha + \bar{\alpha}, \tag{3.1}$$

so that $d\beta = \omega$.

Suppose s is a non-vanishing holomorphic section of $\mathbf{L}$. Then

$$\gamma = \sqrt{-1}\frac{\nabla s}{s} \tag{3.2}$$

is a complex 1-form, and, by the definition of curvature, $d\gamma = \omega$. This means that $\gamma - \beta$ is closed. Since K is semi-simple,

$$H^1(X, \mathbf{C}) = H^1(KA, \mathbf{C}) = H^1(K, \mathbf{C}) = 0.$$

Therefore, there exists a complex-valued function h such that

$$\gamma - \beta = dh.$$

From (3.2), we see that

$$\sqrt{-1}\nabla s = (\beta + dh)\, s. \tag{3.3}$$

Let J be the almost complex structure on X. Since s is holomorphic,

$$\nabla_{\sqrt{-1}\xi - J\xi} s = 0 \tag{3.4}$$

for every real vector field ξ. Combining (3.3) and (3.4), we get

$$(\beta + dh, \xi + \sqrt{-1}J\xi)s = 0.$$

But s is non-vanishing, so $(\beta + dh, \xi + \sqrt{-1}J\xi) = 0$. Since $\xi + \sqrt{-1}J\xi$ is anti-holomorphic,

$$\beta + dh \in \Omega^{1,0}(X, \mathbf{C}).$$

From (3.1),

$$\alpha + \bar{\alpha} + \partial h + \bar{\partial} h \in \Omega^{1,0}(X, \mathbf{C}).$$

We end up with

$$\alpha + \bar{\partial} h \in \Omega^{1,0}(X, \mathbf{C}),$$

where α is a *(0,1)*-form that is not $\bar{\partial}$-exact. This is a contradiction, and hence the proposition. □

Using this result, we now show that if ω is not right T invariant, then the trivial (one dimensional) representation does not occur in $\mathcal{O}(\mathbf{L})$ as a subrepresentation. This is because, if the trivial representation occurs in $\mathcal{O}(\mathbf{L})$, then it contains some K-invariant holomorphic sections other than the zero section. However:

Proposition 3.2. *Suppose ω is not right T invariant. Then the only K-invariant holomorphic section of $\mathbf{L}$ is the zero section.*

Proof. Suppose s is a K-invariant holomorphic section. By the previous proposition, $s_p = 0$ for some $p \in X$. Let Kp denote the K orbit through p. Then s, being K-invariant, vanishes on Kp. However, the orbit Kp contains some totally real subspace of X, as can be seen from (2.12) and (2.27). Hence s, being holomorphic, has to be the zero section. □

This completes the proof of Theorem II. We conclude that $\mathcal{O}(\mathbf{L})$ serves the purpose of geometric quantization best when X is a $K \times T$-invariant Kaehler manifold.

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