SIAM J. Applied Dynamical Systems
Vol. 7, No. 4, pp. 1445–1476

Stable Synchrony in Globally Coupled Integrate-and-Fire Oscillators*

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Abstract. A model of integrate-and-fire oscillators is studied. In the special case of identical oscillators, the model was first proposed and analyzed by Mirollo and Strogatz [*SIAM J. Appl. Math.*, 50 (1990), pp. 1645–1662]. We assume, as in Mirollo and Strogatz's model, that each oscillator x_i evolves according to a map f_i. Our main results are to demonstrate that the concavity structure of f_i plays an important role in determining whether Peskin's second conjecture holds true. Specifically, the following statements are proved. First, the system of convex oscillators (i.e., $f_i'' < 0$ for all i), in general, synchronizes when the oscillators are not quite identical. Second, the system of a certain class of concave oscillators (i.e., $f_i'' > 0$ for all i) will not achieve synchrony for initial conditions in a set of positive measure when the oscillators are nearly identical. Third, the system of concave oscillators may achieve synchrony under certain sufficient conditions, provided that the oscillators are not quite nonidentical and that its concavity is small.

Key words. stable synchrony, nonidentical oscillators, integrate-and-fire, concavity

AMS subject classifications. 92A09, 34C15, 58F40

DOI. 10.1137/070709220

1. Introduction.

Large assemblies of oscillator units can spontaneously evolve to a state of large scale organization. Synchronization is the best known phenomenon of this kind, where after some transient regime a coherent oscillatory activity of the set of oscillators emerges. This interesting phenomenon is quite common in many different disciplines such as engineering [62], physics [15, 35, 51], chemistry [36], as well as biology [61]. For example, in southeastern fireflies, thousands of individuals gathered on trees may flash in unison. Other examples of biological oscillators are the rhythmic activity of cells of the heart pacemaker [29, 40, 43, 55], of cells of the pancreas [48, 49], and of neural networks [9, 13, 20, 43, 45, 50]. Synchronization of oscillators has been studied in both phase-coupled models [3, 4, 5, 6, 11, 16, 17, 18, 19, 30, 33, 37, 38, 39, 42, 44, 52, 53, 55, 56, 57, 60, 58, 63], where the interaction between the oscillators is smooth and continuous in time, and pulse-coupled models [1, 7, 10, 12, 23, 24, 25, 27, 28, 31, 32, 36, 41, 46, 47, 57, 59], where the membrane voltage is discontinuously reset to a fixed value once it reaches a certain threshold. It should be noted that pulse-coupled models are of greater relevance for neuroscience applications since synaptic coupling is often spike mediated.

This paper deals with a population of integrate-and-fire oscillators with all-to-all pulse coupling. We begin with describing Peskin's model of n integrate-and-fire oscillators. Let the state of the ith oscillator be denoted by x_i, where x_i are subject to the dynamics $\frac{dx_i}{dt} =$

*Received by the editors November 25, 2007; accepted for publication (in revised form) by D. Terman July 17, 2008; published electronically December 3, 2008.
http://www.siam.org/journals/siads/7-4/70922.html

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$-r_ix_i+I_i$, $0 \le x_i \le 1$, $i = 1, 2, \dots, n$, with input $I_i > 0$, a normalized threshold 1, and leakiness $r_i \ge 0$. When $x_i = 1$, the ith oscillator fires and x_i jumps back to zero. As a consequence of the firing of the ith oscillator, the activation of any other oscillator j is incremented by the coupling $\omega_{i,j}$. Should no confusion arise, we write $\omega_{i,j}$ as ω_{ij}. This model was later generalized by Mirollo and Strogatz [41]. It was assumed that the state variable x_i evolves according to a map f_i. When x_i reaches the threshold, the oscillator fires and x_i jumps back instantly to zero, and the activation of any other oscillator j is incremented by the positive coupling ω_{ji}. Specifically, x_i evolve according to $x_i = f_i(\phi_i)$, where $f_i : [0, 1] \to [0, 1]$ is smooth and strictly increasing, i.e., $f_i' > 0$ on $(0, 1)$. Here ϕ_i is a phase variable so that (i) $\frac{d\phi_i}{dt} = \frac{1}{T_i}$, where T_i is the cycle period for oscillator x_i when evolving freely; (ii) $\phi_i = 0$ when the oscillator is at its lowest state $x_i = 0$; and (iii) $\phi_i \equiv 1$ at the end of cycle when the oscillator reaches the threshold $x_i = 1$. Therefore, f_i satisfy $f_i(0) = 0$, $f_i(1) = 1$. These maps f_i are to be called evolution maps. The inverses of f_i are to be denoted by g_i. If $f_i \equiv f$, $T_i \equiv T$, and $\omega_{ij} \equiv \omega$ for all i, j, then the corresponding system is called identical. Otherwise, it is called nonidentical. To describe the dynamics of the model, let $\Phi^0 = (\phi_1^0, \phi_2^0, \dots, \phi_n^0) \in \mathbb{R}^n$ be the initial condition of the oscillators. Here $0 = \phi_1^0 \le \phi_2^0 \le \cdots \le \phi_n^0 < 1$. Further, $\Phi^k = (\phi_{k_1}^k, \phi_{k_2}^k, \dots, \phi_{k_n}^k)$, where $0 = \phi_{k_1}^k \le \phi_{k_2}^k \le \cdots \le \phi_{k_n}^k < 1$, is the state of n oscillators after the kth firing. Denote by $V_k(\Phi^0)$ the set of the indexes of oscillators reaching threshold simultaneously and thus firing the kth time at the same instance. After the $(k-1)$th firing, there will be at least one oscillator ready to fire at the next instance. Such an index set $V_k(\Phi^0)$ of the next firing oscillators is called the trigger set with respect to the initial condition Φ^0 at the kth stage. Let $U_k(\Phi^0)$ be the index set of oscillators which reach the threshold at the kth stage. Note that $U_k(\Phi^0) \supset V_k(\Phi^0)$. Hence, $U_k(\Phi^0)$ may contain the index of the oscillators which reach the threshold after receiving activations from other oscillators in $V_k(\Phi^0)$. Such a set $U_k(\Phi^0)$ is to be termed the spike set with respect to the initial condition Φ^0 at the kth stage. The terms for sets U_k and V_k were first used in [57]. Should no confusion arise, we shall write $V_k(\Phi^0)$ and $U_k(\Phi^0)$ as V_k and U_k, respectively. Immediately after the first firing, the resulting state $\Phi^1 = (\phi_{1_1}^1, \phi_{1_2}^1, \dots, \phi_{1_n}^1)$, $0 = \phi_{1_1}^1 \le \phi_{1_2}^1 \le \cdots \le \phi_{1_n}^1 < 1$, is given by

$$\begin{aligned} \phi_{1_\ell}^1 &= g_{1_\ell}\left(f_{1_\ell}\left(\frac{T_{i_0}}{T_{1_\ell}}(1-\phi_{i_0}^0)+\phi_{1_\ell}^0\right)+\sum_{j\in U_1}\omega_{1_\ell,j}\right) \\ &=: g_{1_\ell}(f_{1_\ell}(\delta_{1_\ell})+\omega_{1_\ell}), \qquad i_0 \in V_1 \text{ and } 1_\ell \in \{1,2,\dots,n\} - U_1 =: S_n - U_1. \end{aligned} \tag{1.1}$$

Note that the first firing consists of firings due to some oscillators reaching threshold simultaneously as well as any other oscillators then reaching threshold due to chain reaction of the earlier firings that are infinitesimally apart. All those chains of firings can be lumped into one set of "simultaneously firing" oscillators. The states $\Phi^k = (\phi_{k_1}^k, \phi_{k_2}^k, \dots, \phi_{k_n}^k)$ of n oscillators after the kth firing can then be defined accordingly. If the cardinality of the spike set U_k, $k = 1, 2, \dots, n$, is one, then we shall say that the system of n oscillators undergoes one *whole* cycle of firings or no *absorption* occurs for the system of n oscillators within one cycle of firings. For Peskin's model, $f_i(\phi) = \frac{I_i}{r_i}(1 - e^{-r_iT_i\phi})$ and $T_i = \ln(\frac{I_i}{I_i - r_i})/r_i$. Peskin conjectured that, first, for identical oscillators, the system approaches a state in which all oscillators are firing synchronously for almost all initial conditions and that, second, this remains true even when the oscillators are not quite identical. The first part of the conjecture was essentially proved by Mirollo and Strogatz [41] with convex oscillators (i.e., $f_i'' < 0$). The second part of Peskin's conjecture was verified by Urbanczik and Senn [57] with flat oscillators (i.e., $f_i'' \equiv 0$). The key feature in those proofs relies on the nonconcavity of the evolution functions f_i. However, Bottani [8] numerically showed that even concave oscillators (i.e., $f_i'' > 0$) can synchronize, provided that the concavity is not too large. The purpose of this paper is two-fold. First, we prove the second part of Peskin's conjecture for the system of convex oscillators. Second, we prove Bottani's numerical results and more. Specifically, we shall show that for the system of n "identical" concave oscillators, no synchronization occurs for initial values in a set of positive measure, provided that $n = 3$ or n is even or phase responding curve $h(x) = g(f(x) + \omega)$ is concave upward. That is to say, in general, concave oscillators may synchronize for almost all initial conditions only if the concavity of the evolution maps is small. Indeed, we prove that the imbalance between the speeds and/or coupling strengths of the oscillators induces the synchronization of the system, provided that the concavity of the evolution maps is sufficiently small.

Since the work of Mirollo and Strogatz, current research into pulse-coupled or integrate-and-fire oscillators has become motivated by more elaborate questions (see, e.g., [25, 32, 47]). There have been many papers [7, 13, 25, 26, 39, 45, 46, 47] discussing those more advanced and complicated models. Some progress has also been made for more realistic biophysical models such as oscillators subject to small noise [36], constant delays [21], or a finite duration of synaptic response [2, 14, 22, 26].

We conclude this introductory section by mentioning the organization of the paper. Section 2 is devoted to the stability conditions for systems of two or more oscillators. In section 3, we derive the absorption conditions for systems of two or more oscillators. In particular, the necessary and sufficient condition for the absorption of two oscillators is given. This, in turn, provides some insight into the role that concavity of the evolution maps plays in determining the absorption process for systems of more than two oscillators. Some sufficient conditions for the absorption conditions for systems of more than two oscillators are derived. The main results of the paper are also recorded in this section.

2. Stable partial and full synchrony.

Before beginning the analysis, we give an intuitive account of the way that synchrony develops as the system evolves: oscillators begin to clump together in "groups" that fire at the same time. For nonidentical oscillators, such groups of oscillators when they reach partial/full synchrony may break up again as the system continues to evolve. Consequently, it is desirable to find *stability conditions* for which a group of oscillators reaching the threshold at the same time will remain coordinated in the future. Such stable partial synchrony then gives rise to a positive feedback process, and thereby tends to grow by "absorbing" other oscillators. Absorptions reduce the number of groups until ultimately only one group remains—at that point the population is synchronized. The scenario above was first pointed out for a different system by Winfree [60], and the phrase "absorption" was coined by Mirollo and Strogatz [41]. With the characteristic of constant speed and equal coupling strengths, the system of identical oscillators always has the stability conditions satisfied. In this section, we shall derive stability conditions. The absorption conditions of the system are to be derived in section 3.

Unless otherwise stated, throughout this paper, the system of oscillators under consideration is either one of two types: convex or concave oscillators.

2.1. Stability conditions for two oscillators. We begin with the study of the system of two oscillators, which provides some insight as to why the system may or may not synchronize. The stability condition for two oscillators is to be derived in this subsection. To this end, we first need certain common properties shared by f and its inverse g.

Lemma 2.1. *Let $h_i : [0,1] \to [0,1]$ be smooth and strictly increasing maps with $h_i(0) = 0$ and $h_i(1) = 1$. Moreover, we assume that h_i have no inflection points and that $\lim_{x\to 0^+} x h_i'(1-x) = 0$ and $\lim_{x\to 0^+} x h_i'(x) = 0$. For each i, let two points, $A = (a_1, a_2)$ and $B = (b_1, b_2)$, be on $y = h_i(x)$ with $b_1 - a_1 \ge \omega_{\min}$. Here $\omega_{\min}$, the minimum of coupling strength, is defined to be*

$$\omega_{\min} = \min_{i,j} \omega_{ij}. \tag{2.1a}$$

Let m_h and M_h be, respectively, the minimum and maximum slope of the secant to h_i with the difference in x being at least $\omega_{\min}$. They are, respectively, defined as follows:

$$m_h = \min_i \left\{ \min \left\{ \frac{h_i(\omega_{\min})}{\omega_{\min}}, \frac{1 - h_i(1-\omega_{\min})}{\omega_{\min}} \right\} \right\} \tag{2.1b}$$

and

$$M_h = \max_i \left\{ \max \left\{ \frac{h_i(\omega_{\min})}{\omega_{\min}}, \frac{1 - h_i(1-\omega_{\min})}{\omega_{\min}} \right\} \right\}. \tag{2.1c}$$

Then

$$M_h \ge \frac{h_i(b_1) - h_i(a_1)}{b_1 - a_1} \ge m_h, \qquad m_h \le 1 \text{ and } M_h \ge 1. \tag{2.2}$$

The equalities hold only if $b_1 - a_1 = \omega_{\min}$ and $a_1 = 0$ or $b_1 = 1$.

Proof. We illustrate only the case that $h_i''(x) > 0$ on $(0,1)$. Clearly, $\frac{h_i(a+x)-h_i(a)}{x} \ge \frac{h_i(x)}{x}$ for any $a \ge 0$, $x > 0$, and $1 \ge a + x \ge 0$. Moreover, $\frac{h_i(x)}{x}$ is increasing and bounded above by 1, and $\frac{1-h_i(1-x)}{x}$ is decreasing and bounded below by 1. Consequently, $M_h \ge \frac{1-h_i(1-\omega_{\min})}{\omega_{\min}} \ge \frac{h_i(b_1)-h_i(a_1)}{b_1-a_1} \ge \frac{h_i(\omega_{\min})}{\omega_{\min}} \ge m_h$. ∎

Remark 2.1.

1. *The geometric and physical meanings of m_h and M_h can be roughly interpreted as follows. Let the difference of two points in the vertical axis be the sum $\sum \omega_{ij}$ of certain coupling strengths due to the firings of certain oscillators; then the resulting difference in h is no smaller than $m_h \sum \omega_{ij}$ and no better than $M_h \sum \omega_{ij}$. See Figure 1.*
2. *Let $\omega_{\max} = \max_{i,j} \omega_{ij}$. An immediate application to Lemma 2.1 and Remark 2.1.1 is the following interpretation of the meaning of the quantities $M_g \omega_{\max}$ and $m_g \omega_{\min}$.*
   (a) *If an oscillator is within the distance $m_g \omega_{\min}$ of the threshold, then it will reach the threshold whenever it receives an activation jump due to the firings of other oscillators. On the other hand, if an oscillator is at least $M_g \omega_{\max}$ away from the threshold, then it will not reach the threshold whenever it receives an activation jump due to a single firing of another oscillator.*

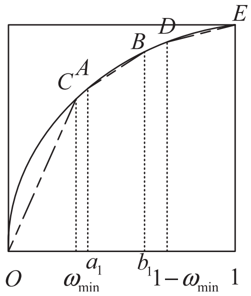


Figure 1. *Points O, C, A, B, D, and E are on a convex map $y = h(x)$. In this situation, $m_{\overline{OC}}$ is defined as the slope of $\overline{OC} = M_h$ and $m_{\overline{DE}} = m_h$. The assertions of Lemma 2.1 can easily be seen from the figure.*

(b) *If the ith oscillator has just received an impulse of strength ω_{ij} at x from the jth oscillator, then its phase jump, $g_i(f_i(x)+\omega_{ij})-x$, is at least $m_g\omega_{\min}$ and at most $M_g\omega_{\max}$ away from the origin.*

Theorem 2.2. *Let*

$$t_{\max} = \max_{i,j}\frac{T_i}{T_j}, \quad \Delta T = t_{\max}-1, \quad \textit{and} \quad \omega_{\min} = \min_{i,j}\omega_{ij}. \tag{2.3}$$

Suppose that f_i satisfy the same assumption as those maps h_i in Lemma 2.1. Let

$$m_g\omega_{\min} \geq \Delta T. \tag{2.4}$$

Then the system of two oscillators is stable.

Proof. Let $\Phi^0 = (\phi_1^0 = 0,\ \phi_2^0 = 0) \in \mathbb{R}^2$. We may assume that ϕ_2^0 has a greater speed $\frac{1}{T_2}$ and, hence, is the one that first reaches the threshold. Thus, $\phi_1^1 = g_1(f_1(\frac{T_2}{T_1})+\omega_{12})$. Therefore, $\phi_1^1 < 1$ if and only if $\frac{1-g_1(1-\omega_{12})}{\omega_{12}}\omega_{12} < 1-\frac{T_2}{T_1}$. If $f_i''(x) > 0$, or equivalently, $g_i''(x) < 0$, and (2.4) holds, then we conclude, via (2.2), that $\phi_1^1 \geq 1$. Consequently, the assertion of the theorem holds. Suppose that $f_i''(x) < 0$, or equivalently, $g_i''(x) > 0$, and that (2.4) is satisfied. Then $\frac{1-g_1(1-\omega_{12})}{\omega_{12}}\omega_{12} \geq \frac{g_1(\omega_{12})}{\omega_{12}}\omega_{12} \geq m_g\omega_{\min} \geq \Delta T \geq 1-\frac{T_2}{T_1}$. We have completed the proof of the theorem. ■

The quantity ΔT is the phase difference between the fastest and slowest oscillators when evolving freely from their lowest state 0 toward the threshold. Therefore, if (2.4) holds, then two oscillators will remain firing synchronously according to Remark 2.1.2(a). To derive the stability condition and the absorption condition of the system, we make use of Lemma 2.1. From here on, we shall consider only the evolution maps that cannot turn "too" sharply at both ends. That is, the evolution maps f_i under consideration have the property that $\lim_{x\to 0^+} xf_i'(1-x) = 0$ and $\lim_{x\to 0^+} xf_i'(x) = 0$. It should be noted that each of the inverses of maps f_i cannot turn too sharply at both ends either.

2.2. Stable partial synchrony for n oscillators. To derive stable partial synchrony for n oscillators, we first need to derive conditions to exclude the possibility that one oscillator will run "too fast." The following proposition gives conditions that will prevent any oscillator from running too fast.

Proposition 2.3. *Let h_i be given as in Lemma 2.1, and let Δh and $\Delta\omega$ be given as follows:*

$$\Delta h = \max_{i,j} \max_{0\le\phi\le 1} |h_i(\phi) - h_j(\phi)| \tag{2.5a}$$

and

$$\Delta\omega = \max_{\substack{i,j\\ i\neq j}} \max_{T} \left(\sum_{\ell\in T} |\omega_{i\ell} - \omega_{j\ell}| \right), \tag{2.5b}$$

where $T \subset S_n - \{i,j\}$. If $n = 2$, then $\sum_{\ell\in T} |\omega_{i\ell} - \omega_{j\ell}|$ is to be interpreted as $|\omega_{ij} - \omega_{ji}|$.

1. *Let $\Phi^0 = (\phi_1^0, \ldots, \phi_n^0)$ with ϕ_1^0 just reaching the threshold and being reset to zero. Assume $U_{k'}$, $k' = 1, 2, \ldots, k$, are mutually exclusive and that $1, i \in S_n - \bigcup_{k'=1}^{k} U_{k'}$ with $\phi_i^0 \neq 0$. Suppose*

$$\begin{aligned} m_g^2 m_f \omega_{\min} &\ge \left(\sum_{j=0}^{k-1} \frac{1}{(m_f m_g)^j} \right) (M_g \Delta\omega + \Delta g + M_g(M_f(\Delta T + 1)\Delta T + \Delta f)) \\ &=: \left(\sum_{j=0}^{k-1} \frac{1}{(m_f m_g)^j} \right) \Delta. \end{aligned} \tag{2.6a}$$

   Then $\phi_i^{k'} \ge \phi_1^{k'}$, $k' = 1, 2 \ldots, k$.
2. *Let*

$$m_g^2 m_f \omega_{\min} \ge \left(\sum_{j=0}^{n-1} \frac{1}{(m_f m_g)^j} \right) \Delta. \tag{2.6b}$$

   Suppose an oscillator has just reached the threshold. Then such an oscillator will not reach the threshold again until every other oscillator does. Moreover, suppose that the system of n oscillators undergoes one whole cycle of firings. Let the resulting phase of the system of oscillators be $\Phi^n = (\phi_{i_1}^n, \phi_{i_2}^n, \ldots, \phi_{i_n}^n)$. Then the firing order for the next cycle with respect to the new initial condition Φ^n is preserved. That is, $\phi_{i_{k_2}}$ fires no earlier than $\phi_{i_{k_1}}$ does whenever $k_1 > k_2$.
3. *Let ϕ_i^m and ϕ_j^m be any two oscillators with $\phi_i^m = \phi_j^m < 1$ and $i, j \notin U_{m+1}$. Then the quantity Δ represents the maximum phase difference between these two oscillators after the next firing. That is, $|\phi_i^{m+1} - \phi_j^{m+1}| < \Delta$.*

Proof. Let δ_i and ω_i be given as in (1.1). Applying the mean value theorem, we get that

$$\begin{aligned} f_i(\delta_i) - f_i(\delta_1) &= f_i'(\xi) \left((1 - \phi_{i0}^0) \frac{T_{i0}}{T_i} \left(1 - \frac{T_i}{T_1} \right) + \phi_i^0 \right) \\ &\ge f_i'(\xi) \left((1 - \phi_{i0}^0) \frac{T_{i0}}{T_i} \left(1 - \frac{T_i}{T_1} \right) + g(\omega_{i1}) \right) \\ &\ge m_f m_g \omega_{\min} - M_f t_{\max} \Delta T. \end{aligned} \tag{2.7a}$$

Here $f_i'(\xi) = \frac{f_i(\delta_i)-f_i(\delta_1)}{\delta_i-\delta_1}$. The assumption that ϕ_1^0 just reach the threshold and Lemma 2.1 have been used to justify the inequalities in (2.7a). Using (2.5a), (2.5b), (2.6a), and (2.7a), we get that

$$\begin{aligned}\phi_i^1 - \phi_1^1 &= [g_i(f_i(\delta_i)+\omega_i) - g_i(f_i(\delta_i)+\omega_1)] + [g_i(f_i(\delta_i)+\omega_1) - g_i(f_i(\delta_1)+\omega_1)]\\ &\quad + [g_i(f_i(\delta_1)+\omega_1) - g_i(f_1(\delta_1)+\omega_1)] + [g_i(f_1(\delta_1)+\omega_1) - g_1(f_1(\delta_1)+\omega_1)]\\ &\geq \left(\sum_{j=1}^{k-1} \frac{1}{(m_f m_g)^j}\right)\Delta. \end{aligned} \tag{2.7b}$$

Inductively, we have that

$$\begin{aligned} &\phi_i^{k'} - \phi_1^{k'} \geq \left(\sum_{j=k'}^{k-1} \frac{1}{(m_f m_g)^j}\right)\Delta, \quad k' = 1, 2, \dots, k-1,\\ &\text{and } \phi_i^k - \phi_1^k \geq 0, \end{aligned} \tag{2.8}$$

and the first part of the proposition follows. It should be noted that on the induction part, ϕ_i^0 in (2.7a) is to be replaced by $\phi_i^{k'-1} - \phi_1^{k'-1}$. Other parts of the estimates remain the same. Let Φ^0 be given. Suppose that the second assertion of the proposition were false. Then there exists a pair of indexes (i,j) such that the ith oscillator is the first oscillator reaching the threshold and the jth oscillator is the index of the first nonzero state oscillator that is outrun by the ith oscillator. To save notation, let the resulting phase state when the ith oscillator reaches the threshold be reset as ϕ_1^0, and the old index j be reset as j again. That is, ϕ_1^0 has just arrived at the threshold. Let k be the number of firings needed for ϕ_1^0 to reach the threshold. From how the indexes of 1 and j are chosen, we conclude that $k \leq n-1$ and that the spike sets associated with those firings are mutually disjoint. It follows from the first part of the proposition that if $\phi_1^k \geq 1$, then $\phi_i^k \geq \phi_1^k \geq 1$, a contradiction. We have just completed the proof of the first assertion of the second part of the proposition, and the second assertion of the second part of the proposition follows. To complete the proof of the last assertion of the proposition, we see that $\phi_i^{m+1} - \phi_j^{m+1}$ can be similarly expressed as those in (2.7b). The corresponding four terms in the brackets of (2.7b) are, respectively, bounded by $M_g\Delta\omega$, $M_g M_f t_{\max}\Delta T$, $M_g\Delta f$, and Δg. ■

We are now ready to state the stability conditions for synchrony.

Theorem 2.4. *Assume that the following stability condition holds:*

$$m_g^2 m_f \omega_{\min} \geq \max\left\{\sum_{j=0}^{n-1} \frac{1}{(m_f m_g)^j}, \sum_{j=0}^{n-2} (M_f M_g)^j\right\}\Delta. \tag{2.9}$$

Then any group of oscillators which reaches the threshold simultaneously at some point will keep doing so in the future.

Proof. Let the ith and the jth oscillators be any two oscillators in the group spiking synchronously. Now reset both oscillators as $\phi_1^0 = \phi_2^0 = 0$. Suppose $1 \in U_{k+1}$ and $2 \notin \bigcup_{k'=1}^{k+1} U_{k'}$. It then follows from Proposition 2.3.2 that $U_{k'}$, $k' = 0, 1, \dots, k+1$, are mutually disjoint and that $k \leq n-2$. Following from Proposition 2.3.3, we conclude that $|\phi_1^1 - \phi_2^1| \leq \Delta$ and, inductively, $|\phi_1^k - \phi_2^k| \leq (\sum_{j=0}^{k-1} \frac{1}{(M_f M_g)^j})\Delta$. Since $\phi_2^{k+1} = g_2(f_2(\frac{T_1}{T_2}(1-\phi_1^k) + \phi_2^k) + \sum_{\ell \in U_{k+1}} \omega_{2\ell})$, the index 2 being not in the set $\bigcup_{k'=1}^{k+1} U_{k'}$ implies that $\frac{T_1}{T_2}(1-\phi_1^k) + \phi_2^k < g_2(1 - \sum_{\ell \in U_{k+1}} \omega_{2\ell})$. Upon using (2.2), we conclude that $m_g \omega_{\min} \leq g_2'(\xi) \sum_{\ell \in U_{k+1}} \omega_{2\ell} < 1 - \frac{T_1}{T_2}(1-\phi_1^k) - \phi_2^k \leq \Delta T + (\sum_{j=0}^{k-1} M_f M_g)\Delta \leq (\sum_{j=0}^{n-2} M_f M_g)\Delta$, a contradiction to (2.9). ■

Each of the terms in (2.9) can be verified analytically. Moreover, the inequality in (2.9) gives a measurement as to how not quite identical the system can be to get the stability condition. Roughly speaking, stability condition (2.9) amounts to saying that the total "weighted" measurements in how "nearly" identical the system is should be less than the minimum of the coupling strengths of the oscillators. In particular, the system of identical oscillators is always stable.

3. Absorption conditions. In this section, we shall derive the conditions for which the absorption process of the system will forge ahead. In fact, we will show that the absorption process always occurs for a system of convex oscillators satisfying stability condition (2.9). On the other hand, the absorption process generally will not occur for a "nearly" identical system of concave oscillators. However, for a system of concave oscillators whose concavity is small, the absorption process is made possible by inducing an imbalance between the speeds and coupling strengths of the oscillators.

3.1. Absorption conditions for two oscillators. We begin with the study of two oscillators. Let $\Phi^0 = (\phi_1^0, \phi_2^0)$ with $0 \leq \phi_1^0 < \phi_2^0 < 1$. Assume that $U_1 = \{2\}$ and $U_2 = \{1\}$. Letting $\phi_2^0 = \phi$, the return map $R_2(\phi)$ is defined to be ϕ_2^2, the phase of the second oscillator immediately after the second firing. Specifically,

(3.1a) $$\phi_1^1 = g_1\left(f_1\left(\frac{T_2}{T_1}(1-\phi_2^0) + \phi_1^0\right) + \omega_{12}\right) =: h_1(\phi),$$

(3.1b) $$\phi_2^2 = g_2\left(f_2\left(\frac{T_1}{T_2}(1-\phi_1^1)\right) + \omega_{21}\right) =: h_2(\phi_1^1),$$

(3.1c) $$\phi_2^2 = h_2 h_1(\phi) =: R_2(\phi).$$

Define the absorption map $A_2(\phi)$ as

(3.1d) $$A_2(\phi) = R_2(\phi) - \phi.$$

The domain of the return map is the set of points for which $U_1 = \{2\}$ and $U_2 = \{1\}$. That is, no absorption occurs within one cycle of the firings whenever the initial values are in the domain of the return map. Now, $U_1 = \{2\}$ if and only if

(3.2a) $$\phi_2^0 > \ell_{12}, \quad \text{where } \ell_{ij} =: 1 - \frac{T_i}{T_j} g_i(1-\omega_{ij}),$$

and $U_2 = \{1\}$ if and only if

(3.2b) $$\phi_1^1 > \ell_{21}.$$

It should be noted that the positivity of ℓ_{ij} can be guaranteed by (2.4). The inequalities (3.2a) and (3.2b) amount to saying that there are limitations as to how close ϕ_2^0 can be to $0(=\phi_1^0)$ and $1(=\phi_1^0)$, respectively. To see why the second observation holds true, let

$$\gamma_{ij} = g_j(\omega_{ji}) - \ell_{ij}. \tag{3.3}$$

Note first that (3.2b) is equivalent to

$$f_1\left(\frac{T_2}{T_1}(1-\phi_2^0)+\phi_1^0\right)+\omega_{12} > f_1(\ell_{21}). \tag{3.4a}$$

If

$$\omega_{12} > f_1(\ell_{21}) \quad \text{or, equivalently,} \quad \gamma_{21} > 0, \tag{3.4b}$$

then ϕ_2^0 can be taken arbitrarily close to 1 from the left and (3.4a) still be satisfied. On the other hand, if $\gamma_{21} < 0$, then ϕ_2^0 cannot get too close to 1. In fact, $\phi_2^0 < h_1^{-1}(\ell_{21}) < 1$. Thus, the sign of γ_{21} determines how close ϕ_2^0 can be to 1 and therefore determines what is the boundary of the domain of the return map at the right end, which, in turn, influences the direction of the flow of the return map near the boundary of the domain. Such direction of the flow then determines whether the absorption process for the system of concave oscillators is to occur. (See Proposition 3.3.) We next show that for "nearly" identical oscillators the signs of γ_{ij} are determined by the concavity structure of the evolution maps.

Lemma 3.1. *Let ∇g be a measurement for the concavity of g_i, which is defined as follows:*

$$\nabla g = \min_i \left|\frac{g_i(\omega_{\max}) + g_i(1-\omega_{\max}) - 1}{\omega_{\max}}\right|. \tag{3.5}$$

Let $\widetilde{\Delta}\omega = \max_{i\neq j}|\omega_{ij} - \omega_{ji}|$. Assume that (2.4) and the following inequality, which is to be called the nearly identical condition, hold:

$$\omega_{\min}\nabla g > \Delta g + M_g\widetilde{\Delta}\omega + \Delta T. \tag{3.6}$$

Then $\gamma_{ij} < 0$ (resp., > 0) for all $i \neq j$, provided that $f_i'' < 0$ (resp., > 0) for all i.

Proof. Let $\widetilde{h}(x) =: \frac{h(x)+h(1-x)-1}{x}$. Here h is a map satisfying the assumptions of the maps given in Lemma 2.1. Then $\widetilde{h}(x)$ is increasing (resp., decreasing) on $(0,1)$, provided that $h''(x) > 0$ (resp., < 0). To see this, we have that $\widetilde{h}'(x) = \frac{x(h'(x)-h'(1-x))-(h(x)+h(1-x)-1)}{x^2} =: \frac{\widetilde{h_1}(x)}{x^2}$ and $\widetilde{h_1}'(x) = x(h''(x)+h''(1-x)) > 0$. Therefore, $\lim_{x\to 0^+}\widetilde{h_1}(x) = 0$, and so $\widetilde{h}(x)$ is increasing on $(0,1)$. The case for $h''(x) < 0$ can be similarly obtained. It is also clear that $\widetilde{h}(x) \leq 0$ (resp., ≥ 0) whenever $h''(x) > 0$ (resp., < 0). Consequently,

$$|-1+g_1(\omega_{12})+g_1(1-\omega_{12})| = \left|\frac{-1+g_1(\omega_{12})+g_1(1-\omega_{12})}{\omega_{12}}\omega_{12}\right| \leq \nabla g\omega_{\min}.$$

Suppose (3.6) holds. Then

$$\begin{aligned}\gamma_{ij} &= -1+g_j(\omega_{ji})+g_j(1-\omega_{ji})+g_i(1-\omega_{ji})-g_j(1-\omega_{ji})\\ &\quad + g_i(1-\omega_{ij})-g_i(1-\omega_{ji})+\left(\frac{T_i}{T_j}-1\right)g_i(1-\omega_{ij}) < 0 \quad (\text{resp.}, > 0),\end{aligned} \tag{3.7}$$

provided that $f_i''(x) < 0$ (resp., > 0), and the assertions of the lemma now follow. ■

Remark 3.1.

1. *The consequences of Lemma 3.1 give that if the system of two oscillators is "nearly" identical in the sense that (3.6) are satisfied, then the domain of the absorption map* A_2 *is* $(\frac{T_1}{T_2}\phi_1^0+\ell_{12}, h_1^{-1}(\ell_{21}))$ *(resp.,* $(\frac{T_1}{T_2}\phi_1^0+\ell_{12}, 1)$*), provided that* $f_i'' < 0$ *(resp.,* $f_i'' > 0$*) for all* i*.*
2. *If* ϕ_2^0 *is not in the domain of the absorption map, then the two oscillators must fire simultaneously within one cycle of the firings. The corresponding system then will stay firing synchronously, provided that stability condition (2.4) is satisfied.*

The domain and monotonicity of the absorption map A_2 play an important role in determining whether the system is to forge ahead in the absorption process. The following lemma shows that the monotonicity of the absorption map depends on the concavity structure of f.

Lemma 3.2. $\frac{\partial A_2}{\partial \phi} > 0$ *(resp.,* < 0*) on its domain, provided that* $f_i'' < 0$ *(resp.,* > 0*) on* $[0,1]$ *for all* i*.*

Proof. We illustrate only the case that $f_i'' < 0$. The other cases can be similarly obtained. Applying the chain rule, we get

$$\begin{aligned}\frac{\partial R_2}{\partial \phi} &= \frac{\partial \phi_2^2}{\partial \phi}\\ &= g_2'\left(f_2\left(\frac{T_1}{T_2}(1-\phi_1^1)\right)+\omega_{21}\right) f_2'\left(\frac{T_1}{T_2}(1-\phi_1^1)\right)\\ &\quad \cdot g_1'\left(f_1\left(\frac{T_2}{T_1}(1-\phi_2^0)+\phi_1^0\right)+\omega_{12}\right) f_1'\left(\frac{T_2}{T_1}(1-\phi_2^0)+\phi_1^0\right).\end{aligned}$$

Using the facts that $g_i'' > 0$ and $g_i'(f_i(x)), f_i'(x) = 1$, $i = 1, 2$, we see immediately that $\frac{\partial R_2}{\partial \phi} > 1$, and hence $\frac{\partial A_2}{\partial \phi} > 0$. ■

Proposition 3.3. *Assume that (2.4) is satisfied. Then the following statements hold:*

1. *Let (3.6) hold or* $\gamma_{21} < 0$*. Then* $R_2(\phi)$ *has a repelling fixed point, provided that* $f_i'' < 0$ *for all* i*. If* $\gamma_{12} - \frac{T_1}{T_2}\phi_1^0 > 0$*, then* $R_2(\phi) - \phi > 0$ *for all* ϕ *in its domain.*
2. *If* $f_i'' > 0$ *for all* i *and* $\gamma_{21} < 0$*, then* $R_2(\phi) - \phi > 0$ *for all* ϕ *in its domain.*
3. *Let* $f_i'' > 0$ *for all* i*. Assume that (3.6) holds. If* $\phi_1^0 < \frac{T_1}{T_2}\gamma_{12}$*, then* $R_2(\phi)$ *has a stable fixed point. If* $\phi_1^0 > \frac{T_2}{T_1}\gamma_{12}$*, then* $R_2(\phi) - \phi < 0$ *for all* ϕ *in its domain.*

Proof. Let $\phi = \frac{T_1}{T_2}\phi_1^0 + \ell_{12}$. Then

$$A_2(\phi) = \gamma_{12} - \frac{T_1}{T_2}\phi_1^0. \tag{3.8a}$$

Thus $A_2(\phi) < 0$, provided that $\gamma_{12} < 0$. On the other hand,

$$A_2(h_1^{-1}(\ell_{21})) = h_2(\ell_{21}) - h_1^{-1}(\ell_{21}) = 1 - h_1^{-1}(\ell_{21}) > 0, \tag{3.8b}$$

and the first part of the proposition now follows. The second part of the proposition is a direct consequence of Lemma 3.1, Lemma 3.2, and (3.8a). To complete the last part of the proposition, it remains to show that $A_2(1) < 0$ or, equivalently, $f_2(\frac{T_1}{T_2}(1 - g_1(\omega_{12}))) + \omega_{21} < 1$

or, equivalently, $\gamma_{12} > 0$, which follows from Lemma 3.1. We have just completed the proof of the proposition. ■

Theorem 3.4.

1. *Assume that* (2.4) *holds. Then we have the following:*
   (a) *The system of two convex oscillators, in general, fires synchronously. Specifically, if* $\gamma_{21} > 0$*, then the synchrony of the system occurs for all initial values. Otherwise, that is, if* $\gamma_{21} \leq 0$*, it synchronizes for almost all initial values. Consequently, for such a system, stability alone implies synchronization.*
   (b) *The system of two concave oscillators converges for all initial values to synchronous firing if and only if*
   $$\gamma_{21} < 0 \quad \text{or} \quad \gamma_{12} < 0. \tag{3.9}$$
   The inequalities in (3.9) *are to be called the absorption condition for the system of two concave oscillators.*
2. *Assume that* (2.4) *and* (3.6) *hold. Let* $\phi_1^0 = 0$*. Then the system of two concave oscillators will settle into a fixed nonfiring state if and only if* ϕ_2^0 *is in the domain of the absorption map* A_2*, that is, if* $\ell_{12} < \phi_2^0 < 1$*.*

Proof. To discuss synchrony for the system of two oscillators, we may just assume $\phi_1^0 = 0$. The statement 1(a) now follows from Proposition 3.3.1. The statement 2 follows easily from Proposition 3.3.3 and Lemma 3.1. It remains to prove statement 1(b). Consider the worst possible cases: (i) $\gamma_{21} < 0$ and $\gamma_{12} > 0$ or (ii) $\gamma_{21} > 0$ and $\gamma_{12} < 0$. The system will achieve synchronization at finite time for all initial conditions. To see this, we consider the case (ii). Let $\Phi^0 = (\phi_1^0, \phi_2^0)$ with $0 \leq \phi_1^0 < \phi_2^0 < 1$. Then either Φ^1 is in synchrony or $\Phi^1 = (\phi_2^1, \phi_1^1)$ with $0 = \phi_2^1 < \phi_1^1 < 1$. Consequently, if no synchrony is achieved after the first firing, then the return map R_2 with respect to the initial phase state Φ^0 has a stable fixed point, while the return map R_2 with respect to the initial phase state Φ^1 has the property that $R_2(\phi) - \phi > 0$. However, the latter case will win out because it takes ϕ_1^1 finite time to reach the threshold and it takes ϕ_2^0 infinite time to reach the fixed point. On the other hand, if both γ_{21} and γ_{12} are nonnegative, then the corresponding return map has a stable fixed point. ■

For the system of two convex oscillators, the associated return map is (volume) expanding; i.e., there exists some $r > 1$ such that $|A_2(\phi) - A_2(\overline{\phi})| > r|\phi - \overline{\phi}|$ for all $\phi \neq \overline{\phi}$ in the domain. Thus, the absorption is bound to happen except for the initial value being the fixed point of the absorption map. The sign of γ_{12} (or γ_{21}) then plays the role of determining whether the absorption map has a (repelling) fixed point or not. On the other hand, for the system of concave oscillators, the corresponding return map is (volume) contracting. If the flow of the return map at both ends of the domain points inward, which is the case for a nearly identical system (see Proposition 3.3.3), then its return map has a stable fixed point. As a result, the corresponding system converges to a nonfiring state. To make the system of concave oscillators fire synchronously, the flow of the return map at both ends has to point in the same direction, which in turn makes the absorption process go forward. The above scenario occurs whenever there is a certain degree of imbalance between oscillators (i.e., $\gamma_{12} < 0$ or $\gamma_{21} < 0$). To see this, note that $\gamma_{12} < 0$ is equivalent to $g_2(\omega_{21}) + \frac{T_1}{T_2} g_1(1 - \omega_{12}) < 1$. For identical concave oscillators, the inequality above will not be satisfied. Thus, to drive such a system into synchrony, the variations in the speed and/or the coupling strength cannot be too small.

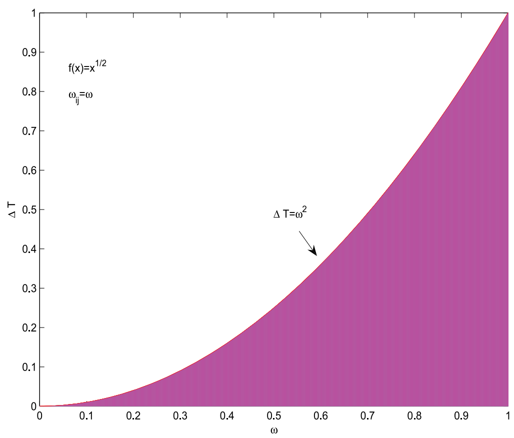


Figure 2. *The shaded area is the set of parameters satisfying* (3.10).

3.2. Feasible parameter and examples. For practical purposes, we consider how feasible it is to verify those stability and absorption conditions. Some numerical results are also provided to support the validity of the theorem. To simplify our calculations, we consider the following three cases: (i) $f_i(x) = \sqrt{x}$, $g_i(x) = x^2$, and $\omega_{12} = \omega_{21} = \omega$; (ii) $f_i(x) = x^2$, $g_i(x) = \sqrt{x}$, and $\omega_{12} = \omega_{21} = \omega$; (iii) $f_i(x) = x^2$, $g_i(x) = \sqrt{x}$, and $T_1 = T_2$.

Case (i): Since $m_g = \omega$, (2.4) becomes

$$\omega^2 \geq \Delta T. \tag{3.10}$$

In the $\omega - \Delta T$ plane, the equality in (3.10) is a parabola. As shown in Theorem 3.4, no absorption condition is needed to achieve synchrony for the system considered here. By choosing parameters randomly from the feasible region (see Figure 2), the numerical results (see Figure 3) indeed support our theory.

Case (ii): For case (ii), if (3.6) is satisfied, then no absorption occurs. Thus, the system in general will not fire synchronously unless ϕ_2^0 is too close to $\phi_1^0 = 0$. To see this, note that $\nabla g = \frac{\sqrt{\omega}+\sqrt{1-\omega}-1}{\omega}$, $m_g = \frac{1-\sqrt{1-\omega}}{\omega}$, and $\Delta g = \Delta\omega = 0$. The stability condition and (3.6) for the associated system then reduce to

$$(1 - \sqrt{1-\omega}) \geq \Delta T \tag{3.11}$$

and

$$\sqrt{\omega} + \sqrt{1-\omega} - 1 > \Delta T, \tag{3.12}$$

respectively. The feasible parameters region in the $\omega - \Delta T$ plane is nonempty (see Figure 4). Picking parameters from this region, we see, via Figure 5, that if $0 \leq \phi_2^0 < \ell_{12}$, then each of the corresponding systems will fire synchronously. Otherwise, they will settle into a nonfiring state. In fact, we choose various sets of parameters from different locations of the region, and all the corresponding systems behave as predicted in Theorem 3.4.2 (see Figure 5).

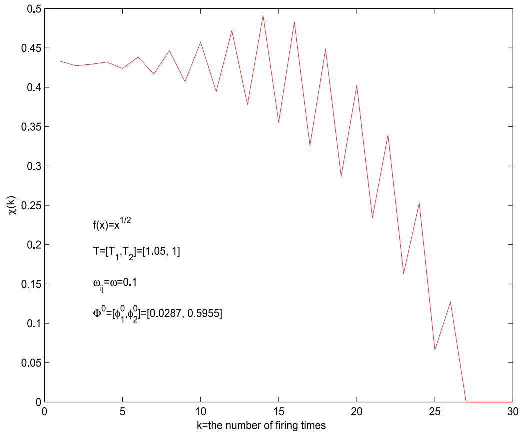


Figure 3. *The evolution of the synchronization order parameter $\chi(k)$ is defined as the sum of the minimum distances between any two oscillators at the kth stage $= \sum_{i=1}^{n}\sum_{j=i+1}^{n} d(\phi_i^k, \phi_j^k)$, where $d(x,y) = \min(|x-y|, |x-y+1|, |x-y-1|)$. If $\chi(k) = 0$ for some large k, then the system fires synchronously at finite time. If $\lim_{k\to\infty}\chi(k) = 0$, then the system fires synchronously eventually or asymptotically.*

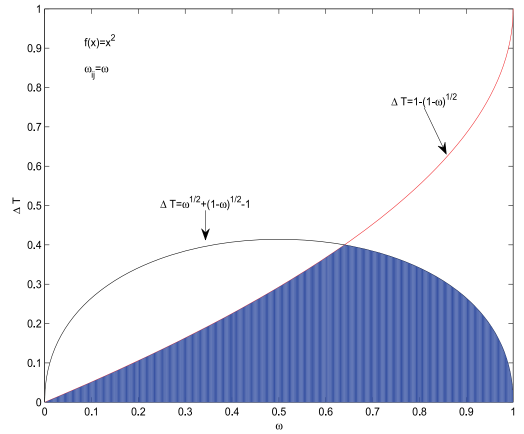


Figure 4. *The shaded area is the set of parameters satisfying (3.11) and (3.12).*

Case (iii): The absorption condition studied here is (3.9). Since $\Delta T = 0$, the stability condition is automatically satisfied. Moreover, (3.9) becomes

$$(\omega_{12})^{\frac{1}{2}} + (1-\omega_{21})^{\frac{1}{2}} < 1. \tag{3.13}$$

The feasible parameters region in the ω_{21}-ω_{12} plane, as given in Figure 6, shows the "imbalance" between parameters ω_{12} and ω_{21}. The numerical results, as demonstrated in Figure 7, also support our theory.

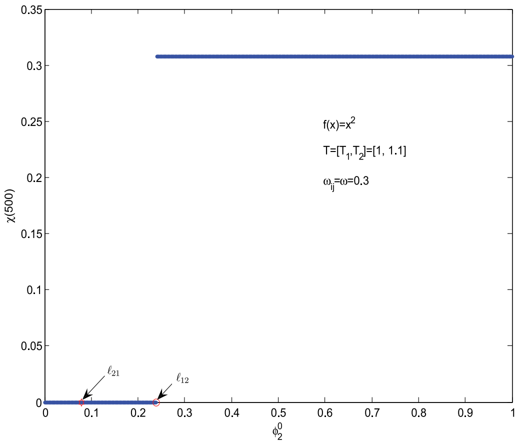


Figure 5. *Choosing parameters T_i and ω from the shaded part in Figure 4, we see that after 500 firings, the synchronization order parameter $\chi(500)$ is a step function with respect to the initial state ϕ_2^0. As predicted, if $\ell_{12} < \phi_2^0 < 1$, then the system settles into a nonfiring state. Otherwise, it fires synchronously.*

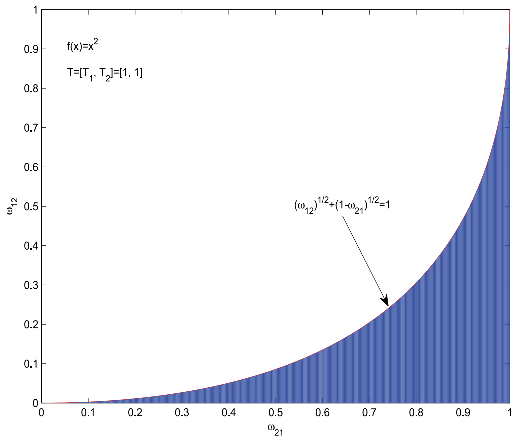


Figure 6. *The shaded area is the stability region for case* (iii).

3.3. Absorption conditions. To understand the absorption process of a system of more than two oscillators, we begin with defining the return map, which was originally defined in [41]. Throughout this section, we shall assume that stability condition (2.9) holds. Unlike the system of two oscillators, the corresponding return map under study in this section is now a high-dimensional map. Let the system of n oscillators undergo one whole cycle of firings. Assume that the resulting phase is denoted by $(\phi_1^0 = 0, \phi_2^0, \ldots, \phi_n^0)$. Let $\Phi^0 = (\phi_2^0, \ldots, \phi_n^0)$. Then the return map $R_n : \mathrm{Domain}(R_n) =: A_n \subset \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is defined to be

$$R_n(\Phi^0) = \Phi^n = (\phi_2^n, \phi_3^n, \ldots, \phi_n^n) =: (r_{2,n}(\Phi^0), \ldots, r_{n,n}(\Phi^0)). \tag{3.14a}$$

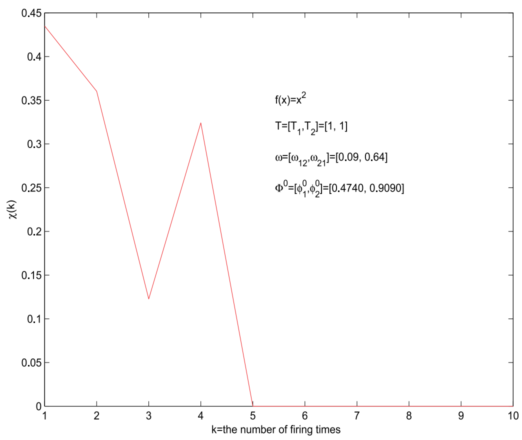


Figure 7. *Let the speed of the oscillators be* 1*. Pick the parameters* ω_{ij} *from the shaded region in Figure* 6*. The synchronization order parameter* $\chi(k)$ *reaches zero after* 5 *firings. The imbalance of parameters in activation gives the synchrony of the system.*

It should be noted, via Proposition 2.3.2, that the maps in (3.14a), (3.14b) are well defined. Moreover,

$$R_n(\Phi^0) = H_n \cdots H_2 H_1(\Phi^0), \tag{3.14b}$$

where

$$H_i = \tau_i \Sigma(\Phi). \tag{3.14c}$$

Here $\Phi = (\phi_2, \phi_3, \dots, \phi_n)$,

$$\begin{aligned}\Sigma(\Phi) &= (\sigma_2, \sigma_3, \dots, \sigma_n)\\ &=: \left(\frac{T_n}{T_1}(1-\phi_n), \frac{T_n}{T_2}(1-\phi_n)+\phi_2, \dots, \frac{T_n}{T_{n-1}}(1-\phi_n)+\phi_{n-1}\right),\end{aligned}$$

and

$$\tau_i(\sigma_2, \sigma_3, \dots, \sigma_n) = (g_1(f_1(\sigma_2)+\omega_{1,n}), \dots, g_{n-1}(f_{n-1}(\sigma_{n-1})+\omega_{n-1,n})).$$

Note that we have implicitly relabeled the oscillators, so each of the image vectors $H_i(\Phi)$ represents the phases of the oscillators $1, 2, \dots, n-1$. That is, the original oscillator 1 has become 2, oscillator 2 has become 3, ..., and oscillator n has become oscillator 1. It also follows from Proposition 2.3.3, Remark 2.1.2(b), and stability condition (2.9) that domain $(R_n) \subset S$, where $S = \{\Phi^0 = (\phi_2^0, \dots, \phi_n^0) \in \mathbb{R}^{n-1} : 0 < \phi_2^0 < \phi_3^0 < \cdots < \phi_n^0 < 1\}$. In fact, the domain of the return map R_n is the set of points in S so that the spike sets $U_i = \{n-i+1\}$, $i = 1, 2, \dots, n$. Having such spike sets is equivalent to the following inequalities:

$$\phi_{n-i+1}^{i-1} - \frac{T_{n-i}}{T_{n-i+1}}\phi_{n-i}^{i-1} > \ell_{n-i,\,n-i+1}, \qquad i = 1, 2, \dots, n, \tag{3.15}$$

where $\ell_{n-i,n-i+1}$ are defined as in (3.2a) and T_0, $\ell_{0,1}$, and ϕ_0 are interpreted as T_n, $\ell_{n,1}$, and ϕ_n, respectively. Consequently, the domain A_n of the return map is

$$A_n = \{\Phi^0 \subset S : \text{the inequalities in (3.15) hold}\}. \tag{3.16a}$$

Since A_n is the finite intersection of open sets, it is open. Moreover, the domain A_k of H_k is the set of initial points satisfying the inequalities in (3.15) for $i = 1, 2, \ldots, k$. So A_i is the set of initial values that will have at least i firings before an absorption occurs. Then

$$A = \bigcap_{i=1}^{\infty} A_i = \text{the set of initial values that live forever without any absorptions.} \tag{3.16b}$$

We next state some properties of the return map $R_n : A_n \to S$. The first assertion of the theorem below is essentially due to Mirollo and Strogatz (see Theorem 3.1 of [41]).

Theorem 3.5. *Assume that stability condition* (2.9) *holds for a system of n oscillators. The following hold true:*

1. *Let $f_i'' < 0$ for all i. Then R_n is volume-expanding on A_n. Consequently, the set A has Lebesgue measure zero.*
2. *Let $f_i'' > 0$ for all i. Then R_n is volume-contracting on A_n.*

Proof. To prove the first assertion of the theorem, it suffices to show that the Jacobian determinant of R_n has absolute value greater than one. From (3.14b) and (3.14c) and the definitions of τ_i and Σ, $\det(DR_n) = \prod_{i=1}^n \det(DH_i) = \prod_{i=1}^n \det(D\tau_i)\det(D\Sigma)$. The map Σ is affine and satisfies $\sigma^n = I$, so $\det(D\Sigma) = \pm 1$. Note that $D\tau_i$ is a diagonal matrix; thus it is easily seen that $\det D\Sigma > 1$ under the assumption that each of the evolution maps is convex. Hence $|\det(DR)| > 1$. The arguments for proving the second assertion of the theorem are similar to those of the first. ■

Since the return map of the system of convex oscillators is volume-expanding, the set of initial values that live forever without any absorptions has measure zero. Hence, it is the nature of the system of convex oscillators to grow by absorbing other oscillators. On the other hand, if the flow of the return map of the system of concave oscillators near the boundary of the domain points inward, such as that of identical concave oscillators, then the system converges to a fixed point, which is a nonfiring state. Hence, to break such a natural tendency of the system one has to introduce some imbalance between the parameters so as to make the direction of the flow point outward near a certain portion of the boundary, as in the case for two oscillators, where a necessary and sufficient condition has been established. Due to the technical difficulty of this, only sufficient conditions are established for systems of more than two oscillators. Such a result is stated in the following.

Theorem 3.6. *Let the number of concave oscillators under consideration be no less than three. Assume the following absorption condition, which is to say that the imbalance measurement is greater than or equal to the concavity of the inverse of the evolution maps:*

$$\frac{M_g}{m_g} \le \max_{0\le i\le n-1}\left(\frac{T_i\omega_{i,i+1}}{T_{i+1}\omega_{i+1,i}}\right). \tag{3.17}$$

Suppose that (2.9) *and* (3.17) *hold. Then the absorption of the system must occur.*

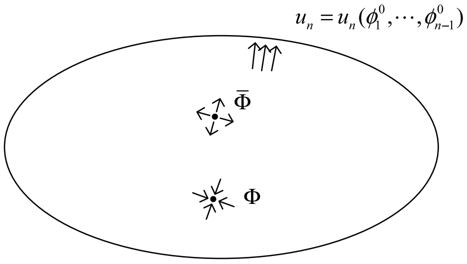


Figure 8. *A visualization of the claim in Step* 3 *of Theorem* 3.6.

Proof. Let $\max_{0\le i\le n-1} \frac{T_i\omega_{i,i+1}}{T_{i+1}\omega_{i+1,i}} = \frac{T_m\omega_{m,m+1}}{T_{m+1}\omega_{m+1,m}}$ for some m. Suppose that no absorption occurs after the first $(n-m)$ firings; then we may relabel oscillators so that the indexes $m+1,\dots,n,1,\dots,m$ of the oscillators become $1,2,\dots,n$, respectively. We then may assume that $\max_{0\le i\le n-1} \frac{T_i\omega_{i,i+1}}{T_{i+1}\omega_{i+1,i}} = \frac{T_n\omega_{n,1}}{T_1\omega_{1,n}}$. The proof the theorem then breaks into three steps. The first part is to prove that sufficient condition (3.17) is so given that the inequality in (3.15) with $i=n$ is violated whenever ϕ_n^0 is sufficiently close to 1 from the left. Consequently, if the system is to undergo one whole cycle of firings, ϕ_n^0 must stay away from 1. That is, $\phi_n^0 < u_n = u_n(\phi_1^0,\dots,\phi_{n-1}^0) < 1$ for some u_n depending on $\phi_1^0,\dots,\phi_{n-1}^0$ and being away from 1. Here $u_n = u_n(\phi_1^0,\dots,\phi_{n-1}^0)$ is a portion of the boundary of the domain of the return map described by $\phi_1^{n-1} - \frac{T_n}{T_1}\phi_n^{n-1} = \ell_{n,1}$. The second step of the proof is to show that the direction of the flow points outward to the boundary whenever ϕ_n^0 is sufficiently close to u_n from the left. Finally, to complete the proof the theorem, we need to show that the return map has no periodic points.

Step 1. Let ϕ_n^0 be sufficiently close to 1 from the left so that $\phi_1^1 - \phi_n^1 (=0) < M_g\omega_{\min}$. We have used Lemma 2.1 to ensure that the above assertion can be done. Note that each of $g_i(f_i(\phi)+\omega) - \phi$, the phase jump at ϕ, is decreasing in ϕ. Hence, the phase jump is greater when the phase position ϕ is closer to the origin. Upon using Lemma 2.1, we conclude that

$$\begin{aligned}\phi_1^{n-1} - \frac{T_n}{T_1}\phi_n^{n-1} &< \phi_1^1 - \phi_n^1 + \left(1 - \frac{T_n}{T_1}\right)\\ &< M_g\omega_{1,n} + \left(1 - \frac{T_n}{T_1}\right) \le \frac{T_n}{T_1} m_g\omega_{n,1} + \left(1 - \frac{T_n}{T_1}\right) < \ell_{n,1}.\end{aligned}$$

We just proved that the boundary of the domain of the return map cannot get arbitrarily close to $\phi_n^0 = 1$. Note that if $n=2$, then $\phi_2^1 = 0$, and so the first inequality above is not necessarily true.

Step 2. Suppose that ϕ_n^0 is close to u_n. Then $\phi_1^{n-1} - \frac{T_n}{T_1}\phi_n^{n-1}$ is close to $\ell_{n,1}$. Consequently, ϕ_n^n is close to 1. Therefore, $\phi_n^n > \phi_n^0$ whenever ϕ_n^0 is sufficiently close to u_n.

Step 3. Since R_n is volume-contracting, any of its periodic points, if one exists, must be stable. Assume, to the contrary, that there exists a periodic point Φ with period k. Let $\overline{R} = R_n^k$. Then Φ becomes a stable fixed point of $\overline{R}$. Moreover, the direction of the flow under $\overline{R}$ near the boundary of the domain still points outward. Consequently, there must exist a unstable fixed point $\overline{\Phi}$ of $\overline{R}$, a contradiction (see Figure 8).

Using Steps 1–3, we conclude that the direction of the flow of the return map points outward to the boundary u_n. Hence, the absorption must occur. We have just completed the proof of the theorem. ∎

From the proof of the above theorem as well as that of Theorem 3.4.1(b), it is easily concluded that for the system of concave oscillators to undergo the absorption process, the domain of the return map contains only the points for which their ϕ_n^0's must stay away from 1. This, in turn, makes the direction of flow near the boundary $u_n = u_n(\phi_1^0, \dots, \phi_{n-1}^0)$ point outward. While the best possible condition to ensure such a scenario for the system of two concave oscillators can be obtained, it is not clear whether the condition that $\min_{1\le i\le n} \gamma_{i-1,i} < 0$ (here $\gamma_{0,1}$ is to be interpreted as $\gamma_{n,1}$) is the best absorption condition for the system of more than two oscillators. Nevertheless, if the concavity of a system is small, then the inequalities in (3.17) can be satisfied by inducing an imbalance between the speeds and weights of oscillators, which will be demonstrated in Proposition 3.7.

We next discuss the dynamics under iteration of the absorption maps. Assume an initial value Φ^0, not necessarily in the domain of the return map. Suppose after initial firings that the system forms k partially synchronous groups. Let the ith group, $1 \le i \le k$, contain k_i oscillators, where $\sum_{i=1}^k k_i = n$, and let these be treated as one new oscillator, denoted by $\overline{\phi}_i$. Clearly, when oscillator $\overline{\phi}_i$ is firing, the activation of each oscillator ϕ_j in the $(i+1)$th synchronous group, where $(\sum_{\ell=1}^i k_\ell) + 1 \le j \le \sum_{\ell=1}^{i+1} k_\ell =: \sigma_{i+1}$, is incremented by the positive coupling $\sum_{k=\sigma_{i-1}+1}^{\sigma_i} \omega_{jk} =: \widetilde{\omega}_{ji}$. For each j, $\sigma_{i-1}+1 \le j \le \sigma_i$, we may define $\widetilde{\omega}_{ji+1}$ similarly. Since the ith and $(i+1)$th synchronous groups may contain more than one oscillator, the new cycle periods $\overline{T}_i$ and $\overline{T}_{i+1}$ of the new oscillators $\overline{\phi}_i$ and $\overline{\phi}_{i+1}$ are chosen as the minimum cycle periods among the oscillators in each group, i.e., $\overline{T}_i = \min_{\sigma_{i-1}+1\le i\le \sigma_i} T_i$ and $\overline{T}_{i+1} = \min_{\sigma_i+1\le i\le\sigma_{i+1}} T_i$. That is, the speed of each group is chosen to be the fastest speed among oscillators in the group. With $\overline{T}_i$ and $\overline{T}_j$ now being fixed, the corresponding new coupling strengths $\overline{\omega}_{i,i+1}$ and $\overline{\omega}_{i+1,i}$ are so chosen that

$$\max_{\sigma_{i-1}+1\le \ell\le\sigma_i}\left(\max_{\sigma_i+1\le j\le\sigma_{i+1}} \frac{\overline{T}_i \widetilde{\omega}_{\ell,i+1}}{\overline{T}_{i+1}\widetilde{\omega}_{j,i}}\right) = \frac{\overline{T}_i\overline{\omega}_{i,i+1}}{\overline{T}_{i+1}\overline{\omega}_{i+1,i}}. \tag{3.18}$$

The idea for such choices is to make the inequality (3.17) as easy as possible to satisfy. Due to the presence of the stability condition, we are allowed to make such choices. For these newly formed synchronous groups to continue their absorption process, we need to further assume that for any permissible set $\{k, k_1, k_2, \dots, k_k\}$, where $2 < k \le n$ and $\sum_{i=1}^k k_i = n$,

$$\frac{M_g}{m_g} \le \max_{0\le i\le k-1}\left(\frac{\overline{T}_i\overline{\omega}_{i,i+1}}{\overline{T}_{i+1}\overline{\omega}_{i+1,i}}\right). \tag{3.19}$$

The right-hand side of the inequality above is to be called the imbalance measurement for the system of more than two oscillators. Note that the quantity $\frac{M_g}{m_g}$ is a measurement for the concavity of g. The closer $\frac{M_g}{m_g}$ is to 1, the more flat the g is. With such an absorption condition, the system continues to grow by absorption until it reaches full synchrony or reduces to two synchronous groups of oscillators. To ensure that these two synchronous groups continue to grow by absorption, we need to have a modified absorption condition for these two groups.

To this end, we assume that the first group consists of old oscillators $\phi_{\ell_1}, \ldots, \phi_{\ell_2}$, where $1 \le \ell_1 < \ell_2 < n$ or $1 < \ell_1 < \ell_2 \le n$, while the second group contains the remaining oscillators. Then the parameters in γ_{12} and γ_{21}, as given in (3.3), need to be updated as well. Let $N_1 = \{\ell_1, \ldots, \ell_2\}$ and $N_2 = \{1, 2, \ldots, n\} - N_1$. Set $\widetilde{\omega}_{j1} = \sum_{i \in N_1} \omega_{ji}$, $j \in N_2$, and $\widetilde{\omega}_{j2} = \sum_{i \in N_2} \omega_{ji}$, $j \in N_1$. Define the new cycle periods of groups N_1 and N_2 to be the minimum cycle periods among the oscillators in each group. Denote such new periods by $\overline{T}_1$ and $\overline{T}_2$. Let

$$\gamma_{12}(\ell_1, \ell_2) = \min_{\substack{i \in N_1 \\ j \in N_2}} \left(g_j(\widetilde{\omega}_{j1}) - 1 + \frac{\overline{T}_1}{\overline{T}_2} g_i(1 - \widetilde{\omega}_{i2}) \right) \tag{3.20a}$$

and

$$\gamma_{21}(\ell_1, \ell_2) = \min_{\substack{j \in N_1 \\ i \in N_2}} \left(g_j(\widetilde{\omega}_{j2}) - 1 + \frac{\overline{T}_2}{\overline{T}_1} g_i(1 - \widetilde{\omega}_{i1}) \right). \tag{3.20b}$$

Then the absorption condition for any two sizes of synchronous groups of oscillators is

$$\min \left\{ \max_{\ell_1, \ell_2} \gamma_{12}(\ell_1, \ell_2), \max_{\ell_1, \ell_2} \gamma_{21}(\ell_1, \ell_2) \right\} < 0. \tag{3.20c}$$

The left-hand side of the inequality in (3.20c) is to be called the imbalance measurement for the system of two oscillators. With those absorption conditions on hand, one would expect the full synchrony of the system. The drawback of absorption conditions (3.19) and (3.20c) is that when n is large, there are enormously many cases needing to be checked. As a consequence, the question of nonemptiness of the set of parameters satisfying the constraints (3.19) and (3.20c) has to be addressed.

Proposition 3.7. *Let the coupling strengths $\omega_{ij} (= \omega)$ of a system of n oscillators all be equal. Let the period cycles of oscillators all be different. Assume that $\omega < \frac{2}{n}$ and that*

$$\frac{[\frac{n}{3}] + 1}{[\frac{n}{3}]} > t_{\max}. \tag{3.21}$$

Then the absorption conditions (3.19) and (3.20c) are satisfied, provided that the concavity of the evolution maps is sufficiently small.

Proof. With the speed of oscillators being all different, $t_{\max} > 1$. Suppose that the absorption occurs after the initial firings. Assume that the system evolves into k, $k > 2$, synchronous groups with sizes of groups being $k_1, k_2, \ldots,$ and k_k. If $k_1 = k_2 = \cdots = k_k$, then the system continues to grow by absorption, provided that $\frac{M_g}{m_g}$ is sufficiently close to 1. Suppose that the sizes of k synchronous groups are not all equal. Then there must exist an index i for which $\frac{\overline{\omega}_{i,i+1}}{\overline{\omega}_{i+1,i}} \ge ([\frac{n}{3}] + 1)/[\frac{n}{3}] > t_{\max}$. Here $[x]$ is the greatest integer that is equal to or less than x. Consequently, the imbalance measurement for this system is greater than one. The system then must reach full synchrony or reduce to the system of two synchronous groups, provided that the concavity of the evolution maps is small. In the case of the latter, we assume that the sizes of these two groups N_1 and N_2 are ℓ and $n-\ell$, respectively, and let $f_i(x) = x$ for all i. Then (3.20c) reduces to

$$\gamma_{12}(\ell, n-\ell) = \ell\omega - \left(\frac{\overline{T}_i}{\overline{T}_j}\right)(n-\ell)\omega + \frac{\overline{T}_i}{\overline{T}_j} - 1, \qquad i \in N_1,\ j \in N_2,$$

$$\gamma_{21}(\ell, n-\ell) = (n-\ell)\omega - \left(\frac{\overline{T}_j}{\overline{T}_i}\right)\ell\omega + \frac{\overline{T}_j}{\overline{T}_i} - 1, \qquad i \in N_1,\ j \in N_2.$$

If n is even and $\ell = n - \ell$, then

$$\gamma_{12}(\ell, n-\ell) = \left(\frac{\overline{T}_i}{\overline{T}_j} - 1\right)\left(1 - \frac{n\omega}{2}\right) \quad \text{and} \quad \gamma_{21}(\ell, n-\ell) = \left(\frac{\overline{T}_j}{\overline{T}_i} - 1\right)\left(1 - \frac{n\omega}{2}\right).$$

Since $t_{\max} > 1$, either $\gamma_{12}(\ell, n-\ell)$ or $\gamma_{21}(\ell, n-\ell)$ is negative. If $n - \ell = \ell + \ell_1$, where $\ell_1 \geq 1$, then $\gamma_{12}(\ell, n-\ell) = (\frac{\overline{T}_i}{\overline{T}_j} - 1)(1 - \ell\omega) - \frac{\overline{T}_i}{\overline{T}_j}\ell_1\omega$. Suppose $\frac{\overline{T}_i}{\overline{T}_j} \leq 1$. Then $\gamma_{12}(\ell, n-\ell) < 0$. If $\frac{\overline{T}_i}{\overline{T}_j} > 1$, then $\gamma_{12}(\ell, n-\ell) < \Delta T - \omega_{\min} \leq 0$. The last inequality is justified by stability condition (2.4). The case that $\ell = (n-\ell) + \ell_1$, where $\ell_1 \geq 1$, can be similarly addressed. Therefore, the remaining two synchronous groups will achieve full synchrony, provided that the concavity of the evolution maps is small. ■

The result of the proposition supports the numerical observation of Bottani [8]. We next define phase responding function $h(x)$ and phase difference function $D(x)$. Both functions are helpful in determining the direction of the flow of the system near the boundary of the return map whenever the number of oscillators is greater than three. Assume that an oscillator receives an activation ω at x. Let the resulting phase $g(f(x) + \omega)$ be denoted by $h(x)$, and define $D(x)$ as

$$D(x) = h(x+a) - h(x). \tag{3.22}$$

Here $a > 0$ is a constant.

Proposition 3.8.

1. *Consider an identical system of three concave oscillators. That is,* $f_i \equiv f$, $g_i \equiv g$, $T_i \equiv T$, *and* $\omega_{ij} = \omega$. *Then the direction of the flow near the boundary of the domain of the return map points inward.*
2. *Suppose* $h''(x) > 0$. *Then* $D(x)$ *is increasing in* x.
3. *Consider an identical system of* n *concave oscillators. If* $h''(x) > 0$, *then*

$$\phi_n^0 - \phi_{n-1}^0 < \phi_n^n - \phi_{n-1}^{n-1} \tag{3.23}$$

whenever $\phi_n^0 - \phi_{n-1}^0$ *is sufficiently close to* $\ell_{ij} = 1 - g(1-\omega)$ *from the left. Consequently, the direction of the flow of the system points inward near the boundary of the domain of the return map.*

Proof. The boundary of the domain of the return map consists of three pieces of curves Γ_1, Γ_2, and Γ_3 defined by $\phi_{n-i+1}^{i-1} - \phi_{n-i}^{i-1} = 1 - g(1-\omega)$, $i = 1, 2, 3$, respectively. To prove the first part of the proposition, it suffices to show that for $i = 1, 2, 3$

$$\phi_{n-i+1}^{i-1} - \phi_{n-i}^{i-1} < \phi_{n-i+1}^{n+i-1} - \phi_{n-i}^{n+i-1} \tag{3.24}$$

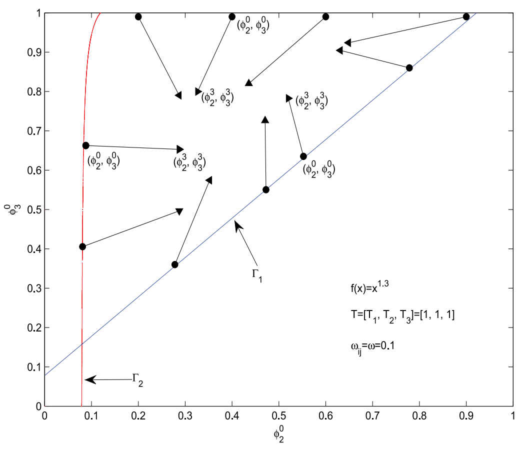


Figure 9. *The initial position and ending position of each arrow are (ϕ_2^0, ϕ_3^0) and (ϕ_2^3, ϕ_3^3), respectively. The direction of the flow near the boundary of the domain of the return map indeed points inward as predicted.*

whenever $\phi_{n-i+1}^{i-1} - \phi_{n-i}^{i-1}$ are sufficiently close to $1-g(1-\omega)$. The inequalities in (3.24) amount to saying that $R(\phi_2^0, \phi_3^0) = (\phi_2^3, \phi_3^3)$ are moving further away from their respective boundaries whenever (ϕ_2^0, ϕ_3^0) are near Γ_1, Γ_2, and Γ_3, respectively (see Figure 9). To this end, we first prove that Γ_3 can be interpreted as $\phi_3^0 = 1$. For any $\phi_3^0 < 1$, we have that $\phi_1^1 > g(\omega)$. And so, for any $\phi_2^0 + 1 - g(1-\omega) < \phi_3^0 < 1$, we see that $\phi_1^2 - \phi_3^2 = \phi_1^1 - (\phi_3^2 - (\phi_1^2 - \phi_1^1)) > g(\omega) - (g(\omega) - (1 - g(1-\omega))) = 1 - g(1-\omega)$. We have used the fact that the phase jump function $h(x) - x$ is decreasing to justify the above inequality. Hence, Γ_3 can be interpreted as claimed. Now, if $\phi_3^0 - \phi_2^0 \approx (1 - g(1-\omega))^-$, then $\phi_2^1 \approx 1^-$, and so $\phi_3^2 - \phi_2^2 \approx (g(\omega))^+$. Here $\phi_2^2 = 0$. Consequently, $\phi_3^3 - \phi_2^3 = \phi_3^2 - ((\phi_2^3 - \phi_2^2) - (\phi_3^3 - \phi_3^2)) > g(\omega) - (g(\omega) - (1 - g(1-\omega))) = 1 - g(1-\omega)$. To prove (3.24) for $i = 1$, it remains to show that there exists an $\varepsilon > 0$ such that $\phi_3^3 - \phi_2^3 = 1 - g(1-\omega) + \varepsilon$ whenever (ϕ_2^0, ϕ_3^0) is near the boundary of Γ_1. To prove this, we need to make sure that $R(\phi_2^0, \phi_3^0)$ stay away from $1 - g(1-\omega)$ whenever (ϕ_2^0, ϕ_3^0) are near $\Gamma_1 \cap \Gamma_2$ and $\Gamma_1 \cap \Gamma_3$ (see Figure 9). Suppose that (ϕ_2^0, ϕ_3^0) is near the boundaries of Γ_1 and Γ_2. Then $\phi_1^2 \approx 1^-$. Thus, $\phi_3^3 - \phi_3^2 \approx g(2\omega) - g(\omega) = 1 - g(1-\omega) + \varepsilon$, where $\varepsilon > 0$. Similarly, if (ϕ_2^0, ϕ_3^0) is near the boundaries of Γ_1 and Γ_3, $\phi_3^3 - \phi_2^3$ is also bounded away from $1 - g(1-\omega)$. Hence, $\phi_3^3 - \phi_2^3$ is bounded away from $1 - g(1-\omega)$ whenever (ϕ_2^0, ϕ_3^0) is near the boundary of Γ_1. Similarly, one can prove that (3.24) holds for $i = 2, 3$. We have completed the first assertion of the proposition. The second assertion of the proposition is obvious. Suppose $\phi_n^0 - \phi_{n-1}^0 \approx (1 - g(1-\omega))^-$. Then $\phi_{n-1}^1 \approx 1^-$. Since $D(x)$ is increasing in x, $\phi_n^2 - \phi_{n-1}^2 \geq h(\omega) - h(0) = g(\omega)$. Inductively, we see that

$$\phi_n^n - \phi_{n-1}^n \geq g((n-1)\omega) - g((n-2)\omega) > 1 - g(1-\omega).$$

The second assertion of the proposition has been used repeatedly to justify the first inequality above. The second inequality above follows from (2.2). Therefore, (3.23) holds whenever ϕ_n^0 is sufficiently close to $1 - g(1-\omega)$. Hence, the direction of the flow of the system near the

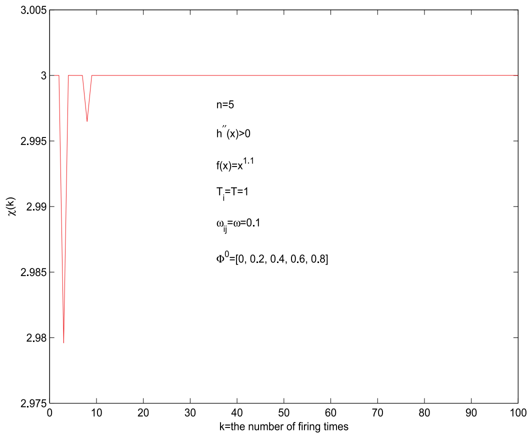


Figure 10. *For the choice of f, its phase responding function $h(x)$ is concave upward. The system in general does not synchronize as predicted in Proposition 3.8.3 and Theorem 3.9.2(b).*

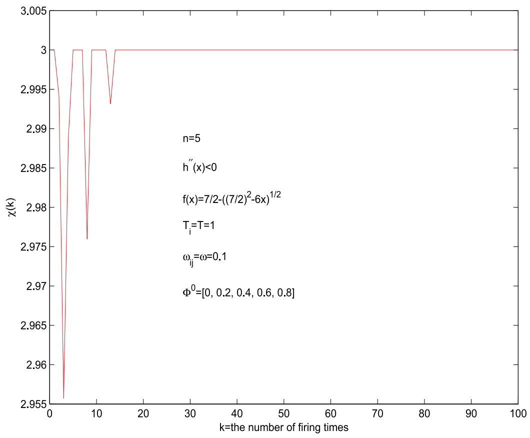


Figure 11. *For the choice of f, its phase responding function $h(x)$ is concave downward. Nevertheless, the system in general does not synchronize either.*

piece of boundary defined by $\phi_n^0 - \phi_{n-1}^0 = 1 - g(1-\omega)$ points inward. Similarly,

$$\phi_{n-i+1}^{i-1} - \phi_{n-i}^{i-1} < \phi_{n-i+1}^{n+i-1} - \phi_{n-i}^{n+i-1}, \qquad i = 2, \dots, n,$$

whenever $\phi_{n-i+1}^{i-1} - \phi_{n-i}^{i-1}$ is close to $1 - g(1-\omega)$. We have just completed the proof of the proposition. ■

Two questions naturally arise from the proposition above. First, is the restriction $h''(x) > 0$ necessary for the validity of the second assertion of Proposition 3.8? Second, what kind of evolution maps with $f'' > 0$ satisfy the constraint $h''(x) > 0$? For the first question, we expect that the answer should be no (see Figures 10 and 11). However, we are unable to prove this.

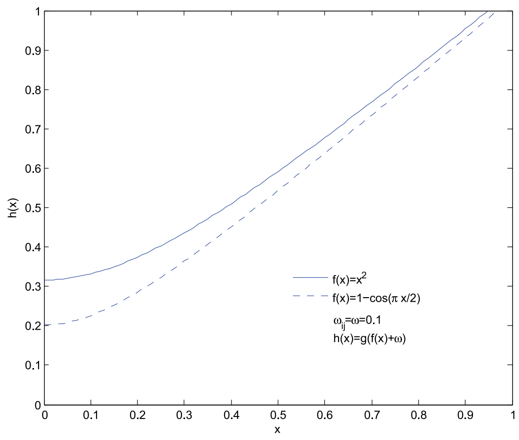


Figure 12. *Two graphs of $h(x)$ with two different f's are shown above. Their graphs are all concave upward.*

For the second question, we see in Figure 12, via the help of the computer, that $f(x) = x^r$, $r > 1$, and $f(x) = 1 - \cos(\pi x/2)$ satisfy $h''(x) > 0$.

We are now ready to state the main result of the paper.

Theorem 3.9.

1. *Suppose that stability condition (2.9) holds. Then the system of convex oscillators will achieve synchrony for all initial values, except possibly for those in a set of measure zero. In particular, the system of identical convex oscillators is to fire synchronously for all initial values, expect for those in a set of measure zero.*
2. (a) *The identical system with an even number of concave oscillators will not achieve full synchrony for certain initial values in a set of positive measure.*
   (b) *Suppose that the phase responding function $h(x)$ is concave upward. Then the identical system of concave oscillators will not synchronize for all initial values in the domain of its return map.*
   (c) *The identical system of three concave oscillators will not synchronize for all initial values in the domain of its return map.*
3. *Suppose that stability condition (2.9) and the absorption conditions (3.19) and (3.20c) are satisfied. Then the system of concave oscillators will achieve synchrony for all initial values.*

Proof. As shown in Theorem 3.5.1, the natural tendency of the system of convex oscillators is to grow by absorption regardless of their coupling strengths and speeds. Therefore, the system will continue to grow by absorption even though we need to update the new coupling strengths and speeds at each stage. The assertion of the first part of the theorem now follows. The third assertion of the theorem is now obvious. It remains to prove the second assertion of the theorem. To this end, let the number of oscillators be $2k$, and let ω and T be the constant coupling strength and constant cycle period, respectively. Pick $\Phi^0 = (\phi_1^0, \phi_2^0, \dots, \phi_n^0)$

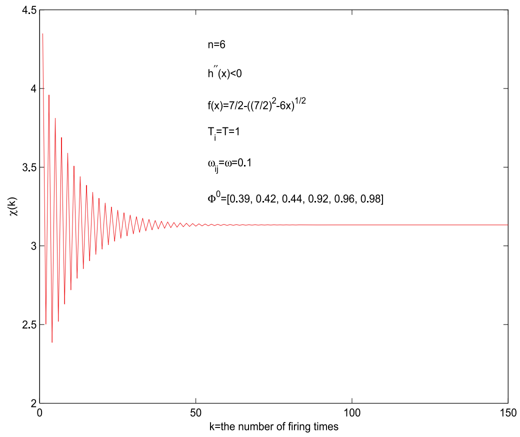


Figure 13. *The choice of f as above has the properties that $f''(x) > 0$ and $h''(x) < 0$. Since the number of oscillators chosen in this case is even, the numerical result demonstrated as above is consistent with the result of Theorem 3.9.2(a).*

to satisfy that

$$\begin{aligned} &\phi_j^0 \in (1 - m_g\omega, 1),\ j = k+1, \ldots, n, \text{ and} \\ &\phi_1^0, \ldots, \phi_k^0 \in (M_g(k+1)\omega - m_g\omega,\ M_g(k+1)\omega). \end{aligned} \tag{3.25}$$

It then follows from Remark 2.1.2(a) that the system will reduce to two synchronous groups after initial firings. In fact, the first group contains oscillators $\phi_1^1, \ldots, \phi_k^1$. The new coupling strengths for these two groups are equal. Denote by $\widetilde{\gamma}_{21}$ and $\widetilde{\gamma}_{12}$ the new corresponding γ_{21} and γ_{12}, respectively. Then $\widetilde{\gamma}_{21} = \widetilde{\gamma}_{12} = g(k\omega) + g(1 - k\omega) - 1 > 0$. Therefore, such a set of the initial values, which has a positive measure, will converge to a nonfiring state (see Figure 13). The assertions in 2(b) and 2(c) are now direct consequences of Proposition 3.8. We have just completed the second part of the theorem. ■

The numerical stimulation suggests that a "nearly" identical system of any number of oscillators in general will not synchronize with or without the requirement that the phase responding curve be concave upward. Such a conjecture remains to be completed.

3.4. Examples and discussion. For the illustration of Theorem 3.9, the following three cases of systems of three oscillators are considered: (i) $f_i(x) = \sqrt{x}$, $\omega_{ij} = \omega$; (ii) $f_i(x) = x^{1.3}$ or $f_i(x) = \frac{7}{2} - \sqrt{(\frac{7}{2})^2 - 6x}$, $T_i = T$, and $\omega_{ij} = \omega$; (iii) $f_i(x) = x^r$, where $r > 1$, and $\omega_{ij} = \omega$.

Case (i): For this case, $m_g = \omega$, $m_f = \frac{1-\sqrt{1-\omega}}{\omega}$, $M_g = 2 - \omega$, and $M_f = \frac{1}{\sqrt{\omega}}$. Moreover, we have that $\frac{1}{m_g m_f} \geq M_g M_f$. Thus, as $n = 3$, equation (2.9) becomes

$$\frac{m_g^4 m_f^3 \omega}{(1 + m_f m_g + m_f^2 m_g^2) M_f M_g} \geq \Delta T(\Delta T + 1). \tag{3.26}$$

The corresponding feasible parameters region in $\omega - \Delta T$ is plotted in Figure 14. In the

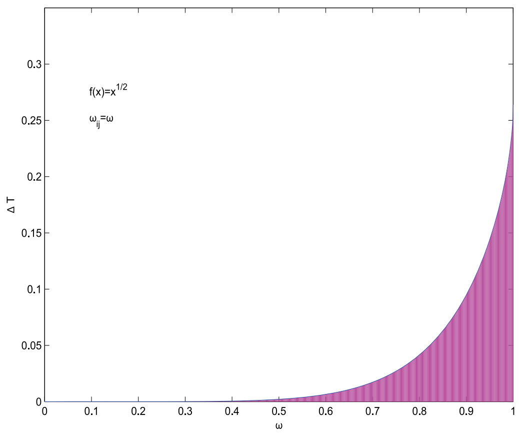


Figure 14. *The shaded part of the region is the set of parameters* $(\omega, \Delta T)$ *satisfying stability condition* (3.26).

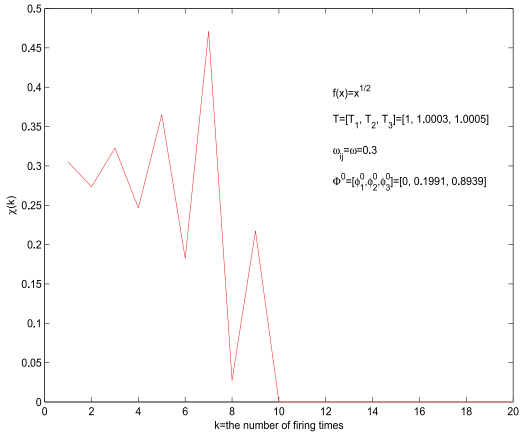


Figure 15. $\chi(t)$, *the synchronization order parameter, is defined in Figure* 3. *The parameters* ω_{ij} *and* T_i *are chosen so as to be from the stability region, Figure* 14. *With initial state being given as above, the system reaches full synchrony in* 10 *firings.*

numerical simulations, we pick randomly more than 20 sets of parameters with various sets of initial values; all the numerical results suggest the synchrony of the system. One such set of parameters and initial values and its corresponding numerical results are recorded in Figure 15.

Case (ii): The identical system is considered here. Let the number of oscillators be three. Figures 16 and 17 give the set of initial values not reaching synchrony, which contains the domain of the return map. Γ_3 is interpreted as $\phi_3^0 = 1$.

Case (iii): The case under consideration is the system of concave oscillators satisfying stability condition (2.9) and a modified absorption condition, which is stronger but easier to

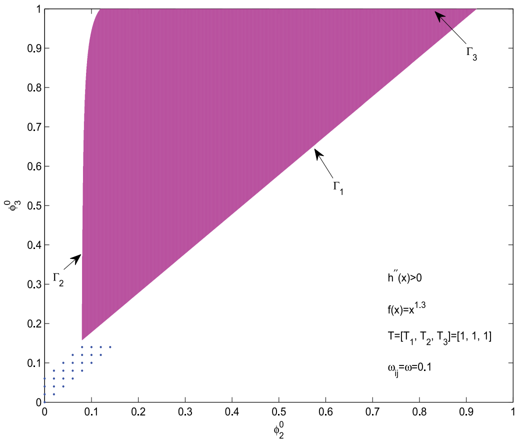


Figure 16. *The set of initial values reaching synchrony numerically is denoted by the dotted region. The points not in the dotted region, including the shaded region, will not acquire synchrony. In fact, the shaded region is the domain of the return map. This figure is consistent with the assertion of Theorems* 3.9.2(b) *and* 3.9.2(c).

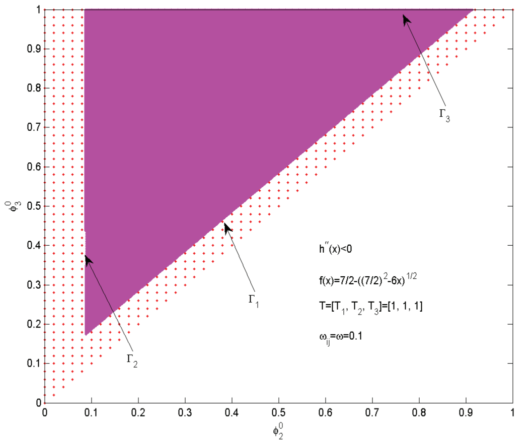
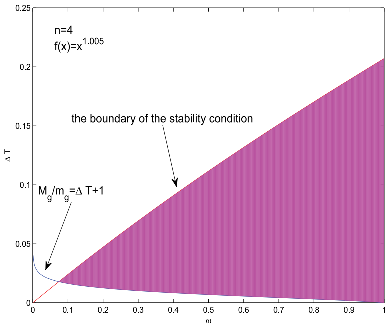


Figure 17. *The set of initial values reaching synchrony numerically is denoted by the dotted region. The points not in the dotted region, including the shaded region, will not acquire synchrony. In fact, the shaded region is the domain of the return map. This figure supports the assertion of Theorem* 3.9.2(c).

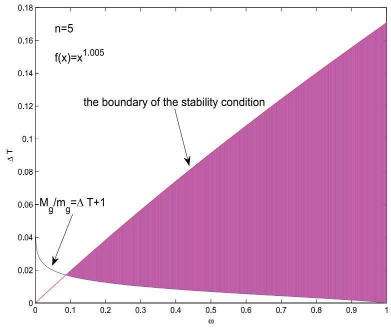
verify. Specifically, we consider the following absorption condition:

$$\frac{n}{n-2} > t_{\max} = \Delta T + 1 > \frac{M_g}{m_g}. \tag{3.27}$$

With such a stronger condition, the system will achieve full synchrony or reduce to two synchronous groups. However, in the case of the latter, to acquire full synchrony, the concavity of the evolution maps is still required to be sufficiently small. Numerically, we have that the



(a) The shaded part above is the region satisfied by both stability condition (2.9) and absorption conditions (3.27) for $n = 4$.



(b) The shaded part above is the region satisfied by both stability condition (2.9) and absorption conditions (3.27) for $n = 5$.

Figure 18.

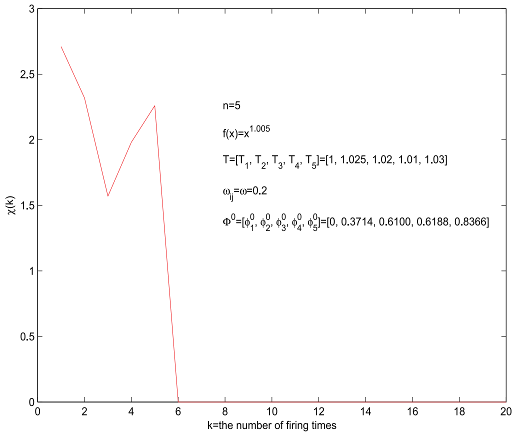


Figure 19. *Let $f(x) = x^{1.005}$. $\chi(t)$, the synchronization order parameter, is defined in Figure 3. The parameters ω_{ij} and T_i are so chosen to be in the stability region, Figure 18. Note that 5 is a prime number. Hence, when absorption occurs, the system breaks into a number of synchronous groups with their sizes being not all equal. Such an imbalance in coupling strength speeds the process of full synchrony. With initial state being given as above, the system reaches full synchrony in 6 firings.*

line $\left\{(\omega, \Delta T) : \Delta T = \frac{[\frac{n}{3}]+1}{[\frac{n}{3}]}\right\}$ does not intersect with the boundary of the stability condition. The parameter regions in the $\omega - \Delta T$ space satisfying (2.9) and (3.27) are, respectively, shown in the shaded regions in Figure 18(a) and (b). Picking the parameters from these regions, we see, in Figures 19 and 20, that the systems of both five and four concave oscillators reach full synchrony after a number of firings, provided that the concavity of the evolution maps is small. It should be mentioned that if n is a prime number, whenever the absorption occurs the system will acquire full synchrony in a short period of time. In this scenario, the imbalance in coupling strengths for the newly formed system is significant. In fact, it needs only six

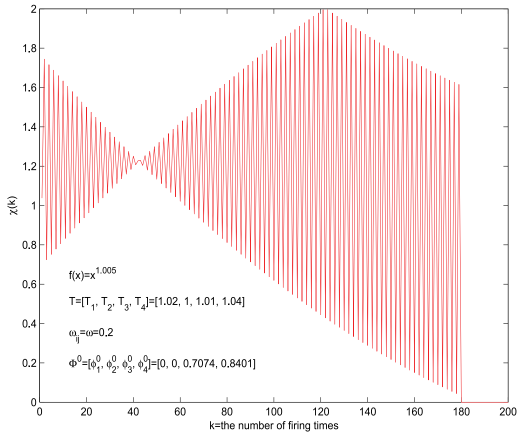


Figure 20. *With the evolution map, parameters, and initial state being given as above, the system reaches full synchrony in* 180 *firings. The reason that it takes so long for the system to synchronize is because n is an even number. When the absorption occurs, each of the synchronous groups may still have equal coupling strengths. Consequently, it takes longer for the system to synchronize since the imbalance in speed is insignificant.*

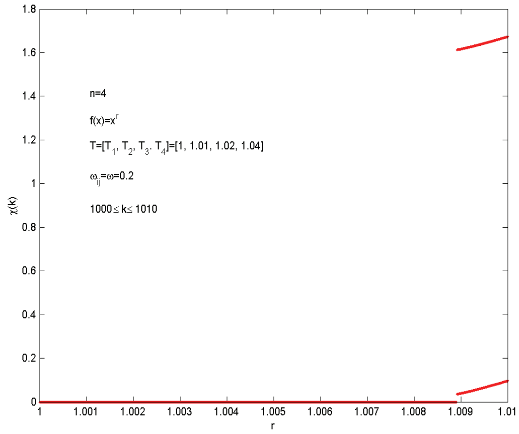


Figure 21. *The horizontal axis is the exponent of the evolution map of the form $f(x) = x^r$, $r > 1$. We plot $\chi(k)$, $1000 \leq k \leq 1010$, on the vertical axis.* 1000 *firings are needed to determine whether the corresponding system will achieve full synchrony or not. From the computer simulation, we see that the system will reach full synchrony, provided that r is roughly less than* 1.009.

firings to achieve full synchrony for $n = 5$. As for $n = 4$, the number of firings is 180 to secure synchrony. (See Figures 19 and 20, respectively.) To further support the validity of Proposition 3.7, we consider the evolution maps of the form $f(x) = x^r$, $r > 1$. The smaller r is, the smaller its concavity is. Treat r as a bifurcation parameter; Figures 21–23 show how we determine the smallest r that will make its corresponding system synchronize with various

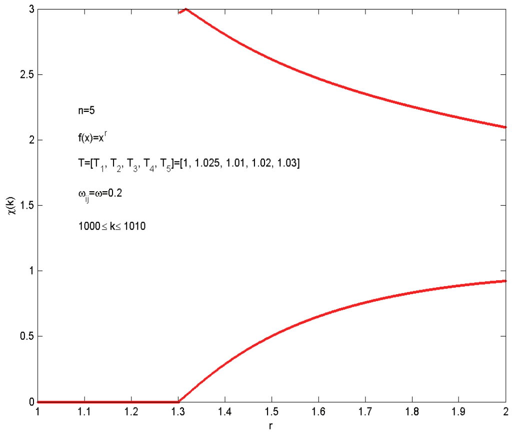


Figure 22. *Since 5 is a prime number, the imbalance measurement is "relatively" large if and when the system reduces to two synchronous groups. Therefore, the system is allowed to have a "larger" concavity at $r \approx 1.3$.*

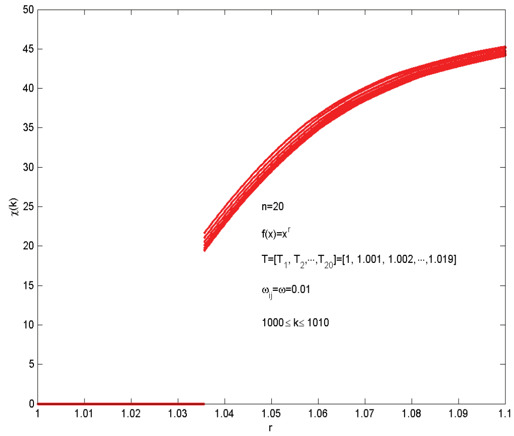


Figure 23. *With a greater number of oscillators present in the system, the computer simulation is consistent with the theory predicted in Proposition 3.7.*

choices of sizes of oscillators.

In conclusion, we prove stable synchrony for an integrate-and-fire model provided by Mirollo and Strogatz. Our results include the proof of Peskin's second conjecture. The next question is whether the results obtained here can be generalized to higher dimensional oscillators such as conductance-based models of neurons and/or phase-coupled networks via phase-response curves (see, e.g., [25] and the work cited therein). Note that the system presented here is just a special case for the phase-response curves approach. Nevertheless, the key ingredients for proving the full synchrony for those more current and advanced models should remain the same even though new technical difficulties might arise. For instance, we still need to derive stability conditions so that the nonidentical system behaves like the identical system. We also need to have some kind of absorption conditions. For example, if the underlining model is dissipative, i.e., its time T-map decreases volume for all $T > 0$, then the natural tendency of the system would be to settle into a nonfiring state unless the direction of the flow of the "associated" return map points outward. If, on the other hand, the underlining model is volume-expanding, then the absorption process of the system tends to occur. It is certainly worthwhile to work on those problems.

Acknowledgments. The authors would like to thank the editor and referees for their helpful comments.

REFERENCES

[1] L. F. Abbott, *A network of oscillators*, J. Phys. A, 23 (1990), pp. 3835–3859.

[2] L. F. Abbott and C. van Vreeswijk, *Asynchronous states in networks of pulse-coupled oscillators*, Phys. Rev. E, 48 (1993), pp. 1483–1490.

[3] V. N. Belykh, N. N. Verichev, L. J. Kocarev, and L. O. Chua, *Chua's Circuit: A Paradigm for Chaos*, World Scientific, Singapore, 1993.

[4] V. N. Belykh, I. V. Belykh, K. V. Nevidin, and M. Hasler, *Hierarchy and stability of partially synchronous oscillations of diffusively coupled dynamical systems*, Phys. Rev. E, 62 (2000), pp. 6332–6345.

[5] V. N. Belykh, I. V. Belykh, K. V. Nevidin, and M. Hasler, *Cluster synchronization in three-dimensional lattices of diffusively coupled oscillators*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 13 (2003), pp. 755–799.

[6] V. N. Belykh, I. V. Belykh, and M. Hasler, *Connection graph stability method for synchronized coupled chaotic systems*, Phys. D, 195 (2004), pp. 159–187.

[7] I. Belykh, E. de Lange, and M. Hasler, *Synchronization of bursting neurons: What matters in the network topology*, Phys. Rev. Lett., 94 (2005), paper 188101.

[8] S. Bottani, *Pulse-coupled relaxation oscillators: From biological synchronization to self-organized criticality*, Phys. Rev. Lett., 74 (1995), pp. 4189–4192.

[9] A. Brailove, *The dynamics of two pulse-coupled relaxation oscillators*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 2 (1992), pp. 341–352.

[10] P. C. Bressloff and S. Coombes, *Synchrony in an array of integrate-and-fire neurons with dendritic structure*, Phys. Rev. Lett., 78 (1997), pp. 4665–4668.

[11] P. C. Bressloff and S. Coombes, *Desynchronization, mode locking, and bursting in strongly coupled integrate-and-fire oscillators*, Phys. Rev. Lett., 81 (1998), pp. 2168–2171.

[12] P. C. Bressloff and S. Coombes, *Symmetry and phase-locking in a ring of pulse-coupled oscillators with distributed delays*, Phys. D, 126 (1999), pp. 99–122.

[13] J. Buck, *Synchronous rhythmic flashing of fireflies.* II, Quart. Rev. Biol., 63 (1988), pp. 265–289.

[14] C. Chow, *Phase-locking in weakly heterogeneous neuronal networks*, Phys. D, 118 (1998), pp. 343–370.

[15] H. Daido, *Lower critical dimension for populations of oscillators with randomly distributed frequencies: A renormalization-group analysis*, Phys. Rev. Lett., 61 (1988), pp. 231–234.

[16] H. Daido, *Intrinsic fluctuation and its critical scaling in a class of populations of oscillators with distributed frequencies*, Progr. Theoret. Phys., 81 (1989), pp. 727–731.

[17] H. Daido, *Intrinsic fluctuations and a phase transition in a class of large populations of interacting oscillators*, J. Statist. Phys., 60 (1990), pp. 753–800.

[18] G. B. Ermentrout and N. Kopell, *Frequency plateaus in a chain of weakly coupled oscillators*, SIAM J. Math. Anal., 15 (1984), pp. 215–237.

[19] G. B. Ermentrout, *Synchronization in a pool of mutually coupled oscillators with random frequencies*, J. Math. Biol., 22 (1985), pp. 1–9.

[20] G. Ermentrout, *An adaptive model for synchrony in the firefly Pteroptyx malaccae*, J. Math. Biol., 29 (1991), pp. 571–585.

[21] U. Ernst, K. Pawelzik, and T. Geisel, *Delay-induced multistable synchronization of biological oscillators*, Phys. Rev. E, 57 (1998), pp. 2150–2162.

[22] W. Gerstner, J. L. van Hemmen, and J. D. Cowan, *What matters in neuronal locking*, Neural Computation, 8 (1996), pp. 1653–1676.

[23] W. Gerstner and W. M. Kistler, *Spiking Neuron Models. Single Neurons, Populations, Plasticity*, Cambridge University Press, Cambridge, UK, 2002.

[24] W. Gerstner, R. Ritz, and J. L. van Hemmen, *A biologically motivated and analytically soluble model of collective oscillations in the cortex:* I. *Theory of weak locking*, Biolog. Cybernet., 68 (1993), pp. 363–374.

[25] P. Goel and B. Ermentrout, *Synchrony, stability, and firing patterns in pulse-coupled oscillators*, Phys. D, 163 (2002), pp. 191–216.

[26] D. Hansel, G. Mato, and C. Meunier, *Synchrony in excitatory neural networks*, Neural Computation, 7 (1995), pp. 307–337.

[27] D. Hansel and G. Mato, *Existence and stability of persistent states in large neuronal networks*, Phys. Rev. Lett., 86 (2001), pp. 4175–4178.

[28] E. Izhikevich, *Class* 1 *neural excitability, conventional synapses, weakly connected networks, and mathematical foundations of pulse-coupled models*, IEEE Trans. Neural Networks, 10 (1999), pp. 499–507.

[29] J. Jalife, *Mutual entrainment and electrical coupling as mechanisms for synchronous firing of rabbit sinoatrial pacemaker cells*, J. Physiol., 356 (1984), pp. 221–243.

[30] J. Juang, C. L. Li, and Y. H. Liang, *Global synchronization in lattices of coupled chaotic systems*, Chaos, 17 (2007), paper 033111.

[31] J. Juang and Y.-H. Liang, *Synchronous chaos in coupled map lattices with general connectivity topology*, SIAM J. Appl. Dyn. Syst., 7 (2008), pp. 755–765.

[32] N. Kopell and G. B. Ermentrout, *Mechanisms of phase-locking and frequency control in pairs of coupled neural oscillators*, in Handbook of Dynamical Systems, Vol. 3, Towards Applications, B. Fiedler, G. Iooss, and N. Kopell, eds., Elsevier, New York, 2000.

[33] Y. Kuramoto, *Self-entrainment of a population of coupled non-linear oscillators*, in International Symposium on Mathematical Problems in Theoretical Physics, Lecture Notes in Physics 39, H. Araki, ed., Springer, Berlin, 1975, pp. 420–422.

[34] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence*, Springer-Verlag, New York, 1984.

[35] Y. Kuramoto and I. Nishikawa, *Statistical macrodynamics of large dynamical systems. Case of a phase transition in oscillator communities*, J. Statist. Phys., 49 (1987), pp. 596–605.

[36] Y. Kuramoto, *Collective synchronization of pulse-coupled oscillators and excitable units*, Phys. D, 50 (1991), pp. 15–30.

[37] J. Lü, X. Yu, and G. Chen, *Chaos synchronization of general complex dynamical networks*, Phys. A, 334 (2004), pp. 281–302.

[38] J. Lü, X. Yu, and G. Chen, *A time-varying complex dynamical network model and its controlling synchronization criteria*, IEEE Trans. Automat. Control, 50 (2005), pp. 841–846.

[39] P. C. Matthews and S. H. Strogatz, *Phase diagram for the collective behavior of limit-cycle oscillators*, Phys. Rev. Lett., 65 (1990), pp. 1701–1704.

[40] D. C. Michaels, E. P. Matyas, and J. Jalife, *Mechanisms of sinoatrial pacemaker synchronization: A new hypothesis*, Circulation Res., 61 (1987), pp. 704–714.

[41] R. E. Mirollo and S. H. Strogatz, *Synchronization of pulse-coupled biological oscillators*, SIAM J. Appl. Math., 50 (1990), pp. 1645–1662.

[42] T. Nishikawa, A. E. Motter, Y. C. Lai, and F. C. Hoppensteadt, *Heterogeneity in oscillator networks: Are smaller worlds easier to synchronize?*, Phys. Rev. Lett., 91 (2003), paper 014101.

[43] C. S. Peskin, *Mathematical Aspects of Heart Physiology*, Courant Institute of Mathematical Sciences, New York University, New York, 1975.

[44] A. Pogromsky and H. Nijmeijer, *Cooperative oscillatory behavior of mutually coupled dynamical systems*, IEEE Trans. Circuits Systems I Fund. Theory Appl., 48 (2001), pp. 152–162.

[45] J. Rubin and D. Terman, *Geometric analysis of population rhythms in synaptically coupled neuronal networks*, Neural Computation, 12 (2000), pp. 597–645.

[46] J. Rubin and D. Terman, *Analysis of clustered firing patterns in synaptically coupled networks of oscillators*, J. Math. Biol., 41 (2000), pp. 513–545.

[47] J. Rubin and D. Terman, *Synchronized activity and loss of synchrony among heterogeneous conditional oscillators*, SIAM J. Appl. Dyn. Syst., 1 (2002), pp. 146–174.
[48] A. Sherman, J. Rinzel, and J. Keizer, *Emergence of organized bursting in clusters of pancreatic beta-cells by channel sharing*, Biophys. J., 54 (1988), pp. 411–425.
[49] A. Sherman and J. Rinzel, *Collective properties of insulin secreting cells*, in Cell to Cell Signalling: From Experiments to Theoretical Models, A. Goldbeter, ed., Academic Press, London, 1989, pp. 61–75.
[50] S. Strogatz, *Norbert Wieners brain waves*, in Frontiers in Mathematical Biology, Lecture Notes in Biomathematics 100, Springer, Berlin, 1994, pp. 122–138.
[51] S. H. Strogatz, C. M. Marcus, R. M. Westervelt, and R. E. Mirollo, *Simple model of collective transport with phase slippage*, Phys. Rev. Lett., 61 (1988), pp. 2380–2383.
[52] S. H. Strogatz and R. E. Mirollo, *Phase-locking and critical phenomena in lattices of coupled nonlinear oscillators with random intrinsic frequencies*, Phys. D, 31 (1988), pp. 143–168.
[53] S. H. Strogatz and R. E. Mirollo, *Collective synchronisation in lattices of nonlinear oscillators with randomness*, J. Phys. A, 21 (1988), pp. L699–L705.
[54] D. Terman, N. Kopell, and A. Bose, *Dynamics of two mutually coupled slow inhibitory neurons*, Phys. D, 117 (1998), pp. 241–275.
[55] V. Torre, *A theory of synchronization of heart pace-maker cells*, J. Theoret. Biol., 61 (1976), pp. 55–71.
[56] M. Tsodyks, I. Mitkov, and H. Sompolinsky, *Pattern of synchrony in inhomogeneous networks of oscillators with pulse interactions*, Phys. Rev. Lett., 71 (1993), pp. 1280–1283.
[57] R. Urbanczik and W. Senn, *Similar nonleaky integrate-and-fire neurons with instantaneous couplings always synchronize*, SIAM J. Appl. Math., 61 (2000), pp. 1143–1155.
[58] C. van Vreeswijk, L. Abbott, and G. Ermentrout, *When inhibition not excitation synchronizes neural firing*, J. Comp. Neurosci., 1 (1994), pp. 313–321.
[59] C. van Vreeswijk, *Partially synchronized states in networks of pulse-coupled neurons*, Phys. Rev. E, 54 (1996), pp. 5522–5537.
[60] A. T. Winfree, *Biological rhythms and the behavior of populations of coupled oscillators*, J. Theoret. Biol., 16 (1967), pp. 15–42.
[61] A. T. Winfree, *The Geometry of Biological Time*, Springer-Verlag, New York, 1980.
[62] C. W. Wu, *Synchronization in Coupled Chaotic Circuits and Systems*, World Sci. Ser. Nonlinear Sci. Ser. A Monogr. Treatises 41, World Scientific, Singapore, 2002.
[63] J. Yang, G. Hu, and J. Xiao, *Chaos synchronization in coupled chaotic oscillators with multiple positive Lyapunov exponents*, Phys. Rev. Lett., 80 (2003), pp. 496–499.