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Synchronous Chaos in Coupled Map Lattices with General Connectivity Topology*

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Abstract. The purpose of the paper is to address the synchronous chaos in coupled map lattices with general connectivity topology. Our main results contain the following. First, the master stability functions also hold for general connectivity topology with coupling through a nonlinear function that needs to be exactly the individual chaotic map. Second, the synchronization curve, composed of pieces of transverse Lyapunov exponent curves, is constructed. Third, necessary and sufficient conditions on coupling strength for yielding the synchronous chaos of the system are given. Moreover, the coupling strength d_c giving the fastest convergence rate of the initial values toward the synchronous state is explicitly obtained. It is also proved that such d_c is independent of the choice of the individual map. Finally, our results here can be applied to address questions of wavelength bifurcations and size instability.

Key words. stable synchronization, Lyapunov exponents, wavelength bifurcation, coupled map lattices

AMS subject classifications. 34C15, 37N35

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1. Introduction.

A particularly interesting form of dynamical behavior occurs in networks of coupled systems or oscillators when all of the individual systems or oscillators acquire identical chaotic behavior. Such behavior of a network models many systems of interest in physics, biology, and engineering. A central dynamical question is: When is such synchronous behavior stable, especially in regard to coupling strengths in the network? Much progress in this direction has been made in lattices of coupled chaotic systems. Indeed, many results [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14] give analytical criteria for determining the range of coupling strength to acquire locally or even globally stable synchronization. On the other hand, to the best of our knowledge, there are no general results for global synchronization in coupled map lattices (CMLs). There are, however, globally synchronous results for some special cases (see, e.g., [15]). As to the study of local synchronization in CMLs, the notion of master stability functions (MSFs) that allows one to isolate the contribution of the network structure in terms of the eigenvalues of the coupling matrix was introduced in [8], [16], [17], [18], [19] to determine the possible range of coupling strength. This function then defines a region of stably synchronous state in terms of the coupling strength and the eigenvalues of the coupling matrix. Most of the work done in finding such a region of stability of the synchronous state is numerical. In a few certain cases, such as when the coupling matrix is symmetric, the MSFs can be further reduced to a number of inequalities [20], [21], [22], [23]

$$h_{\max} + \ln|1 + d\lambda_i| < 0, \quad i = 2, \dots, m. \tag{1.1}$$

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Here $h_{\max}$ is the largest Lyapunov exponent of the individual map, λ_i are the nonzero eigenvalues of the $m \times m$ coupling matrix, and d is the coupling strength. The Gershgorin disk theory is then applied to obtain some sufficient conditions [23] on the coupling strength for local synchronization. The reason for the huge gap between the theory developed in the lattices of coupled chaotic systems and that of CMLs lies mostly in the fact that it is more natural to have a nonlinear coupling between oscillators in the CMLs. This is because a nonlinear coupling within suitable range of the coupling strength tends to yield an invariant region for the corresponding CMLs while linear coupling cannot. It should be noted that there is no such problem for the lattices of coupled chaotic systems. It should also be mentioned that all the analytical results of the lattices of the coupled chaotic systems stated above are linearly coupled.

The purpose of this paper is to give the best possible results for the local synchronization of the CMLs. Indeed, we first prove that (1.1) holds true for general connectivity topology with a limitation that the nonlinear coupling needs to be the individual chaotic map as well. Second, the synchronization curve, composed of pieces of transverse Lyapunov exponent curves, is derived. With the help of the synchronization curve, we give necessary and sufficient conditions on yielding synchronization of the CMLs. Such conditions then lead to the identification of the optimal coupling strength interval for acquiring synchronization of the CMLs. The optimal interval is to be termed the *synchronization interval* of the CMLs. Moreover, the coupling strength d_c, called the *center* of the *synchronization interval* and giving the fastest convergence rate of the initial values toward the synchronous state, can be identified. Such d_c is independent of the choice of the individual map. Like the applications, our work here can also be used to analytically quantify how the small-world scheme improves the synchronizability of the network [24], [25], [26], [27]. Furthermore, our results here can be applied to address questions of wavelength bifurcations [28], [29], [30], [31], [32] and size instability [32]. For CMLs or coupled chaotic systems, the following four scenarios are possible as the coupling varies: (i) no synchronization; (ii) the presence of short wavelength bifurcations (SWBs); (iii) the presence of intermediate wavelength bifurcations (IWBs); and (iv) the presence of long wavelength bifurcations (LWBs). Our main results give the following. First, if the coupling matrix has only real eigenvalues, then only (i) and (ii) are possible. Second, if the coupling matrix has complex eigenvalues, then all four scenarios are possible. Third, the critical values for which wavelength bifurcations occur as well as the exact number of oscillators capable of sustaining stably synchronous chaos can be explicitly computed. Finally, the minimum coupling value where all wavelength modes become de-excited enough to induce the stability of the synchronous state is also explicitly given.

We conclude this introductory section by mentioning the organization of the paper. The main results are contained in section 2. Three types of coupling matrices are provided in section 3 as illustrations and applications to our main results. Some concluding remarks about future research are addressed in section 4.

2. Main results.

Consider a network of CMLs consisting of m oscillators. The equations of the motion then read

$$\boldsymbol{x}_i(n+1) = \boldsymbol{f}(\boldsymbol{x}_i(n)) + d\left(\sum_{k=1}^{m} g_{ik}\boldsymbol{h}(\boldsymbol{x}_k(n))\right), \quad i = 1, \ldots, m. \tag{2.1}$$

Here $\boldsymbol{f} : \mathbb{R}^l \to \mathbb{R}^l$, $l \geq 1$, represents the individual chaotic map, and $\boldsymbol{h} : \mathbb{R}^l \to \mathbb{R}^l$ is an arbitrary nonlinear function describing how each oscillator's variables are used in the coupling. The quantities g_{ij} are the coupling weights between the oscillators i and j. To consider the notion of synchronization, we assume that $\sum_{k=1}^{m} g_{ik} = 0$ for each i, and 0 is the simple eigenvalue of the coupling matrix $\boldsymbol{G} = (g_{ij})$. The quantity d represents the coupling strength of the CMLs (2.1). To have an invariant region for CMLs (2.1), one usually chooses $\boldsymbol{h}$ as $\boldsymbol{f}$. Such nonlinear coupling between oscillators is what makes (2.1) harder to treat analytically. In vector-matrix form with $\boldsymbol{h} = \boldsymbol{f}$, (2.1) becomes

$$\boldsymbol{x}(n+1) = \boldsymbol{F}(\boldsymbol{x}(n)) + d(\boldsymbol{G} \otimes \boldsymbol{I})\boldsymbol{F}(\boldsymbol{x}(n)), \tag{2.2}$$

where $\otimes$ denotes the Kronecker product, $\boldsymbol{x}(n) = (\boldsymbol{x}_1(n), \ldots, \boldsymbol{x}_m(n))^T$, and $\boldsymbol{F}(\boldsymbol{x}(n)) = (\boldsymbol{f}(\boldsymbol{x}_1(n)), \ldots, \boldsymbol{f}(\boldsymbol{x}_m(n)))^T$.

To study the stability of the synchronous state $\{\boldsymbol{x}_i = \boldsymbol{s}\ \forall i\}$ of CML (2.2), we consider the variational equation of (2.2):

$$\begin{aligned}\boldsymbol{\xi}(n+1) &= D\boldsymbol{F}(\boldsymbol{s}(n))\boldsymbol{\xi}(n) + d(\boldsymbol{G} \otimes \boldsymbol{I})D\boldsymbol{F}(\boldsymbol{s}(n))\boldsymbol{\xi}(n)\\ &= \left[\boldsymbol{I} \otimes D\boldsymbol{f}(\boldsymbol{s}(n)) + d\left(\boldsymbol{G} \otimes \boldsymbol{I}\right)\left(\boldsymbol{I} \otimes D\boldsymbol{f}\left(\boldsymbol{s}(n)\right)\right)\right]\boldsymbol{\xi}(n),\end{aligned} \tag{2.3a}$$

where $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m)$ and each $\boldsymbol{\xi}_i$ is the perturbation to the ith oscillator. Let $\boldsymbol{J} = \boldsymbol{P}^{-1}\boldsymbol{G}\boldsymbol{P}$, where $\boldsymbol{J} = [0] \oplus \boldsymbol{J}_1 \oplus \cdots \oplus \boldsymbol{J}_p$ is the real Jordan canonical form of $\boldsymbol{G}$. Applying the change of variables $\boldsymbol{\eta} = (\boldsymbol{P}^{-1} \otimes \boldsymbol{I})\boldsymbol{\xi}$, we get

$$\boldsymbol{\eta}(n+1) = \left[(\boldsymbol{I} + d\boldsymbol{J}) \otimes D\boldsymbol{f}(\boldsymbol{s}(n))\right]\boldsymbol{\eta}(n),$$

or, equivalently, in block diagonal form,

$$\begin{aligned}\boldsymbol{\eta}_i(n+1) &= \left[(\boldsymbol{I} + d\boldsymbol{J}_i)^n \otimes D\boldsymbol{f}^n(\boldsymbol{s}(1))\right]\boldsymbol{\eta}_i(1)\\ &=: \boldsymbol{A}_i(n)\boldsymbol{\eta}_i(1).\end{aligned} \tag{2.3b}$$

Let $\sigma(\boldsymbol{A})$ denote the spectrum of $\boldsymbol{A}$. Then $\sigma(\boldsymbol{A}_i(n)\boldsymbol{A}_i^*(n))$ equals

$$\begin{aligned}&\sigma(\left[(\boldsymbol{I} + d\boldsymbol{J}_i)^n \otimes D\boldsymbol{f}^n(\boldsymbol{s}(1))\right]\left[(\boldsymbol{I} + d\boldsymbol{J}_i^*)^n \otimes (D\boldsymbol{f}^n(\boldsymbol{s}(1)))^*\right])\\ &= \sigma(\left[(\boldsymbol{I} + d\boldsymbol{J}_i)^n\left(\boldsymbol{I} + d\boldsymbol{J}_i^*\right)^n\right] \otimes \left[D\boldsymbol{f}^n(\boldsymbol{s}(1)) \cdot (D\boldsymbol{f}^n(\boldsymbol{s}(1)))^*\right])\\ &= \sigma((\boldsymbol{I} + d\boldsymbol{J}_i)^n\left(\boldsymbol{I} + d\boldsymbol{J}_i^*\right)^n) \cdot \sigma(D\boldsymbol{f}^n(\boldsymbol{s}(1)) \cdot (D\boldsymbol{f}^n(\boldsymbol{s}(1)))^*)\\ &= \sigma((\boldsymbol{I} + d\bar{\boldsymbol{J}}_i)^n\left(\boldsymbol{I} + d\bar{\boldsymbol{J}}_i^*\right)^n) \cdot \sigma(D\boldsymbol{f}^n(\boldsymbol{s}(1)) \cdot (D\boldsymbol{f}^n(\boldsymbol{s}(1)))^*),\end{aligned}$$

where $\bar{\boldsymbol{J}} = [0] \oplus \bar{\boldsymbol{J}}_1 \oplus \cdots \oplus \bar{\boldsymbol{J}}_p$ is the Jordan canonical form of $\boldsymbol{G}$. Consequently, the Lyapunov exponents of (2.2) are

$$h_j + \lim_{n\to\infty} \frac{\ln\sqrt{\lambda_{k,i}}}{n}.$$

Here h_j are the Lyapunov exponents of the individual system $\boldsymbol{f}$, and $\lambda_{k,i}$ are the eigenvalues of $(\boldsymbol{I} + d\boldsymbol{J}_{\lambda_i})^n(\boldsymbol{I} + d\boldsymbol{J}_{\lambda_i}^*)^n$, where $\boldsymbol{J}_{\lambda_i}$ is a Jordan block of matrix $\boldsymbol{G}$ and λ_i is an eigenvalue of

$\boldsymbol{G}$. Let the size of matrix $\boldsymbol{J}_{\lambda_i}$ be $k_i \times k_i$, and let $\boldsymbol{N} = \boldsymbol{J}_{\lambda_i} - \lambda_i \boldsymbol{I}$. It should be noted that for sufficiently large n,

$$\begin{aligned}(\boldsymbol{I} + d\boldsymbol{J}_{\lambda_i})^n &= ((1 + d\lambda_i)\boldsymbol{I} + d\boldsymbol{N})^n = (1 + d\lambda_i)^n(\boldsymbol{I} + \alpha\boldsymbol{N})^n \\ &= (1 + d\lambda_i)^n \left(\boldsymbol{I} + \sum_{j=1}^{k_i-1} \binom{n}{j} \alpha^j \boldsymbol{N}^j\right) \\ &=: (1 + d\lambda_i)^n \boldsymbol{T}_i,\end{aligned}$$

where $\alpha = d/(1 + d\lambda_i)$. Clearly, the order of the magnitude of each entry of $\boldsymbol{T}_i\boldsymbol{T}_i^*$ is at most $\mathrm{O}(n^{2k_i-2})$. We conclude, via the Gershgorin disk theorem, that all eigenvalues of $\boldsymbol{T}_i\boldsymbol{T}_i^*$ are of the order $\mathrm{O}(n^{2k_i-2})$. Consequently, the Lyapunov exponents of (2.2) are

$$h_j + \ln|1 + d\lambda_i|. \tag{2.4}$$

We summarize the above as follows.

Theorem 2.1. *Let $\boldsymbol{G} = (g_{ij})$ be the coupling matrix satisfying that all its row sums are zero and zero is a simple eigenvalue. Then the synchronous state of CML (2.2) is (locally) stable provided that*

$$h_{max} + \ln|1 + d\lambda_i| < 0, \quad i = 2, \dots, m, \tag{2.5}$$

where $h_{\max}$ is the largest Lyapunov exponent of the individual map $\boldsymbol{f}$ and $\lambda_i \in \sigma(\boldsymbol{G}) - \{0\}$, $i = 2, \dots, m$. That is to say, if d satisfies the inequalities in (2.5), then for any initial values of (2.2) that are sufficiently close to the synchronous state $\{\boldsymbol{x}_i = \boldsymbol{s}\ \forall i\}$, we have that each of the oscillators $\boldsymbol{x}_i(n)$ tends to the same state as n goes to infinity. Otherwise, CML (2.2) will not acquire local synchronization.

Remark. (i) The decoupling form (2.3b) of variational equation (2.3a) was first observed and proposed by Pecora and Carroll [8]. (ii) If the identity matrix $\boldsymbol{I}$ in (2.2) is replaced by a diagonal matrix $\boldsymbol{D}$ with some but not all diagonal elements being zero, then the corresponding system (2.2) is called a partial-state coupling. The partial-state coupling also finds applications in various fields. For instance, in self-pulsating laser diode equations (see, e.g., [33]), only the photon density can be coupled with the electron density of the active region. Moreover, in the case of coupled chaotic systems, the systems that are partial-state coupled may exhibit different dynamic behavior. For instance, it is well known (see, e.g., [7]) that for the coupled Lorentz systems, only if the x-component or y-component is coupled will the resulting system achieve synchronization.

We shall assume from here on that the real parts of the eigenvalues of $\mathbf{G}$ are nonpositive. To find the range of the coupling d so that (2.5) is fulfilled, we need to solve the following min max problem:

$$\begin{aligned}\min_{d\in\mathbb{R}} \max_{2\le i\le m} |1 + d\lambda_i| &= \min_{d>0} \max_{2\le i\le m_1} |1 + d\lambda_i| \\ &=: \min_{d>0} \max_{2\le i\le m_1} r_i(d) =: \min_{d>0} r(d).\end{aligned} \tag{2.6}$$

Here m_1 is the number of eigenvalues lying in upper complex plane or on the real axis. The curves $r_i(d)$ are termed the ith mode of the transverse Lyapunov exponent curves. The equalities above are due to the facts that $|1+d\lambda_i| = |1+d\bar{\lambda}_i|$, the real parts of the eigenvalues of $\mathbf{G}$ are nonpositive, and (2.5) is violated whenever $d \leq 0$. Without loss of generality, we may assume that those distinct nonzero eigenvalues are λ_i, $i = 2, \ldots, m_1$, with $0 < |\lambda_2| \leq \cdots \leq |\lambda_{m_1}|$. The coupling value $d := d_c$ solving the min max problem (2.6) is the optimal choice of the coupling in the sense that it gives the fastest convergence rate of the initial values toward the synchronous state. To understand how $r(d)$ is formed, we need to know the ordering of $r_i(d)$. For $d > 0$, direct computations yield

$$\begin{aligned} r_i(d) &= \left[|\lambda_i|^2 d^2 + 2\,\mathrm{Re}(\lambda_i) d + 1\right]^{\frac{1}{2}} \\ &= \left[|\lambda_i|^2 \left(d - \frac{\mathrm{Re}(-\lambda_i)}{|\lambda_i|^2}\right)^2 + \frac{|\lambda_i|^2 - \mathrm{Re}^2(\lambda_i)}{|\lambda_i|^2}\right]^{\frac{1}{2}} \\ &=: \left[|\lambda_i|^2 \left(d - c_i\right)^2 + \tan^2\theta_i\right]^{\frac{1}{2}}. \end{aligned} \tag{2.7a}$$

Moreover, $r_i(d) \geq r_j(d)$ if and only if

$$\mathrm{Re}(\lambda_i) \geq \mathrm{Re}(\lambda_j) \quad \text{if } |\lambda_i| = |\lambda_j|, \tag{2.7b}$$

and

$$(|\lambda_i|^2 - |\lambda_j|^2)(d - d_{ij}) \geq 0 \quad \text{if } |\lambda_i| \neq |\lambda_j|, \tag{2.7c}$$

where

$$d_{ij} = \frac{2(\mathrm{Re}(-\lambda_i) - \mathrm{Re}(-\lambda_j))}{|\lambda_i|^2 - |\lambda_j|^2}. \tag{2.7d}$$

Let $A_i = \{j : 2 \leq j \leq m_1 \text{ and } |\lambda_i| = |\lambda_j|\}$. Then $\max_{j \in A_i} |1 + d\lambda_j| = |1 + d\lambda_k|$, where k is chosen so that $\mathrm{Re}(\lambda_k) \geq \mathrm{Re}(\lambda_j)\ \forall j \in A_i$. This gives that within each of the index set A_i, their corresponding quantities $|1+d\lambda_i|$ are well ordered for any $d > 0$. Consequently, to solve (2.6), we may assume, without loss of generality, that $0 < |\lambda_2| < \cdots < |\lambda_{m_1}|$ from here on. Using the terminology in [32], we see that the numbers 2 and m_1 correspond to the longest and shortest wavelength modes, respectively. The numbers in between 2 and m_1 are to be called intermediate wavelength modes. Since $d_{ij} = d_{ji}$ for any i and j we consider only d_{ij} with $i > j$. Our reduction process is now complete.

The following procedures are proposed to determine the "actual" node points of $r(d)$ from the candidate set $\{d_{ij} : i > j\}$.

(A) Set $k_0 = 0$, and $k_1 = \max\{l \mid \mathrm{Re}(\lambda_i) \leq \mathrm{Re}(\lambda_l)\ \forall i = 2 \ldots, m_1\}$. Let k_2 be the largest index so that $0 < d_{k_2k_1} \leq d_{kk_1}\ \forall k_1 < k \leq m_1$.

(B) Let k_3 be the largest index so that $d_{k_2k_1} < d_{k_3k_2} \leq d_{kk_2}\ \forall k_2 < k$. The process can be continued until $k_p = m_1$ for some $p \leq m_1$.

The next result shows that $\{k_i\}_{i=1}^{p}$ is the set of "actual" node points of $r(d)$.

Theorem 2.2. *Let $\boldsymbol{G}$ be given as in Theorem 2.1. Assume that the real parts of the eigenvalues of $\boldsymbol{G}$ are nonpositive. Then $r(d) = r_{k_i}(d)$ whenever $d_{k_ik_{i-1}} \le d \le d_{k_{i+1}k_i}$, $i = 1, \dots, p$. Here $d_{k_1,k_0} = 0$ and $d_{k_{p+1}k_p} = \infty$.*

Proof. Denote $I_j = [d_{k_jk_{j-1}}, d_{k_{j+1}k_j}]$. It then follows from (2.7c) that if $i > j$ and $d_{ij} > 0$, then $r_i(d) > r_j(d)$ whenever $d > d_{ij}$ and $r_i(d) < r_j(d)$ whenever $0 < d < d_{ij}$. We then conclude that

(2.8a) (i) the ordering of $r_i(d)$ and $r_j(d)$ remains the same until both curves meet;

(2.8b) (ii) if $r_i(d^*) > r_j(d^*)$ for some $d^* > 0$ with $i > j$, then $r_i(d) > r_j(d)\ \forall d \ge d^*$.

Using the first inequality in (2.7a), we have that $r(d) = r_{k_1}(d)$ for $\epsilon_1 > d \ge 0$. Here ϵ_1 is sufficiently small. It then follows from (2.8a), (2.8b), and procedure (A) that $r(d) = r_{k_1}(d)$ on I_1. Upon using (2.8a), we conclude that $r(d) = r_{k_2}(d)$ for $d \in (d_{k_2k_1}, d_{k_2k_1} + \epsilon_2)$. Here ϵ_2 is sufficiently small. Similarly, $r(d) = r_{k_2}(d)$ on I_2. We omit the proof of the remaining assertions of the theorem due to the similarity. ∎

Note that not all c_{k_i} given in (2.7a) could be critical points of $r(d)$. In fact, the critical points of $r(d)$ may not even come from the set $\{c_{k_i}\}$. We next identify the "actual" critical points of $r(d)$. Our next main result shows that $r(d)$ has exactly one critical point.

Theorem 2.3. *The curve $r(d)$ has a unique critical point d_c that solves the min max problem (2.6). Moreover, the optimal range of coupling d to sustain stably synchronous chaos of (2.2) is (d_l, d_r). Here d_l and d_r, $d_l < d_r$, are the intersection points (if any exist) of the straight line $y = e^{-h_{\max}}$ and the curve $y = r(d)$. Consequently, CML (2.2) acquires local synchronization if and only if $d \in (d_l, d_r)$.*

Proof. We break up the proof of the theorem into the following three steps.

Step I. We first claim that the number of c_{k_i} lying in the interior $\mathring{I}_i$ of I_i is at most one. Indeed, suppose there exist $c_{k_a} \in \mathring{I}_a$ and $c_{k_b} \in \mathring{I}_b$ with $c_{k_a} < c_{k_b}$. Then the following hold true: (i) $r_{k_a}(c_{k_b}) > r_{k_a}(c_{k_a})$. (ii) $r_{k_a}(c_{k_a}) > r_{k_b}(c_{k_a})$. (iii) $r_{k_b}(c_{k_a}) > r_{k_b}(c_{k_b})$. Inequalities (i) and (iii) hold true since c_{k_a} and c_{k_b} are, respectively, the minimum points of $r_{k_a}(d)$ and $r_{k_b}(d)$. The fact that $r_{k_a}(d)$ lies above all other curves on I_a leads to inequality (ii). Combining these inequalities, we have that $r_{k_a}(c_{k_b}) > r_{k_b}(c_{k_b})$, which is in contradiction to the fact that r_{k_b} is the maximum curve on I_{k_b}.

Step II. We next show that if $c_{k_i} \in \mathring{I}_i$, then $r(d)$ is decreasing on $(0, c_{k_i})$ and increasing on (c_{k_i}, ∞). Indeed, for $d \in I_{i+1}$, $r(d) = r_{k_{i+1}}(d) > r_{k_i}(d) > r_{k_i}(d_{k_{i+1}k_i}) = r_{k_{i+1}}(d_{k_{i+1}k_i})$. Using the conclusion in Step I and the fact that $r^2_{k_i}(d)$ is parabolic, we conclude that $r_{k_{i+1}}(d)$ must be increasing on I_{i+1}. On the other hand, $r_{k_{i-1}}(d)$ must be decreasing on I_{i-1} since $r_{k_{i-1}}(d) > r_k(d) > r_{k_i}(d_{k_ik_{i-1}}) = r_{k_{i-1}}(d_{k_ik_{i-1}})$. The monotonicity of $r(d)$ on each interval I_j, $1 \le j \le m_1$, can be similarly determined.

Step III. Since $r(d)$ is decreasing initially on I_1 and increasing eventually on I_p, there must be at least one critical point. If such points do not lie in the set of node points, then $r(d)$ has a unique critical point. Suppose $c_{k_i} \notin \mathring{I}_i\ \forall i = 1, \dots, p$. Then $r(d)$ is monotonic on each interval I_i. Suppose $r(d)$ first changes its monotonicity at $d_{k_{l+1}k_l}$ for some l. Then an argument similar to that given in Step II shows that once $r(d)$ becomes increasing on I_{l+1}, it will stay increasing the rest of the way. We have just completed the proof of the theorem. ∎

Remark. (i) If the straight line $y = e^{-h_{\max}}$ and the curve $y = r(d)$ do not intersect, then CML (2.2) will not achieve synchronization for any coupling strength. Suppose d_r and d_l exist. Then, as soon as d exceeds d_r, a certain wavelength mode is excited, which, in turn, causes the instability of the synchronous state. The illustration in Examples 2 and 3 shows that the excited wavelength mode could be either the shortest wavelength mode, the intermediate wavelength mode, or the longest wavelength mode. In any event, d_r is the exact critical value where wavelength bifurcation occurs. On the other hand, d_l is the exact critical value where all wavelength modes become de-excited enough to induce the stability of the synchronous state. (ii) Such $r(d)$ is called the *synchronization curve* of CML (2.2), and the interval (d_l, d_r), if it exists, is termed the *synchronization interval* of CML (2.2). Clearly, $d_c \in (d_l, d_r)$ and depends only on the connectivity topology.

Theorem 2.4. *Suppose the coupling matrix* $\mathbf{G}$ *has nonpositive real eigenvalues. Denote by* $\{\lambda_i\}_{i=2}^{m_1}$ *the distinct nonzero eigenvalues of* $\mathbf{G}$*. Then*

$$r(d) = \begin{cases} \lambda_2(d), & d \in [0, d_{m_1 2}] = I_1, \\ \lambda_{m_1}(d), & d \in (d_{m_1 2}, \infty) = I_2, \end{cases}$$

and $d_c = d_{m_1 2} = \frac{-2}{\lambda_2 + \lambda_{m_1}}$*. Consequently, depending on the quantity of* $h_{\max}$*, either CML (2.2) achieves no synchronization or SWB occurs as d varies. Furthermore, if* d_l *and* d_r *exist, then the* synchronization interval *of the corresponding CMLs is* $\left(\frac{1-e^{-h_{\max}}}{-\lambda_2}, \frac{1+e^{-h_{\max}}}{-\lambda_{m_1}}\right)$.

Proof. It is easily seen that $k_1 = 2$ and $k_2 = m_1$ since $d_{ij} = \frac{-2}{\lambda_i + \lambda_j}$. Thus, $r(d)$ is as asserted. The proof then follows from the facts that $c_{m_1} = -\frac{1}{\lambda_{m_1}} < \frac{-2}{\lambda_2 + \lambda_{m_1}} < -\frac{1}{\lambda_2} = c_2$ and $d_c = d_{m_1 2}$. Solving equations $y = r(d)$ and $y = e^{-h_{\max}}$, we have that d_l and d_r are as claimed. ∎

3. Illustrations and applications.

We illustrate our theorems with the following examples.

Example 1. Let the oscillators be diffusively coupled with periodic boundary conditions. For such $\mathbf{G}$, $m_1 = m$, $-\lambda_{m_1} = 4\sin^2 \frac{[\frac{m}{2}]\pi}{m}$, and $-\lambda_2 = 4\sin^2 \frac{\pi}{m}$.

Let $f(x) = 4x(1-x)$, $0 \le x \le 1$. Then $h_{\max} = \ln 2$, and the corresponding candidates for d_l and d_r are, respectively, $\frac{1}{8}\sin^{-2}\frac{\pi}{m}$ and $\frac{3}{8}\sin^{-2}\frac{[\frac{m}{2}]\pi}{m}$. However, $d_l \le d_r$ only if $m \le 5$. Hence, we conclude that the maximum number of oscillators to sustain synchronous chaos is 5.

We next compare our results with those obtained in [23], [34]. Their sufficient conditions on the coupling strength for obtaining stable synchronization are, respectively, given as follows: $\frac{1-e^{-h_{\max}}}{m} < d\, g_{ij} < \frac{1+e^{-h_{max}}}{m}$ and $\left(\sum_{k=1,\, k\neq i}^{m} |g_{ki} - g_{ji}|\right) + \left|\frac{1}{d} + g_{ii} - g_{ji}\right| < \frac{1}{d}e^{-h_{\max}}$ $\forall i, j$ with $i \neq j$. However, the first inequality above fails to find any suitable coupling strength provided that $\boldsymbol{G}$ has zero off-diagonal elements. If $\boldsymbol{G}$ is given as above with $m \ge 4$ and $f(x) = 4x(1-x)$, then the second inequality also fails to find any suitable coupling strength.

Example 2. Consider synchronization in a directed ring of $2K$ nearest neighbor coupled oscillators [19] with $K = 2$ and $m = 9$. Specifically, the coupling matrix $\boldsymbol{G}$ under consideration is a circulant matrix of the form

$$\boldsymbol{G} = \text{circ}(-30,\, 13,\, 2,\, 0,\, \ldots,\, 0,\, 5.4,\, 9.6).$$

The spectrum of $\boldsymbol{G}$ is $\{-30+13e^{\frac{2(j-1)\pi}{9}i}+2e^{\frac{4(j-1)\pi}{9}i}+5.4e^{\frac{14(j-1)\pi}{9}i}+9.6e^{\frac{16(j-1)\pi}{9}i}:j=1,\dots,9\}$. Here $\lambda_2 \approx -11.4024+1.1629i$, $\lambda_3 \approx -33.0293+2.1855i$, $\lambda_4 \approx -45+5.8890i$, and $\lambda_5 \approx -45.5683+3.3483i$. Direct computations yield that $d_{42} \approx 0.0348 < d_{52} < d_{32}$, $d_{54} \approx 0.0406$, and $c_5 < c_4 < d_{42} < d_{54} < c_2$ (see Figure 1). Consequently,

$$r(d) = \begin{cases} r_2(d), & d \in I_1 = [0, d_{42}], \\ r_4(d), & d \in I_2 = [d_{42}, d_{54}], \\ r_5(d), & d \in I_3 = [d_{54}, \infty], \end{cases}$$

the node points of $r(d)$ are d_{42} and d_{54}, and the critical point of $r(d)$ occurs at d_{42}. Let $f_\mu(x) = \mu x(1-x)$. For $\mu = 4$, since $e^{-h_{max}} = e^{-\ln 2} = 0.5 < r(d_{42})$, the *synchronization interval* does not exist. As μ varies from $\mu_\infty \approx 3.57$ to $\mu = 4$, scenarios (i), (ii), and (iii) described in the introductory section can be clearly observed from the figure.

On the other hand, the maximum number of oscillators on such connectivity topology to sustain stably synchronous chaos is 7. The claim above is done by checking the intersection of the equations $y = \frac{1}{2}$ and $y = r(d)$ $\forall m \leq 8$.

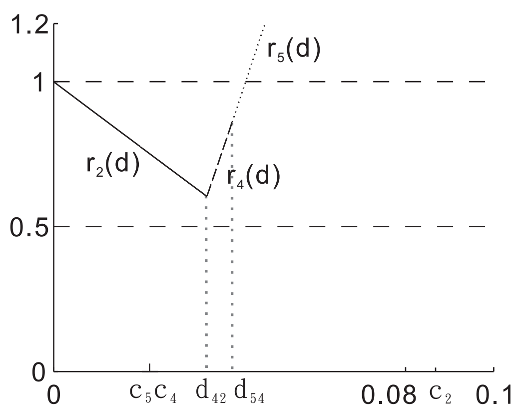

Figure 1. *Graph of $r(d)$ in Example 2. Here $d_{42} \approx 0.0348$, $d_{54} \approx 0.0406$, $r(d_{42}) \approx 0.6040$, $r(d_{54}) \approx 0.8604$, $c_5 \approx 0.02182$, $c_4 \approx 0.02185$, and $c_2 \approx 0.0868$. The critical point of $r(d)$ is d_{42}.*

Example 3. The following example shows that LWB is also possible. Let $\boldsymbol{G}$ be given as follows:

$$\begin{pmatrix} -30 & 3 & 12 & 5 & 10 \\ 10 & -30 & 3 & 12 & 5 \\ 5 & 10 & -30 & 3 & 12 \\ 12 & 5 & 10 & -30 & 3 \\ 3 & 12 & 5 & 10 & -30 \end{pmatrix}.$$

The spectrum of $\boldsymbol{G}$ is $\{0, -35.2639+10.7719i =: \lambda_2, -39.7361+2.5429i =: \lambda_3, \bar{\lambda}_2, \bar{\lambda}_3\}$. Then the graph of $r(d)$ is demonstrated in Figure 2. Consider $f(x) = 4x(1-x)$. Then the longest wavelength mode becomes excited to induce instability as d is increased beyond d_r.

Example 4. To illustrate the accuracy of our theorems, *synchronization intervals* established in Theorem 2.4 are compared with those obtained by the computer simulation. In particular, theoretically and numerically predicted *synchronization intervals* for three exam-

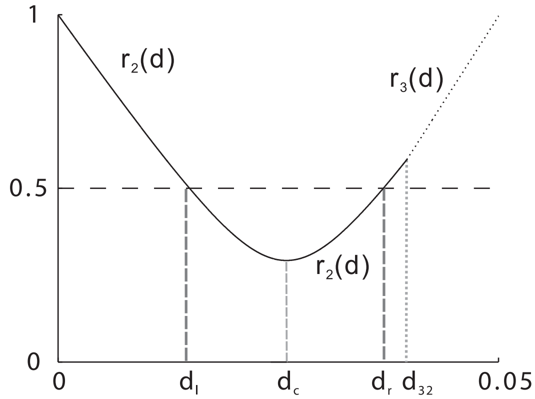

Figure 2. *Graph of $r(d)$ in Example 3. Here $d_{32} \approx 0.0396$, $c_2 \approx 0.0259$, $c_3 \approx 0.0251$, $d_l \approx 0.0149$, and $d_r \approx 0.0369$. The critical point of $r(d)$ is c_2.*

ples above are almost identical. Such comparisons are recorded in Figure 3. They are "almost" identical. This simulation is set up so that the differences between the initial values $\boldsymbol{x}_i$ are within 10^{-5}. Synchronization is achieved when their differences are within 10^{-15}.

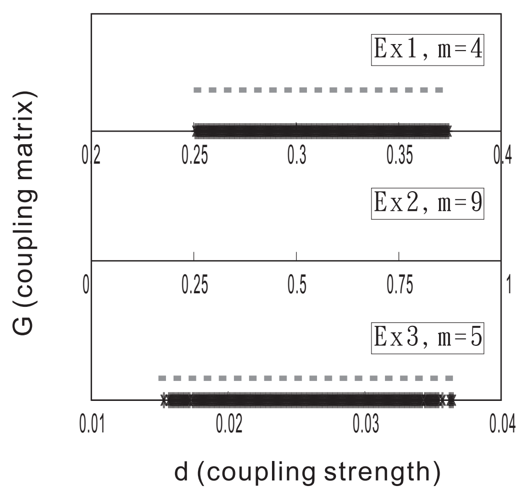

Figure 3. *Three typical* synchronization intervals *for coupled logistic map with various coupling matrices are shown. Solid (bold) lines are synchronization intervals obtained by computer simulation. Dotted (fine) lines are synchronization intervals predicted by our theorems. All are scaled for clear visualization.*

4. Conclusions. We conclude this paper by mentioning the difficulty one might face by applying our methods to more general cases, $\boldsymbol{D} \neq \boldsymbol{I}$ or $\boldsymbol{h} \neq \boldsymbol{f}$, and a possible approach to solving them.

Our main results in this paper are based on the study of the inequalities in (2.5). However, in the case that $\boldsymbol{D} \neq \boldsymbol{I}$ or $\boldsymbol{h} \neq \boldsymbol{f}$, it seems to be a nontrivial matter to find their corresponding inequalities such as (2.5). One possible approach is to find the lower and upper bounds of the Lyapunov exponents of (2.2), where both bounds have expressions similar to those in (2.4).

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