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Performance Analysis of Block Codes in Hidden Markov Channels

Chang-Ming Lee, Yu. T. Su, and Li-Der Jeng

***Abstract*— Most investigations on the effect of channel memory on the performance of block codes use a two-state Gilbert-Elliott (GE) model to describe the channel behavior. As there are circumstances that the channel of concern can not be properly described by the GE model, there are some recent works on coded performance [5-7] that characterize the channel behavior by a general finite-state Markov chain. This letter presents a new efficient systematic approach to analyze the performance of block codes in such a hidden Markov channel (HMC). An application example is given to predict codeword error probability performance of an RS-coded system in a channel with memory. Numerical results are also provided to validate our analytic results.**

***Index Terms*— Gilbert-Elliott (GE) model, hidden Markov channel (HMC), interleaver.**

I. Introduction

The effect of channel memory on the performance of an error-correcting code is often analyzed by the use of some Markovian channel models [1-6]. Most analysis used the Gilbert-Elliott (GE) channel model to evaluate the influence of finite interleaving. However, it is known that the simple GE model can not properly describe some of the important channel characteristics [3]. Performance analysis for block codes in general finite-state Markov channels can be found in [5-7], this letter presents a new systematic and computational-efficient method and derives the associated conditional probabilities to accurately predict the codeword error probability (CEP). Because of space limitation we only present one application example here although our general analysis can be applied to other cases.

In the following section we discuss a communication system in which the corresponding channel is best described as a hidden Markov model (HMM) with at least four states and evaluate the corresponding parameters. Section III focuses on the evaluation of the CEPs. We first give a general CEP expression which is in turn decomposed into some conditional CEPs. Based upon the methods to derive the probability of an error event in an arbitrary s-state Markov chain proposed in [5], we derive the general forms of various conditional CEPs and the specific expressions of each CEP for the the channel of Section II. Finally, some numerical examples and related discussion are provided in Section IV.

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II. Hidden Markov Channels and Models

Consider the system whose (source) data stream is first encoded by an $M-$ary (n,k) block code, e.g., a singly-extended Reed-Solomon code, where $n=p^m$ for a prime p and an integer m, the coded symbols are then interleaved by a block symbol interleaver before being mapped into M-ary orthogonal signals ($M=n$). The carrier of the sequence of the resulting orthogonal signals is hopped periodically according to some predetermined pattern. We assume that the received signal suffers from (i) additive white Gaussian noise (AWGN) whose one-sided power spectral density (PSD) is N_0 W/Hz, (ii) frequency nonselective fading, i.e., the received amplitude remains constant during a symbol period, and (iii) partial band noise jamming whose probability of presence is μ, $0<\mu\leq 1$.

We assume that a hop duration is a multiple of the depth of the block interleaver used, where the interleaver size K_I is equal to the product of its depth (m) and span (n). There will be several hops in one interleaving block and, at the interleaver output, symbols in several adjacent rows will be in the same hop. Let H be the number of hops per $K_I T_s$ seconds and assume $n=HS$ and J out of H hops are jammed. Therefore, the number of jammed symbols in one codeword is JS and the remaining $n-JS$ symbols are free of jamming.

Obviously, the classic two-state GE model does not suffice to characterize this channel; one needs at least four states to indicate the jammer states (jammed versus unjammed) and the fading states (good versus bad), i.e., there are four channel states, namely, the unjammed and good (G_u), unjammed and bad (B_u), jammed and good (G_J) and jammed and bad (B_J) states; see Fig. 1. They represent respectively whether the jammer is present and/or the received average SNR is greater than a certain threshold.

The hopping rate and the interleaver structure impose constraints on the allowable state transitions in the above channel model. Depending on whether the first codeword symbol is jammed, the channel will degenerates to a two-state GE model in the next $S-1$ consecutive symbol intervals (G_u vs. B_u or G_J vs. B_J). The channel will not return to a four-state model again until the $(S+1)$th symbol interval. For convenience, each of the two groups of states is referred to as a superstate (Fig. 1).

A partial band noise jammer (PBNJ) distributes its total power P_J evenly over a continuous spectrum of W_J Hz. Let W_{ss} be the total hopping bandwidth then $\mu=W_J/W_{ss}\leq 1$ is the probability that the PBNJ is present in the signal band.

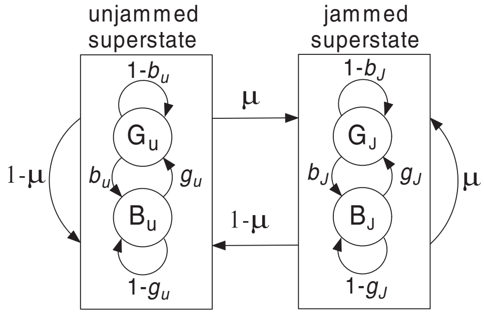

Fig. 1. A four-state model for jammed Rayleigh channels.

Within the jammed band, the transmitted signal is corrupted by an equivalent AWGN whose PSD level N_T is equal to $N_J/\mu + N_0$, where $N_J = P_J/W_{ss}$; otherwise the PSD level is $N_T = N_0$. When a hop is not jammed, the channel behaves just like the GE model [1]. Hence the transition probabilities g_u (B_u to G_u) and b_u (G_u to B_u) are given by

$$g_u = \frac{\rho f_D T_s \sqrt{2\pi}}{e^{\rho^2} - 1} \quad (1a)$$

$$b_u = \rho f_D T_s \sqrt{2\pi} \quad (1b)$$

where f_D is the Doppler frequency, T_s is the symbol duration and $\rho = \gamma_t/\overline{\gamma}$ is the ratio between the channel state threshold and the average SNR of the received signal. Note that the product $f_D T_s$ determines the average fade duration in the sense that a smaller $f_D T_s$ implies smaller transition probabilities and larger probabilities of staying at a given state. The choice of the threshold, as found by [1], has, in many cases, little impact on the accuracy of the model if it is within a reasonable range. On the other hand, when a hop is jammed, the average SNR of the received signal $\overline{\gamma_J}$ is equal to $c\overline{\gamma}$, where $c = N_T/N_0$. The transition probabilities g_J (B_J to G_J) and b_J (G_J to B_J) then can be easily modified by substituting ρ_J for ρ where $\rho_J^2 = \gamma_t/\overline{\gamma_J}$.

III. Codeword Error Probability Analysis

The CEP $P_w(e)$ performance of a block code in an Hidden Markov (HM) channel can be decomposed into a sum of some conditional CEPs. For an s-state HM channel (HMC), if we denote by $E_s = (n_1, \cdots, n_s)$ and $P_n(E_s)$, the event that during an n-symbol period, the channel is in state j for n_j (symbol) times, $j = 1, \cdots, s$, and the corresponding probability then

$$P_w(e) = \sum_{E_s} P_w(e|n_1, \cdots, n_s) P_n(n_1, \cdots, n_s) = \sum_{E_s} P_w(e|E_s) P_n(E_s), \quad (2)$$

where $P_w(e|E_s) = P_w(e|n_1, \cdots, n_s)$ is the conditional CEP given E_s and $P_n(E_s)$ is called the channel state-sequence (CSS) probability.

A. Conditional error probabilities

Besides the CSS probabilities, we still have to compute the conditional CEP $P_w(e|E_s)$ to obtain the unconditional CEP $P_w(e)$. The evaluation of $P_w(e|E_s)$, in turn, relies on the knowledge of the following component conditional probabilities.

$$P_T(t) = P_r\{t \text{ symbols of a codeword are incorrectly detected but not erased}\} \quad (3a)$$

$$P_E(\ell) = P_r\{\ell \text{ erasures in one codeword}\} \quad (3b)$$

$$P_{e|i} = P_r\{\text{a symbol is erased}|\text{channel state} = i\} \quad (3c)$$

$$P_{s|i} = P_r\{\text{a symbol is incorrectly detected}|\text{channel state} = i\} \quad (3d)$$

$$P_{c,\overline{e}|i} = P_r\{\text{a symbol is correctly detected and not erased}|\text{channel state} = i\} \quad (3e)$$

$$P_{s,\overline{e}|i} = P_r\{\text{a symbol is incorrectly detected but not erased}|\text{channel state} = i\} \quad (3f)$$

For a system that employs a block code of length n for transmission over an HMC characterized by an s-state Markov chain, the corresponding CEP expression is

$$P_w(e) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} \cdots \sum_{n_{s-1}=0}^{n-\sum_{j=1}^{s-2} n_j} P_n(E_s) P_w(e|E_s), \quad (4)$$

where $n = n_1 + n_2 + \cdots + n_s$. Let the minimum distance of the code be $d_{min} = 2t + 1$. Then for errors-only (EO) decoding, we have the decomposition

$$P_w(e|E_s) = \sum_{t=\lceil \frac{d_{min}}{2} \rceil}^{n} \sum_{t_1} P_s(t_1|n_1) \sum_{t_2} P_s(t_2|n_2) \cdots \cdot \sum_{t_{s-1}} P_s(t_{s-1}|n_{s-1}) P_s(t_s|n_s), \quad (5)$$

where $t = \sum_{i=1}^{s} t_i$ and $\lceil x \rceil$ represents the smallest integer greater than or equal to x while

$$P_s(t_i|n_i) = \binom{n_i}{t_i} P_{s|i}^{t_i} (1 - P_{s|i})^{n_i - t_i} \quad (6)$$

is the conditional probability that t_i errors occur during the n_i times the channel stays at state i. (6) follows from the facts that (i) a received codeword is associated with a specific (hidden) CSS E_s, and (ii) given E_s, the conditional symbol error probabilities associated with codeword symbols are independent of each other. The upper and lower limits of various summations in (5), (8b), and (8c) given below are listed in Table I.

If an errors-and-erasures (EE) decoder is used, we have

$$P_w(e|E_s) = \sum_{\substack{\ell, t \\ 2t+\ell \geq d_{min}}} P(\ell, t|E_s) = \sum_{\ell=0}^{d_{min}-1} \sum_{t=\lceil \frac{d_{min}-\ell}{2} \rceil}^{n-\ell} P(\ell, t|E_s) + \sum_{\ell=d_{min}}^{n} P_E(\ell|E_s), \quad (7)$$

where $P(\ell, t|E_s)$ is the joint (conditional) probability that there are ℓ erasures and t errors in a codeword and

TABLE I

UPPER LIMITS (UL) AND LOWER LIMITS (LL) OF VARIOUS PARAMETERS IN (5), (8B) AND (8C)

	EO decoding	EE decoding
t_j UL	$\max\left(0, t-\sum_{m=1}^{j-1} t_m - \sum_{m=j+1}^{k} n_m\right)$	$\max\left(0, t-\sum_{m=1}^{j-1} t_m - \sum_{m=j+1}^{k} (n_m - e_m)\right)$
t_j LL	$\min\left(n_j, t-\sum_{m=2}^{j-1} t_m\right)$	$\min\left(n_j - e_j, t-\sum_{m=2}^{j-1} t_m\right)$
e_j UL	none	$\max\left(0, e-\sum_{m=1}^{j-1} e_m - \sum_{m=j+1}^{k} n_m\right)$
e_j LL	none	$\min\left(n_j, e-\sum_{m=2}^{j-1} e_m\right)$

$P_E(\ell|E_s) = P_E(\ell, 0|E_s)$ is the (conditional) probability that there are ℓ erasures in a codeword. It is straightforward to show that

$$P(\ell, t|E_s) = P_E(\ell|E_s) P_T(t|\ell, E_s) \quad (8a)$$

$$P_E(\ell|E_s) = \sum_{\ell_1} P_e(\ell_1|n_1) \sum_{\ell_2} P_e(\ell_2|n_2) \cdots \cdot \sum_{\ell_{s-1}} P_e(\ell_{s-1}|n_{s-1}) P_e(\ell_s|n_s) \quad (8b)$$

$$P_T(t|\ell, E_s) = \sum_{t_1} P_s(t_1|n_1, \ell_1) \sum_{t_2} P_s(t_2|n_2, \ell_2) \cdots \cdot \sum_{t_{s-1}} P_s(t_{s-1}|n_{s-1}, \ell_{s-1}) P_s(t_s|n_s, \ell_s) \quad (8c)$$

where $\sum_i \ell_i = \ell$, $\sum_j t_j = t, \sum_k n_k = n$, and

$$P_e(\ell_i|n_i) = \binom{n_i}{\ell_i} P_{e|i}^{\ell_i} (1 - P_{e|i})^{n_i - \ell_i}, \quad (9a)$$

$$P_s(t_i|n_i, \ell_i) = \binom{n_i - \ell_i}{t_i} P_{s|i}^{t_i} (1 - P_{s|i})^{n_i - t_i - \ell_i} \quad (9b)$$

with $P_{e|i} = 1 - P_{c,\overline{e}|i} - P_{s,\overline{e}|i}$ and $P_{s|i} = \frac{P_{s,\overline{e}|i}}{1-P_{e|i}}$.

The conditional CEP associated with an EO decoder is a function of $P_{s|i}$. If the channel state is defined according to where the received waveform's instantaneous SNR $\stackrel{def}{=}$ γ lies and $f(\gamma)$ is the pdf of the γ while $P_s(\gamma)$ is the codeword error probability conditioned on γ, then $P_{s|i} = \frac{1}{P_i^\infty} \int_{R_i} f(\gamma) P_s(\gamma) d\gamma$, where R_i is the defining region of the channel state i and $P_i^\infty \stackrel{def}{=} \int_{R_i} f(\gamma) d\gamma$. On the other hand, the conditional CEP of an EE decoder depends on $P_{s|i}$ and $P_{e|i}$, which are derived from $P_{c,\overline{e}|i}$ and $P_{s,\overline{e}|i}$. Let $P_{c,\overline{e}}(\gamma)$ be the conditional probability that a received symbol is correctly detected and not erased and $P_{s,\overline{e}}(\gamma)$ be the conditional probability that a received symbol is incorrectly detected but not erased. Then $P_{c,\overline{e}|i} = \frac{1}{P_i^\infty} \int_{R_i} f(\gamma) P_{c,\overline{e}}(\gamma) d\gamma$, and $P_{s,\overline{e}|i} = \frac{1}{P_i^\infty} \int_{R_i} f(\gamma) P_{s,\overline{e}}(\gamma) d\gamma$.

B. Evaluating component probabilities

We have outlined a systematic approach and general expressions for evaluating the CEP performance of an arbitrary block code with minimum distance $d_{min} = 2t + 1$ in an arbitrary s-state HMC. The exact component conditional probabilities in those expressions can be derived only if the channel statistic and decoding method, including the erasure-insertion method (EIM) used, are given.

Assuming perfect carrier de-hopping and defining $\eta = \frac{k \log_2 n}{n}$, we can express the associated symbol error probability for an optimal noncoherent detector (matched-filter bank) as,

$$P_s(\gamma) = \sum_{i=0}^{n-2} \binom{n-1}{i+1} (-1)^i \frac{1}{i+2} e^{-\frac{i+1}{i+2}\eta\gamma}. \quad (10)$$

Using the ratio threshold test (RTT) [9] in which an erasure is inserted when the ratio between the largest and the second largest outputs of the noncoherent matched-filter bank is greater than τ, we can easily derive $P_{c,\overline{e}}(\gamma)$ and $P_{s,\overline{e}}(\gamma)$ from [4]. Based on the above analysis, we now proceed to analyze the performance of the RS-coded system. We first note that when there is no jamming the transmitted signal suffers from both AWGN and Rayleigh fading, and the channel is in the unjammed super-state that consists of G_u and B_u states. It can be shown that within this super-state the SNR pdf is given by $f(\gamma) = \frac{1}{\overline{\gamma}} e^{-\gamma/\overline{\gamma}}, \quad \gamma \geq 0$, and

$$P_{s|B_u} = \frac{1}{1 - e^{-\rho^2}} \sum_{i=0}^{n-2} (-1)^i \binom{n-1}{i+1} \cdot \frac{1 - \exp\left[-\left(\frac{1}{\overline{\gamma}} + \left(\frac{i+1}{i+2}\right)\eta\right)\gamma_t\right]}{(i+2)\left[1 + \left(\frac{i+1}{i+2}\right)\eta\overline{\gamma}\right]}, \quad (11a)$$

$$P_{s|G_u} = \frac{1}{e^{-\rho^2}} \sum_{i=0}^{n-2} (-1)^i \binom{n-1}{i+1} \cdot \frac{\exp\left[-\left(\frac{1}{\overline{\gamma}} + \left(\frac{i+1}{i+2}\right)\eta\right)\gamma_t\right]}{(i+2)\left[1 + \left(\frac{i+1}{i+2}\right)\eta\overline{\gamma}\right]}. \quad (11b)$$

Moreover,

$$P_{c,\overline{e}|B_u} = \frac{1}{1 - e^{-\rho^2}} \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \frac{1}{(\tau i + 1)} \cdot \frac{1 - \exp\left[-\left(\frac{1}{\overline{\gamma}} + \frac{\tau i \eta}{\tau i + 1}\right)\gamma_t\right]}{\left[1 + \left(\frac{\tau i \eta \overline{\gamma}}{\tau i + 1}\right)\right]}, \quad (12a)$$

$$P_{s,\overline{e}|B_u} = \frac{1}{1 - e^{-\rho^2}} \sum_{i=0}^{n-2} (-1)^i \binom{n-1}{i+1} \frac{\tau(i+1)}{(\tau i + 1)}$$

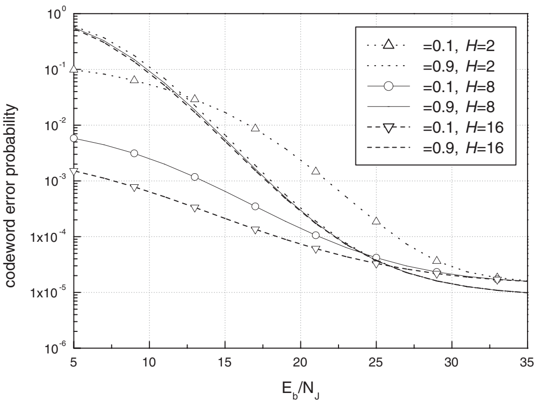

Fig. 2. Analytic CEP performance of a 1T-RTT (16,8) RS decoder as a function of μ and hopping rate (H); average $E_b/N_0 = 20$ dB, RTT thresholds = .6 (unjammed) and .5 (jammed).

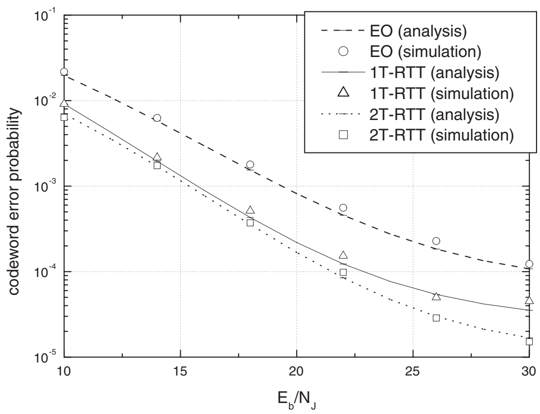

Fig. 3. CEP performance of a (16,8) RS-coded SFH MFSK system in presence of PBNJ and fading; $E_b/N_0 = 20$ dB, $\mu = .2$, $H = 8$.

$$\cdot \quad \frac{\left\{1-\exp\left[-\left(\frac{1}{\overline{\gamma}}+\frac{\tau i\eta}{\tau i+1}\right)\gamma_t\right]\right\}}{[\tau(i+1)+1]\left[1+\left(\frac{(\tau i+1)\eta\overline{\gamma}}{\tau(i+1)+1}\right)\right]}, \tag{12b}$$

$$P_{c,\bar{e}|G_u} = \frac{1}{e^{-\rho^2}}\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\frac{1}{(\tau i+1)} \cdot \frac{\exp\left[-\left(\frac{1}{\overline{\gamma}}+\frac{\tau i\eta}{\tau i+1}\right)\gamma_t\right]}{\left[1+\left(\frac{\tau i\eta\overline{\gamma}}{\tau i+1}\right)\right]}, \tag{12c}$$

$$P_{s,\bar{e}|G_u} = \frac{1}{e^{-\rho^2}}\sum_{i=0}^{n-2}(-1)^i\binom{n-1}{i+1}\frac{\tau(i+1)}{(\tau i+1)} \cdot \frac{\exp\left[-\left(\frac{1}{\overline{\gamma}}+\frac{(\tau i+1)\eta}{\tau(i+1)+1}\right)\gamma_t\right]}{[\tau(i+1)+1]\left[1+\left(\frac{(\tau i+1)\eta\overline{\gamma}}{\tau(i+1)+1}\right)\right]}. \tag{12d}$$

When a PBNJ is present, the channel is in state G_J or B_J, the conditional probabilities can be obtained by replacing $\overline{\gamma}$ and ρ in (11a)–(12d) by $\overline{\gamma}_J$ and ρ_J, respectively.

IV. Numerical Examples and Discussions

By substituting the component conditional probabilities derived in the previous section into the related general CEP and conditional CEP expressions, we will be able to evaluate the performance of these systems analytically and assess the influence of various system and channel parameters. The correlated Rayleigh fading channel used in simulation is generated by the modified Jakes model of [8].

In Fig. 2, we examine the effects of the transmitter's strategy (hops per interleaving block, H) and the jammer's, i.e., the fraction of band jammed μ. The increase of H gives higher hopping rate and makes neighboring symbols less correlated thus improves the decoder performance. When $\mu = .9$, the performance is insensitive to variation of H for the jammer has appeared like a full-band jammer. For all three values of H, $\mu = .9$ yields worse performance at low E_b/N_J, which is consistent with the observation from earlier studies–when the jammer has enough power a better jamming effect is achieved if it distributes its power over a wider bandwidth (larger μ); otherwise it should concentrate its jamming power within a small bandwidth (smaller μ). An ideal anti-jam (AJ) scheme is one that forces the jammer to adopt the full-band jamming strategy at all E_b/N_J. The crossover point of the two curves corresponding to two different μ's with the same H indicates the E_b/N_J threshold at which the jammer should change its jamming strategy. As the crossover point shifts to higher E_b/N_J as H increases, it is clear that a higher hopping rate does enhance the receiver's AJ capability.

Fig. 3 compares the CEP performance of the EO, 1T EE and 2T EE decoders in the presence of PBNJ and fading when the ratio of the average bit energy to noise power level, $\overline{\gamma}_b$, is 20 dB and the threshold γ_t used to separate the two fading states is 2 dB. The CEP performance curves are depicted as a function of $\overline{\gamma}_{bJ} \overset{def}{=} \overline{E}_b/N_J$. Again, we find that our analytic prediction is very close to that predicted by simulation and, as expected, EE decoding is better than EO decoding. At CEP$=10^{-4}$, the EE decoding gain is greater than 7 dB while the corresponding gain is greater than 4 dB at CEP$= 10^{-3}$.

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