



The generalized sequential compound options pricing and sensitivity analysis

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Abstract

This paper proposes a generalized pricing formula and sensitivity analysis for sequential compound options (SCOs). Most compound options described in literatures, initiating by Geske [Geske, R., 1977. The Valuation of Corporate Liabilities as Compound Options. *Journal of Finance and Quantitative Analysis*, 12, 541–552; Geske, R., 1979. The Valuation of Compound Options. *Journal of Financial Economics* 7, 63–81.], are simple 2-fold options. Existing research on multi-fold compound options has been limited to sequential compound CALL options whose parameters are constant. The multi-fold sequential compound options proposed in this study are defined as compound options on (compound) options where the call/put property of each fold can be arbitrarily assigned. In addition, the deterministic time-dependent parameters, including interest rate, depression rate and variance of asset price, make the SCOs more flexible. The pricing formula is derived by the risk-neutral method. The partial derivative of a multivariate normal integration, which is an extension of Leibnitz's Rule, is derived in this study and used to derive the SCOs sensitivities. The general results for SCOs presents in this paper can enhance and broaden the use of compound option theory in the study of real options and financial derivatives.

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1. Introduction

Compound options, initiating by Geske (1977, 1979), are options with other options as underlying assets. The *fold number* of a compound option counts the number of option layers tacked directly onto underlying options. The original closed form of 2-fold compound option is proposed by Geske (1977, 1979) and constitutes as precedents with respect to later works. Specific multi-fold compound option pricing formulas are proposed by Geske and Johnson (1984a) and Carr (1988) while the pricing formula of sequential compound call (SCC) is proved by Thomassen and Van Wouwe (2001) and Chen (2003). Chen (2002) and Lajeri-Chaherli (2002) simultaneously derive the price formula for 2-fold compound options through the risk-neutral method. Agliardi and Agliardi (2003) generalize the results to 2 fold compound calls with time-dependent parameters, while Agliardi and Agliardi (2005) extend the multi-fold compound calls to parameters varying with time.

Financial applications based on compound option theory are widely employed. Geske and Johnson (1984a) use exotic multi-fold compound options for the American put option, while Carr (1988) presents the pricing formula for sequential exchange options. Corporate debt (Chen, 2003; Geske and Johnson, 1984b) and chooser options (Rubinstein, 1992), as well as *capletions* and *floortions* (options on interest rate options) (Musielà and Rutkowski, 1998) are also priced by compound options.

In addition to the pricing of financial derivatives, compound option theory is widely used in the real option study. This approach originates from Myers (1977) and is followed by Brennan and Schwartz (1985), Pindyck (1988), Trigeorgis (1993, 1996) and so forth. Examples include project valuation of new drugs (Casimon et al., 2004), production and inventory (Cortazar and Schwartz, 1993) and capital budget decision (Duan et al., 2003). Compound option methodology turns out to be very common, and the theory is versatile enough to treat many real-world cases (Copeland and Antikarov, 2003).

However, the sophisticated structure of financial derivatives and their wide deployment in the real options field have revealed the limitations of the current compound option methodology. 2-fold compound options cannot be used as further building blocks to model other financial innovations, but results concerning multi-fold compound options so far have focused only on sequential compound calls. Although Remer et al. (2001, p.97) mention that "... in practice, different project phases often have different risks that warrant different discount rates," the important feature of time-dependent (or fold-dependent) parameters is rarely taken into account by current methodologies.

This paper, using vanilla European options as building blocks, extends the compound option theory to multi-fold sequential compound options (SCOs) with time-dependent parameters as well as alternating puts and calls arbitrarily (see Table 1). An SCO is defined as a (compound) option written on another compound option, where the call/put feature of each fold can be assigned arbitrarily. The SCOs presented in this study also allow deterministic parameters (such as interest rate, depression rate and variance of asset price) to vary over time, hence entitle this paper as a "generalized" SCOs and regard the situation of fold-wise parameters as its special case. This study derives an explicit valuation formula for SCOs by the risk-neutral method, and performs the sensitivity analysis on the result. Compared with the P.D.E. method, more financial intuition is gained by the risk-neutral derivation. Moreover, the partial derivative of a multivariate normal integration (an extension of Leibnitz's rule), is also derived here for the sensitivity analysis.

Multi-fold SCOs with alternating puts and calls and time-dependent parameters can greatly enhance the number of practical applications for compound options, especially in the real option field. Real world cases can often be expressed in terms of options, such as expansion, contraction,

Table 1
Evolutions of compound option theory

Reference	Fold Number	Approach	Generalization	
			Put-call alternating	Time-dependent parameters
Geske (1977, 1979) ^a	2	PDE	Put/Call	No
Agliardi and Agliardi (2003)	2	PDE	Call	Yes
Chen (2002), Lajeri-Chaherli (2002)	2	Risk-neutral	Put/call	No
Carr (1988), Chen (2003)	Multiple	Risk-neutral	Call	No
Thomassen and Van Wouwe (2001)	Multiple	PDE	Call	No
Agliardi and Agliardi (2005)	Multiple	Risk-neutral	Call	Yes
This Paper	Multiple	Risk-neutral	Put/call	Yes

^a The seminal compound option paper series.

shutting down, abandon, switch, and/or growth (Trigeorgis, 1993, 1996). These options with different types can be evaluated by the SCOs.

The effect of revenue guarantee, for example, in a build-operate-transfer (BOT) project of utility construction can be evaluated by SCOs. A company signs the BOT contract with the government to build and operate the construction while related revenue belongs to the company during operating period. The guarantee promised by government ensures the company's minimum revenue. If the actual revenue is less than the minimum, the deficit is subsidized by the government. The company hence owns the operating revenue and the put option written by the government. The put option, with the guarantee amount as its strike price, can enhance the incentives for the BOT project. At the preparation period time prior to construction, the put option can be considered as a 2-fold compound option, call on put. The add-in call option, with the construction cost as its strike price, represents the right to participate in the construction and share the potential revenue.

Similarly, the revenue guarantee of the expansion can be regarded as a 3-fold SCO, call on call on put, at the preparation period. Assume the government will offer corresponding revenue guarantee for the expansion if there is an expansion right embedded in the BOT project. The revenue guarantee of the expansion can be viewed as another put option with its own guarantee amount as the strike price. At the main construction time, the put option can be considered as a 2-fold compound option, call on put. This add-in call option, with the expansion cost as its strike price, stands for the expansion right. At the preparation time, the right can be evaluated as a 3-fold SCO: call on call on put. The last add-in call option, with the proportional main construction cost as its strike price, represents the right to participate in the main construction. Note that the main construction cost is divided proportionally as the strike prices of both call options for the guarantee of main and expansion construction. The call on call, stacked on the put option, represents the sequential feature that the expansion right exists only when the main construction is executed. The SCOs discussed in this study make the evaluation of complex options possible.

The SCOs can also be applied to the existing real option applications, such as the competing technology adoption (Kauffman and Li, 2005), joint ventures behavior analysis (Kogut, 1991) and strategic project examination (Bowman and Moskowitz, 2001). Furthermore, the pricing of exotic financial derivatives, such as exotic chooser options and capletions, can also be accomplished using SCO methodology.

This paper is arranged as the follows. Section 2 presents the SCOs pricing formula. Section 3 presents some features of multivariate normal distributions, and derives some comparative statistics as its application. The paper ends with the conclusion.

2. The pricing formula for generalized sequential compound options

This section defines the notation and derives the pricing formula for multi-fold SCOs using the risk-neutral method. An SCO, composed of European options as building blocks, is the (compound) option on another compound option, where the feature of each folds can be assigned arbitrarily. Each fold option may be either call or put. This section begin by providing notation explanation and a fundamental theorem that express a k -variate normal integration in terms of $(k - 1)$ -variate integrations.

Denote the correlation matrix $\mathbf{Q}_k := [Q_{\{k\},g,h}]_{k \times k}$, where $Q_{\{k\},g,h}$ is the symmetric (g, h) entry of the matrix $\mathbf{Q}_k, \forall 1 \leq g \leq h \leq k$. Similarly, $d_{\{k\},g}$ is the g th entry of the vector $[d_{\{k\},g}]_{k \times 1}$. $([Q_{\{k\},g,h}]_{k \times k})^{(-i,-j)}$ is the $(k - 1)$ by $(k - 1)$ matrix which excludes the i th row and the j th column of $[Q_{\{k\},g,h}]_{k \times k}$. Define the function $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$. The k -variate normal integral with upper bound limit vector $[d_{\{k\},g}]_{k \times 1}$ and correlation matrix \mathbf{Q}_k is characterized as

$$\mathbf{N}_k \left\{ [d_{\{k\},g}]_{k \times 1}; \mathbf{Q}_k \right\} = \int_{-\infty}^{d_{\{k\},1}} \int_{-\infty}^{d_{\{k\},2}} \cdots \int_{-\infty}^{d_{\{k\},k}} \frac{1}{(2\pi)^{\frac{k}{2}} \sqrt{|\mathbf{Q}_k|}} e^{-\frac{1}{2} \mathbf{Z}' \mathbf{Q}_k^{-1} \mathbf{Z}} dz_k dz_{k-1} \cdots dz_1,$$

where $\mathbf{Z}' = [z_1, z_2, \dots, z_k]$ and $\mathbf{N}_0 \equiv 1$. The following theorem is the statement about the construction of multivariate normal integrals.

Theorem 1.

(a) *The relationship between the $(k - 1)$ and k -variate normal integrals (Curnow and Dunnett, 1962)*

$$\begin{aligned} & \forall 1 \leq v \leq k, \mathbf{N}_k \left\{ [d_{\{k\},g}]_{k \times 1}; \mathbf{Q}_k \right\} \\ &= \int_{-\infty}^{d_{\{k\},v}} f(z_v) \mathbf{N}_{k-1} \left\{ \left(\left[\frac{d_{\{k\},g} - Q_{\{k\},v,g} z_v}{\sqrt{1 - (Q_{\{k\},v,g})^2}} \right]_{k \times 1} \right)^{(-v)} ; \left(\left[\frac{Q_{\{k\},g,h} - Q_{\{k\},v,g} Q_{\{k\},v,h}}{\sqrt{1 - (Q_{\{k\},v,g})^2} \sqrt{1 - (Q_{\{k\},v,h})^2}} \right]_{k \times k} \right)^{(-v,-v)} \right\} dz_v \end{aligned}$$

(b) *The decomposition of a multivariate normal integral (Schroder, 1989)*

$$\begin{aligned} & \mathbf{N}_k \left\{ [d_{\{k\},g}]_{k \times 1}; \mathbf{Q}_k \right\} \\ &= \int_{-\infty}^{d_{\{k\},v}} \mathbf{N}_{v-1} \left\{ \left[\frac{d_{\{k\},g} - Q_{\{k\},g,v} z_v}{\sqrt{1 - Q_{\{k\},g,v}^2}} \right]_{(v-1) \times 1} ; \left[\frac{Q_{\{k\},g,h} - Q_{\{k\},g,v} Q_{\{k\},h,v}}{\sqrt{1 - Q_{\{k\},g,v}^2} \sqrt{1 - Q_{\{k\},h,v}^2}} \right]_{(v-1) \times (v-1)} \right\} \\ & \times \mathbf{N}_{k-v} \left\{ \left[\frac{d_{\{k\},v+g} - Q_{\{k\},v,v+g} z_v}{\sqrt{1 - Q_{\{k\},v,v+g}^2}} \right]_{(k-v) \times 1} ; \left[\frac{Q_{\{k\},v+g,v+h} - Q_{\{k\},v,v+g} Q_{\{k\},v,v+h}}{\sqrt{1 - Q_{\{k\},v,v+g}^2} \sqrt{1 - Q_{\{k\},v,v+h}^2}} \right]_{(k-v) \times (k-v)} \right\} f(z_v) dz_v \end{aligned}$$

where \mathbf{Q}_k is the correlation matrix, $\forall 1 \leq v \leq k$.

In Theorem 1, (a) reveals that the k -variate normal integral can be constructed from the $(k - 1)$ -variate by adding another dimension to the upper limit vector and correlation matrix. (b) states further that the specific multivariate normal integral can be partitioned into two integrals of lesser variates. This result can extend the current compound option methodology from 2-fold to multi-fold by induction, while Chen (2003) just “observes a pattern” to generalize the SCC. Before applying this theorem to sequential compound option pricing, more pieces of notation are introduced as follows.

Assume $T_{u-1} < T_u$, the time interval from T_{u-1} to T_u is $\tau_u, \forall u \geq 1$. Denote the asset price at time T_u as S_u . Assume the instantaneous variance of asset price, the interest rate and the dividend rate

at time t are given as deterministic $\sigma^2(t)$, $r(t)$ and $q(t)$, respectively. The dividend rate $q(u)$ can also be regarded as the depreciate rate (Remer et al., 2001).

Denote $\Psi_i(T_0)$ as the i -fold SCO price starting at time T_0 and expiring at time T_1 , with strike K_1 . Its underlying asset is the $(i-1)$ -fold SCO $\Psi_{i-1}(T_1)$, which is active from T_1 to T_2 . Under the assumption that the last fold SCO starts from T_0 , the underlying SCO with fold number $(i-u+1)$, $\Psi_{i-u+1}(T_{u-1})$, is valid from T_{u-1} to T_u with strike price K_u . The first fold option, $\Psi_1(T_{i-1})$, is a vanilla option with the asset as its underlying asset. It should be noted that fold numbers come in the reverse order.

The option feature $A_{u,u}$ represents the call or put attribute of the (underlying) SCO with fold number $(i-u+1)$ ranging from T_{u-1} to T_u , $\forall u \geq 1$. If the SCO of this fold is a call, $A_{u,u} = 1$; the feature $A_{u,u} = -1$ is for a put. For example, a call on a put (a 2-fold compound option) starting at T_0 has the option features $A_{1,1} = 1$ and $A_{2,2} = -1$. Denote $A_{h,g} = \prod_{u=g}^h A_{u,u}$, $\forall 1 \leq g \leq h$, and $A_{1,0} \equiv 1$. Fig. 1 shows the notation for an arbitrary i -fold SCO starting from T_0 .

Under the no arbitrage condition and the assumption that the asset price follows a geometric Brownian Motion in the perfect market, the succeeding theorem derives the pricing formula of an i -fold SCO with alternating arbitrarily calls and puts by the risk-neutral method. Although the SCOs presented in later sections can start at any time T_u , the SCO in this theorem is starting from T_0 without loss of generality. The symbol “ $\ast v$ ”, meaning “start from time T_v ”, is used to indicate time shift in the sensitivity derivation.

Theorem 2. Generalized sequential compound option pricing formula

Denote

$$(a) \quad a_{i,g,\ast v} \equiv a_{i,g,\ast v}(S_v) = \frac{\ln\left(\frac{S_v}{S_{\#v+g,v+i}}\right) + \int_{T_v}^{T_{v+g}} \left[r(u) - q(u) + \frac{1}{2}\sigma^2(u)\right] du}{\sqrt{\int_{T_v}^{T_{v+g}} \sigma^2(u) du}}, \forall g \geq 1$$

$$(b) \quad b_{i,g,\ast v} \equiv b_{i,g,\ast v}(S_v) = a_{i,g,\ast v}(S_v) - \sqrt{\int_{T_v}^{T_{v+g}} \sigma^2(u) du}, \forall g \geq 1$$

$$(c) \quad \tilde{\rho}_{g,h,\ast v} = \Lambda_{v+h-1,v+g} \rho_{g,h,\ast v}, \forall h > g \geq 1; \rho_{g,g,\ast v} = 1, \forall g; \rho_{g,h,\ast v} = \rho_{h,g,\ast v}, \forall h, g;$$

$$\rho_{g,h,\ast v} = \begin{cases} \sqrt{\frac{\int_{T_v}^{T_{v+g}} \sigma^2(u) du}{\int_{T_v}^{T_{v+h}} \sigma^2(u) du}}, & \forall l \leq g < h. \end{cases}$$

$$(d) \quad a_{i,g,\#v} \equiv a_{i,g,\ast v}(S_{\#v,i}); b_{i,g,\#v} \equiv b_{i,g,\ast v}(S_{\#v,i})$$

$$(e) \quad a_{i,g} \equiv a_{i,g,\ast 0}; b_{i,g} \equiv b_{i,g,\ast 0}; \rho_{i,g} \equiv \rho_{i,g,\ast 0}; \tilde{\rho}_{g,h} \equiv \tilde{\rho}_{g,h,\ast 0}$$

(f) Equivalent asset price (EAP) of the underlying

$$S_{\#g,i} = \begin{cases} K_i, & \text{for } g = i \\ \text{The asset price which makes } \Psi_{i-g}(T_g) = K_g, & \forall 1 \leq g < i \end{cases}$$

then

$$\Psi_i(T_0) = A_{i,1} e^{-\int_{T_0}^{T_i} q(u)du} S_0 \mathbf{N}_i \left\{ [A_{i,g} a_{i,g}]_{i \times 1}; [\tilde{\rho}_{g,h}]_{i \times i} \right\} - \sum_{j=1}^i A_{j,1} e^{-\int_{T_0}^{T_j} r(u)du} K_j \mathbf{N}_j \left\{ [A_{i,g} b_{i,g}]_{j \times 1}; [\tilde{\rho}_{g,h}]_{j \times j} \right\} \tag{1}$$

under the assumption that the EAP ($S_{g,i}$) exists, $\forall 1 \leq g \leq i$

Proof. see Appendix A. \square

Through the induction, the SCO price in Eq. (1) is derived according to the risk-neutral method, by which the physical probability is changed as the risk-neutral measure. Eq. (A.2) is the key of derivation. It means that the current asset price is the expectation of the future price with interest rate discount under the risk neutral probability measure. The above statement accords with public general intuition. The interest rate is a deterministic function, thus the discount factor $e^{-\int_{T_0}^{T_i} r(u)du}$ can be dealt ignoring the expectation operator. The other part, $\max[A_{1,1} \Psi_i(T_1) - A_{1,1} K_1]$, within the expectation is derived similarly as the cases of 1-fold and 2-fold options.

According to Eq. (1), the price of an i -fold SCO can be expressed as the weighted asset price minus the sum of weighted strike prices. The weights consist of three factors: the cumulative option features, the discount factor and the in-the-money probabilities. The cumulative option feature is obtained by synthesizing the option features from the current fold to the last fold. The discount factor is a deduction made due to interest rate or depreciation rate compounding. The in the money probabilities are assessed by multivariate normal integrals under different probability measures. The factors $a_{i,g}$ and $b_{i,g}$ in the integration are similar to the “ d_1 ” and “ d_2 ” appearing in conventional option pricing formulas. The correlation matrices of SCOs are similar to those of the sequential compound calls, except for a sign change due to the cumulative option features. Within these 3 weighting factors, the parameters of the last fold have the widely impact on the pricing formula.

The pricing formula of SCOs is more general than those of vanilla options, 2-fold compound options, and sequential compound calls, all of which can be regarded as special cases of SCOs. The main difference between SCOs and sequential compound calls lies in the freedom to alternate calls and puts, which is represented by a sign changes in the cumulative option features $A_{h,g}$, $\forall 1 \leq g \leq h$. In other words, the option prices will depend on the fold features $A_{h,g}$. Moreover, allowing the parameters to vary over time makes the integrated variance and discounting factors

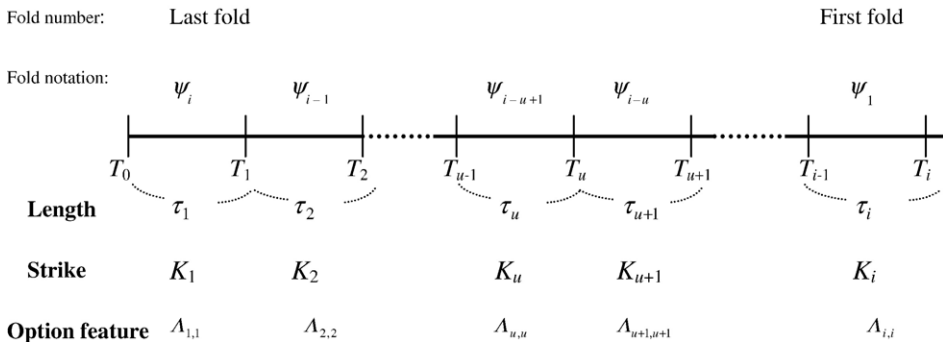


Fig. 1. The Notation of the i -fold generalized sequential compound option.

of an SCO quite different from the constant parameters in Thomassen and Van Wouwe (2001). Setting all $A_{h,g}$ to +1 in an SCO results in a SCC.

The arbitrary put/call alternation of SCOs causes the EAP existence issue. The existence of EAPs is crucial (Frey and Sommer, 1998) because the decision whether exercises the SCO or not is transformed as whether the asset price is greater (or lesser) than the EAP. Similar to the concept of implied volatility, the EAP can be regarded as the “implied asset price”, solving by the known (compound) option price (given as the strike price) and other conventional option parameters except the asset price itself. Thus there is no EAP concern in the 1-fold option computation and it is calculating only for the 2 or more fold compound options. The SCO price (Ψ_i) is monotone with respect to the asset price and hence the equivalent asset price (EAP, $S_{\#g,i}$) is unique if it exists. According to the sensitivity analysis in Theorem 4 (b), the Delta ($\frac{\partial \Psi_i(T_0)}{\partial S(T_0)}$) is a strictly monotone function. Its increasing or decreasing nature depends on the cumulative option feature ($A_{i,1}$). Therefore the EAP, defined as the asset price making the SCOs price equal to a specific strike price, is unique if it exists. For the SCC, its option price increases with respect to the asset price, thus there is no EAP existence concern. However, the EAP may not exist due to the range limitation of a decreasing SCO price. This is another difference between SCO and SCC. The following Lemma describes explicitly the sufficient conditions of the EAP existence.

Denote $\tilde{\Psi}_{i,2\&3}(T_0) = \sum_{j=1}^i A_{j,1} e^{-\int_{T_0}^{T_j} r(u)du} K_j N_j\{[A_{i,g} b_{i,g}]_{j \times 1}; [\hat{\rho}_{g,h}]_{j \times j}\}$, which is the second component of the SCO pricing formula in Eq. (1). Note that $\tilde{\Psi}_{i,2\&3}(T_0)$ may be negative or positive but all SCO prices $\Psi_i(T_0)$ are always nonnegative.

Lemma 1. *The sufficient condition for the existence of existence of equivalent asset price (EAP) Given g ($1 \leq g \leq i-1$), the $S_{\#g,i}$ exists if*

- (a) $S_{\#g,i}$ exists for all $g-1 \leq \ell \leq i-1$, and either the following condition stand.
- (b) $A_{i-g,1} = +1$;
- (c) $A_{i-g,1} = -1$ and $K_g \leq -\tilde{\Psi}_{i-g,2\&3}(T_g)$.

Proof. see Appendix B. □

The condition (a) of Lemma 1 reveals that the existence conditions is also derived based essentially on the induction, by which the multi-fold SCO price is available in Theorem 2. If the EAPs of previous folds exist, the EAP existence of the current fold is discussed according to the different sign of the cumulative option future $A_{i-g,1}$. The condition (c) states that the strike price of the current fold K_g is limited by a maximum because the asset price has opposite direction against the current fold SCO price. The opposite direction is represented by the negative cumulative option feature. For the case of positive cumulative option feature (condition (b)), there is no restriction for the strike price. The non-existing EAP will incur the zero SCO price.

3. The sensitivities of sequential compound options

This section derives some features of multivariate normal integrations and investigates the sensitivities of SCOs based on the derivation.

3.1. Some features of multivariate normal integration

This subsection presents the partial derivatives and separations of multivariate normal integrations. Thomassen and Van Wouwe (2002) use the partial derivatives of multivariate normal integrals, but only for the specific correlation matrices suitable for SCCs only. This study

generalizes their work, and is applicable to multivariate normal integrations with any kind of correlation matrix. This result will pave the way to the sensitivity analysis of SCOs.

Theorem 3. *Partial derivative of the multivariate normal integral*

Let $d_{\{k\},g}(G_1, G_2, \dots, G_p) \equiv d_{\{k\},g}$, representing a function of G_1, G_2, \dots, G_p .

$$\forall 1 \leq k, 1 \leq \ell \leq p, \frac{\partial \mathbf{N}_k([d_{\{k\},g}]_{k \times 1}; [Q_{\{k\},g,h}]_{k \times k})}{\partial G_\ell} = \sum_{j=1}^k f(d_{\{k\},j}) \frac{\partial d_{\{k\},j}}{\partial G_\ell} \mathbf{N}_{k-1} \left\{ \left(\left[\frac{d_{\{k\},g} - d_{\{k\},j} Q_{\{k\},j,g}}{\sqrt{1 - Q_{\{k\},j,g}^2}} \right]_{k \times 1} \right)^{(-j)} ; \left(\left[\frac{Q_{\{k\},g,h} - Q_{\{k\},j,g} Q_{\{k\},j,h}}{\sqrt{(1 - Q_{\{k\},j,g}^2)(1 - Q_{\{k\},j,h}^2)}} \right]_{k \times k} \right)^{(-j,-j)} \right\} \quad (2)$$

where $[Q_{\{k\},g,h}]_{k \times k}$ is a correlation matrix that is not a function of G_ℓ .

Proof. see Appendix C. \square

Theorem 3 shows that the partial derivatives of a $(k+1)$ -variate normal integration can be represented as the $k+1$ weighted sum of k -variate normal integrations. As Eq. (C.1) shows, the Leibnitz’s rule can be used to decompose the partial derivative into two parts. The first term is a k -variate normal integration with a weighting factor. The second part is an integration of a partial derivative of the $(k-1)$ -variate normal. Theorem 3 proves that this second part turns out has the same form as the first term. This means that Theorem 3 extends the Leibnitz’s rule to multivariate normal cases.

The specific partial derivatives presented in Thomassen and Van Wouwe (2002) can be viewed as a special case of Theorem 3. If the elements of the correlation matrix in Eq. (2) as specified as $Q_{\{k\},g,h} = Q_{\{k\},h,g} = \sqrt{\frac{\tau_g}{\tau_h}}$, for $1 \leq g \leq h$, then $\frac{Q_{\{k\},g,h} - Q_{\{k\},j,g} Q_{\{k\},j,h}}{\sqrt{(1 - Q_{\{k\},j,g}^2)(1 - Q_{\{k\},j,h}^2)}} = 0$, for $g < j < h$ or $h < j < g$.

Another feature of multivariate normal integrations will be presented after the following notation has been defined. Let

$$\begin{aligned} \mathfrak{N}_{v-1,b} &\equiv \mathbf{N}_{v-1} \left\{ \left[\frac{A_{i,g} b_{i,g} - A_{i,v} b_{i,v} \tilde{\rho}_{g,v}}{\sqrt{1 - (\tilde{\rho}_{g,v})^2}} \right]_{(v-1) \times 1} ; \left[\frac{\tilde{\rho}_{g,h} - \tilde{\rho}_{v,g} \tilde{\rho}_{v,h}}{\sqrt{[1 - (\tilde{\rho}_{v,g})^2][1 - (\tilde{\rho}_{v,h})^2]}} \right]_{(v-1) \times (v-1)} \right\}, \\ \mathfrak{N}_{j-1,b,-v} &\equiv \mathfrak{N}_{j-1,b,-v}(b_{i,g}) \\ &= \mathbf{N}_{j-1} \left\{ \left(\left[\frac{A_{i,g} b_{i,g} - A_{i,v} b_{i,v} \tilde{\rho}_{v,g}}{\sqrt{1 - (\tilde{\rho}_{v,g})^2}} \right]_{i \times 1} \right)^{(-v)} ; \left(\left[\frac{\tilde{\rho}_{g,h} - \tilde{\rho}_{v,g} \tilde{\rho}_{v,h}}{\sqrt{[1 - (\tilde{\rho}_{v,g})^2][1 - (\tilde{\rho}_{v,h})^2]}} \right]_{i \times 1} \right)^{(-v,-v)} \right\}, \\ \mathfrak{N}_{i-1,a,-v} &\equiv \mathfrak{N}_{i-1,a,-v}(a_{i,g}) \end{aligned}$$

Lemma 2 shows that the multivariate integrals for SCO sensitivities can be factored into two separated normal integrals.

Lemma 2.

$$\begin{aligned} (a) \quad \mathfrak{N}_{i-1,a,-v} &= \mathfrak{N}_{v-1,b} \times \mathbf{N}_{i-v} \left\{ [A_{i,v+g} a_{i,g,\#v}]_{(i-v) \times 1} ; [\tilde{\rho}_{g,h,*v}]_{(i-v) \times (i-v)} \right\} \\ (b) \quad \mathfrak{N}_{j-1,b,-v} &= \mathfrak{N}_{v-1,b} \times \mathbf{N}_{j-v} \left\{ [A_{i,v+g} b_{i,g,\#v}]_{(j-v) \times 1} ; [\tilde{\rho}_{g,h,*v}]_{(j-v) \times (j-v)} \right\} \end{aligned}$$

Proof. See Appendix D. \square

Note that the same factor $\mathfrak{N}_{v-1,b}$ appears on the right-hand side of (a) & (b).

3.2. The sensitivity analysis of SCOs

The sensitivity analysis of SCOs is now possible thanks to the two results demonstrated in the preceding subsection. Thomassen and Van Wouwe (2002) derived the sensitivities of SCCs and Theorem 4 extends their analysis to SCOs with the possibility of alternating calls and puts arbitrarily based on Theorem 3. Theorem 4 also shows the interest rate sensitivity under the special case of interest rate fold-wise.

Theorem 4. *Sensitivities of SCOs*

$$(a) \text{ Delta: } \frac{\partial \Psi_i(T_0)}{\partial S_0} = A_{i,1} e^{-\int_{T_0}^{T_i} q(u) du} \mathbf{N}_i \left\{ [A_{i,g} a_{i,g}]_{i \times 1}; [\tilde{\rho}_{g,h}]_{i \times i} \right\}$$

$$(b) \text{ Gamma: } \frac{\partial^2 \Psi_i(T_0)}{\partial S_0^2} = \sum_{v=1}^i \frac{A_{v-1,1} e^{-\frac{1}{2}a_{i,v}^2 - \int_{T_0}^{T_i} q(u) du}}{S_0 \sqrt{2\pi \int_{T_0}^{T_v} \sigma^2(u) du}} \mathbf{n}_{i-1,a,-v}$$

(c) *Let the interest rate and the variance of asset price be fold-wise constant. In other words, $r(t) = r_w, \sigma(t) = \sigma_w \forall T_{u-1} \leq t < T_u, 1 \leq u \leq i$. Under this simplification, the “underscore” labels are added to the corresponding pieces of notation. The SCO price, the correlation matrix and the two upper limit vectors are denoted as $\underline{\Psi}_i, \underline{\tilde{\rho}}_{g,h}, \underline{a}_{i,g}$ and $\underline{b}_{i,g}$, respectively. Thus, the interest rate sensitivity Rho is: $\forall 1 \leq \ell \leq i$,*

$$\frac{\partial \underline{\Psi}_i(T_0)}{\partial r_\ell} = r_\ell \sum_{j=\ell}^i A_{j,1} K_j e^{-\sum_{u=1}^j r_u \tau_u} \mathbf{N}_j \left\{ [A_{i,g} \underline{b}_{i,g}]_{j \times 1}; [\underline{\tilde{\rho}}_{g,h}]_{j \times j} \right\}.$$

Proof. See Appendix E. □

As SCOs pricing formulas (Theorem 2) generalize previous results such as vanilla options , 2-fold compound options and SCCs, the SCOs sensitivities given in Theorem 4 are also extension of these previous works intuitively. Again, the sequence of option features will affect the signs of the sensitivities. According to Theorem 4 (a), the value of a SCO is monotonic with respect to the current asset price $S(T_0)$, hence the EAP is unique if it exists.

4. Conclusion

The present study defines and derives the pricing formula of sequential compound options (SCOs), where the parameters vary over time and each fold option may have different put/call attribute. The SCO price can be evaluated by a linear combination of the asset and strike prices weighted by different variate normal integrations. The risk-neutral method enriches the SCOs pricing formula derivation with more financial implications than P.D.E. method. The partial derivative of a multivariate normal integration is derived in this paper as an extension of Leibnitz’s Rule, and is used to derive the sensitivities of SCOs. Previous results

have analyzed 2-fold puts/calls-alternating compound options or multi-fold “sequential compound calls” where all options are of call-type. Fold-wise differences are rarely taken into consideration.

The SCOs presented in this paper have the following qualities. First of all, multi-fold SCOs enable arbitrary option feature (call/put) assignments, greatly enhancing the range of practical applications that can be treated by compound option theory. Second, in real-world problems option parameters often vary over time; SCOs enabling time-dependent parameters (interest rate, depression rate and variance of the asset price) can capture the “sequential” features. Third, SCOs can accommodate an arbitrary number of folds.

Furthermore, SCOs can be used to demonstrate some features of multivariate normal integrals, such as their partial derivatives. The Leibnitz’s rule can be used to decompose the partial differential of $(k+1)$ -variate integration into two parts: a k -variate normal integration and an integration with the integrand of a partial derivative. This paper proves that, under the multivariate normal cases, these two parts can be presented in a unified form. Based on the result, sensitivities of SCOs to asset price (and its change) and interest rate (under the case of interest rate fold-wise) are derived.

SCOs generalize the methodology of European Options (Black and Scholes, 1973), 2-fold compound options (Geske, 1977, 1979) and sequential compound calls (Thomassen and Van Wouwe, 2001; Agliardi and Agliardi, 2005), and can be regarded intuitively as multi-dimensional options extending from their work. Moreover, the sensitivities of SCOs can also be expressed explicitly as generalized versions of those of their works. The generalized parameters presented in this study regard the parameters as deterministic time-dependent functions. This kind of parameter setting considers the constant or fold-wise constant situations as their special cases and allows the SCOs more flexible. However, the case of stochastic interest rate for compound options should under an unreasonable and unacceptable condition (Frey and Sommer, 1998). Thus the SCOs are not extended to stochastic cases for realistic consideration.

For the advantages of using 2fold compound options as financial instruments (Bhattcaharya, 2005), such as split-fee, decision postponement and risk management, SCOs can do better. SCOs buyers pay a few premiums at the initial time and own the privilege to pay again while they exercise to gain the next fold SCOs. The SCOs will be discarded while they are not worth holding in sacrificing previous payment. This split-fee property let the SCOs owners to pay proportionally according to available information at that time, instead of sinking option premium at the beginning. Thus the decision-making can be postponed under indefinite environments and more flexibility is offered to SCOs holders. The feature of SCOs with high profit potential under constrained cost can provide greater leverage and yield enhancement for SCOs owners. SCOs can also be tailored for financial institutions as risk management instruments, such as hedging or mortgage pipeline risk.

SCOs can enhance and broaden the use of compound option theory in real option and financial derivative fields. Real options often incorporate multiple options of different types with sophisticated interactions, but such situations can be evaluated by aggregating various SCOs. Some complex options can be regarded as exotic SCOs and can applied the similar derivation in this study to get explicit pricing formulas. Even milestone projects, which must decide whether or not a project has terminated according to the milestone achievement, can be evaluated through the use of SCOs. Compared with the constant variance and interest rate of the SCC assumed in Casimon et al. (2004), allowing parameters to vary with different periods makes this method of project valuation more precise and flexible. Finally, a number of complex financial derivatives can be developed or evaluated using SCOs in the same way that chooser options and capletions can be priced by 2-fold

compound options. These applications of SCOs with real-world cases will be the subject of probable future study.

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Appendix A. Proof of Theorem 2

This theorem is proved by induction. When $i=1$, $\Psi_1(T_0)$ with $A_{1,1}=1$ and $A_{1,1}=-1$ are the vanilla call and put formulas respectively. When $i=2$, $\Psi_2(T_0)$ is the 2-fold compound option, such as call on call ($A_{1,1}=1, A_{2,2}=1$), put on call ($A_{1,1}=-1, A_{2,2}=1$), call on put ($A_{1,1}=1, A_{2,2}=-1$), and put on put ($A_{1,1}=-1, A_{2,2}=-1$). These generalized 2-fold cases can be extended easily from Chen (2002) and Lajeri-Chaherli (2002).

Assuming that Eq. (1) is true for the i -fold compound option $\Psi_i(T_0)$, it will be shown that Eq. (1) is also true for the $(i+1)$ -fold compound option, for any $A_{g,g}, 1 \leq g \leq i+1$.

Because the underlying asset of $\Psi_{i+1}(T_0)$ is $\Psi_i(T_1)$, instead of $\Psi_i(T_0)$, the start time of the i -fold compound option is shifted from T_0 to T_1 . All pieces of notation for the i -fold compound option are changed simultaneously according to this time shift. (In other words, $v=1$).

$$\begin{aligned} \text{Hence } \Psi_i(T_1) = & A_{i+1,2} e^{-\int_{T_1}^{T_{i+1}} q(u)du} S_1 \mathbf{N}_i \left\{ \left[A_{i+1,g+1} a_{i,g,*1} \right]_{i \times 1}; \left[\tilde{\rho}_{g,h,*1} \right]_{i \times i} \right\} \\ & - \sum_{j=1}^i A_{j+1,2} e^{-\int_{T_1}^{T_{j+1}} r(u)du} K_{j+1} \mathbf{N}_j \left\{ \left[A_{i+1,g+1} b_{i,g,*1} \right]_{j \times 1}; \left[\tilde{\rho}_{g,h,*1} \right]_{j \times j} \right\} \end{aligned} \tag{A.1}$$

At T_1 , the maturity time of the $i+1$ -fold compound option, the option price can be expressed as $\Psi_{i+1}(T_1) = \max[A_{1,1} \Psi_i(T_1) - A_{1,1} K_1]$. At its starting time T_0 , the option price is given by

$$\Psi_{i+1}(T_0) = \tilde{E} \left\{ e^{-\int_{T_0}^{T_1} r(u)du} \max[A_{1,1} \Psi_i(T_1) - A_{1,1} K_1] | F_0 \right\}, \tag{A.2}$$

according to the fundamental theory of asset pricing (Baxter and Runie, 1996). \tilde{E} is the expectation operator under the risk-neutral measure, and F_0 denotes the information available at time T_0 from the asset price.

Under the assumption that the asset price follows a geometric Brownian motion, it can be expressed as

$$S_1 = S_0 e^{\int_{T_0}^{T_1} [r(u) - q(u) - \frac{1}{2}\sigma^2(u)] du + z \sqrt{\int_{T_0}^{T_1} \sigma^2(u) du}} \tag{A.3}$$

where z is a standard normal random number $z \sim N(0,1)$, with density function f . $\Psi_{i+1}(T_0)$ is a function of S_θ and hence a function of z . Thus the SCO price can be represented as

$$\Psi_{i+1}(T_0) = e^{-\int_{T_0}^{T_1} r(u)du} \int_{-\infty}^{\infty} \max[A_{1,1}\Psi_i(T_1) - A_{1,1}K_1]f(z)dz.$$

Assume that $S_{\#1,i+1}$ is the equivalent asset price which makes $\Psi_i(T_1) - K_1 = 0$. The condition “ $S_1 = S_{\#1,i+1}$ ” is then equivalent to “ $z = -b_{i+1,1}$ ”, where $b_{i+1,1} = \frac{\ln\left(\frac{S_0}{S_{\#1,i+1}}\right) + \int_{T_0}^{T_1} (r(u) - q(u) - \frac{1}{2}\sigma^2(u))du}{\sqrt{\int_{T_0}^{T_1} \sigma^2(u)du}}$

Because the integration range is either $[-\infty, -b_{i+1,1}]$ or $[-b_{i+1,1}, \infty]$, depending on $A_{i+1,1}$ (the sign of S_1), the compound option can be expressed in the unified form

$$\Psi_{i+1}(T_0) = e^{-\int_{T_0}^{T_1} r(u)du} A_{i+1,1} \int_{-b_{i+1,1}}^{A_{i+1,1}\infty} \{A_{1,1}\Psi_i(T_1) - A_{1,1}K_1\}f(z)dz.$$

Substituting Eq. (A.1) into the previous equation, it can be obtained that

$$\begin{aligned} \Psi_{i+1}(T_0) &= e^{-\int_{T_0}^{T_1} r(u)du} A_{i+1,1} A_{i+1,1} \int_{-b_{i+1,1}}^{A_{i+1,1}\infty} e^{-\int_{T_1}^{T_{i+1}} q(u)du} S_1 \mathbf{N}_i \left\{ \left[A_{i+1,g+1} a_{i,g,*1} \right]_{i \times 1}; \left[\tilde{\rho}_{g,h,*1} \right]_{i \times i} \right\} f(z) dz \\ &\quad - e^{-\int_{T_0}^{T_1} r(u)du} A_{i+1,1} \sum_{j=1}^i A_{j+1,1} \int_{-b_{i+1,1}}^{A_{i+1,1}\infty} e^{-\int_{T_1}^{T_{j+1}} r(u)du} K_{j+1} \mathbf{N}_j \left\{ \left[A_{i+1,g+1} b_{i,g,*1} \right]_{j \times 1}; \left[\tilde{\rho}_{g,h,*1} \right]_{j \times j} \right\} f(z) dz \\ &\quad - e^{-\int_{T_0}^{T_1} r(u)du} A_{i+1,1} A_{1,1} \int_{-b_{i+1,1}}^{A_{i+1,1}\infty} K_1 f(z) dz \\ &\equiv \tilde{\Psi}_{i+1,1} - \tilde{\Psi}_{i+1,2} - \tilde{\Psi}_{i+1,3}. \end{aligned}$$

The following paragraphs derives $\tilde{\Psi}_{i+1,1}$, $\tilde{\Psi}_{i+1,2}$ and $\tilde{\Psi}_{i+1,3}$ explicitly. By Eq. (A.3), S_1 can be substituted by the representation of S_0 and thus $\tilde{\Psi}_{i+1,1} = A_{i+1,1} e^{-\int_{T_0}^{T_{i+1}} q(u)du} S_0 A_{i+1,1} \times \int_{-b_{i+1,1}}^{A_{i+1,1}\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(z - \sqrt{\int_{T_0}^{T_1} \sigma^2(u)du}\right)^2} \mathbf{N}_i \left\{ \left[A_{i+1,g+1} \tilde{a}_{i,g,*1} \right]_{i \times 1}; \left[\tilde{\rho}_{g,h,*1} \right]_{i \times i} \right\} dz$, where $\tilde{a}_{i,g,*1} =$

$$\frac{\ln\left(\frac{S_0}{S_{g+1,i+1}}\right) + \int_{T_0}^{T_{g+1}} [r(u) - q(u) + \frac{1}{2}\sigma^2(u)]du + z\sqrt{\int_{T_0}^{T_1} \sigma^2(u)du} - \int_{T_0}^{T_1} \sigma^2(u)du}{\sqrt{\int_{T_1}^{T_{g+1}} \sigma^2(u)du}}, \forall 1 \leq g \leq i.$$

Let $z_2 = z - \sqrt{\int_{T_0}^{T_1} \sigma^2(u)du}$, so that

the above equation can be written as $\tilde{\Psi}_{i+1,1} = A_{i+1,1} e^{-\int_{T_0}^{T_{i+1}} q(u)du} S_0 A_{i+1,1} \int_{-a_{i+1,1}}^{A_{i+1,1}\infty} \frac{1}{\sqrt{2\pi}} \times e^{-\frac{1}{2}z_2^2} \mathbf{N}_i \left\{ \left[A_{i+1,g+1} \bar{a}_{i,g,*1} \right]_{i \times 1}; \left[\tilde{\rho}_{g,h,*1} \right]_{i \times i} \right\} dz_2$, where $\bar{a}_{i,g,*1} = \frac{a_{i+1,g+1} + z_2 \rho_{1,g+1}}{\sqrt{1 - \rho_{1,g+1}^2}}, \forall 1 \leq g \leq i$.

Then denote $z_3 = -A_{i+1,1} z_2$, hence

$$\begin{aligned} \tilde{\Psi}_{i+1,1} &= A_{i+1,1} e^{-\int_{T_0}^{T_{i+1}} q(u)du} S_0 \int_{-\infty}^{-A_{i+1,1} a_{i+1,1}} \frac{e^{-\frac{1}{2}z_3^2}}{\sqrt{2\pi}} \mathbf{N}_i \left\{ \left[\frac{A_{i+1,g+1} a_{i+1,g+1} - A_{g,1} \tilde{\rho}_{1,g+1} z_3}{\sqrt{1 - (A_{g,1} \tilde{\rho}_{1,g+1})^2}} \right]_{i \times 1}; \left[\tilde{\rho}_{g,h,*1} \right]_{i \times i} \right\} dz_3 \\ &= A_{i+1,1} e^{-\int_{T_0}^{T_{i+1}} q(u)du} S_0 \mathbf{N}_{i+1} \left\{ \left[A_{i+1,g} a_{i+1,g} \right]_{(i+1) \times 1}; \left[H_{0,g,h} \right]_{(i+1) \times (i+1)} \right\} \end{aligned}$$

The last equation is obtained by Theorem 1 (a). The following derivation will demonstrate that $[H_{0,g,h}]_{(i+1) \times (i+1)} = [\tilde{\rho}_{g,h}]_{(i+1) \times (i+1)}$.

According to Theorem 1 (a), $H_{0,1,1}=1$; $H_{0,1,g}=A_{h-1,1}\rho_{1,h}, \forall 2 \leq g \leq i+1$; $H_{0,g,h}=H_{0,h,g}$; and $H_{0,g,g}=1, \forall 2 \leq g \leq i+1$. Thus $\forall 2 \leq g < h \leq i+1$,

$$H_{0,g,h} = A_{g-1,1}\rho_{1,g}A_{h-1,1}\rho_{1,h} + \sqrt{1 - (A_{g-1,1}\rho_{1,g})^2} \sqrt{1 - (A_{h-1,1}\rho_{1,h})^2} \rho_{g-1,h-1,*1}$$

$$= A_{h-1,g} \frac{\sqrt{\int_{T_0}^{T_g} \sigma^2(u)du}}{\sqrt{\int_{T_0}^{T_h} \sigma^2(u)du}} = A_{h-1,g}\rho_{g,h} = \tilde{\rho}_{g,h}.$$

According to the above statements, $[H_{0,g,h}]_{(i+1) \times (i+1)} = [\tilde{\rho}_{g,h}]_{(i+1) \times (i+1)}$ and hence

$$\tilde{\Psi}_{i+1,1} = A_{i+1,1} e^{-\int_{T_0}^{T_{i+1}} r(u)du} S_0 \mathbf{N}_{i+1} \{ [A_{i+1,g} a_{i+1,g}]_{(i+1) \times 1}; [\tilde{\rho}_{g,h}]_{(i+1) \times (i+1)} \}.$$

By a similar method, $\tilde{\Psi}_{i+1,2}$ and $\tilde{\Psi}_{i+1,3}$ can be derived:

$$\tilde{\Psi}_{i+1,2} = \sum_{j=2}^{i+1} A_{j,1} e^{-\int_{T_0}^{T_j} r(u)du} K_j \mathbf{N}_j \{ [A_{i+1,g} b_{i+1,g}]_{j \times 1}; [\tilde{\rho}_{g,h}]_{j \times j} \}.$$

$$\tilde{\Psi}_{i+1,3} = A_{1,1} e^{-\int_{T_0}^{T_1} r(u)du} K_1 \mathbf{N}_1 \{ A_{i+1,1} b_{i+1,1} \}.$$

Eq. (1) is true for any $i+1$ -fold compound option, provided it is true for the i -fold compound option. Consequently, Theorem 2 is proved. \square

Appendix B. Proof of Lemma 1

According to Theorem 2 (f), the $S_{\#g,i}$ will exist only when the EAPs of the previous folds ($S_{\#g,i}, g-1 \leq \ell \leq i-1$) exist. Thus the condition (a) holds. According to Theorem 4 (a), the option price $\tilde{\Psi}_{i-g}(T_g)$ is strict monotone and its sign is decided by $A_{i-g,1}$. Hence it is discussed as the cases of $A_{i-g,1} = +1$ (condition (b)) and $A_{i-g,1} = -1$ (condition (c)), respectively. For condition (b), $\Psi_{i-g}(T_g)$ has the same sign with the asset price and thus can range from zero to infinity to fit any nonnegative K_g . For condition (c), $\Psi_{i-g}(T_g)$ has the opposite sign with the asset price, then $\Psi_{i-g}(T_g)$ will reach the maximum $-\tilde{\Psi}_{i-g,2\&3}(T_g)$ while the asset price is zero. Therefore the strike price K_g can NOT exceed the maximum in order to keep $S_{\#g,i}$ exist. \square

Appendix C. Proof of Theorem 3

The theorem is proved by induction. For $k=1, \frac{\partial \mathbf{N}_1 \{ d_{\{1\},1} \}}{\partial G_\ell} = f(d_{\{1\},1}) \mathbf{N}_0$. The theorem thus stands for $k=1$.

By the result of $k=1$ and Leibnitz's rule, it is obtained that $\frac{\partial \mathbf{N}_2 \{ d_{\{2\},1}, d_{\{2\},2}; [Q_{\{2\},g,h}]_{2 \times 2} \}}{\partial G_\ell} = -f(d_{\{2\},1}) \times \frac{\partial (-d_{\{2\},1})}{\partial G_\ell} \mathbf{N}_1 \left\{ \frac{d_{\{2\},2} - Q_{\{2\},1,2} d_{\{2\},1}}{\sqrt{1 - Q_{\{2\},1,2}^2}} \right\} + \tilde{\mathbf{N}}_{2,1}$, where $\tilde{\mathbf{N}}_{2,1} \equiv \int_{-d_{\{2\},1}}^{\infty} f(z) \frac{1}{\sqrt{2\pi}} \left\{ \exp \left[-\frac{1}{2} \left(\frac{d_{\{2\},2} + Q_{\{2\},1,2} z}{\sqrt{1 - (Q_{\{2\},1,2)}^2}} \right)^2 \right] \right\} \frac{\partial \left(\frac{d_{\{2\},2} + Q_{\{2\},1,2} z}{\sqrt{1 - Q_{\{2\},1,2}^2}} \right)}{\partial G} dz$.

Denote $z_4 := \frac{z + d_{\{2\},2} Q_{\{2\},1,2}}{\sqrt{1 - Q_{\{2\},1,2}^2}}$. Thus $\tilde{\mathbf{N}}_{2,1}$ can be rewritten as

$$\tilde{\mathbf{N}}_{2,1} = f(d_{\{2\},2}) \frac{\partial d_{\{2\},2}}{\partial G_\ell} \int_{\frac{-d_{\{2\},1} + Q_{\{2\},1,2} d_{\{2\},2}}{\sqrt{1 - Q_{\{2\},1,2}^2}}}^{\infty} f(z_4) dz_4 = f(d_{\{2\},2}) \frac{\partial d_{\{2\},2}}{\partial G_\ell} \mathbf{N}_1 \left\{ \frac{d_{\{2\},1} - Q_{\{2\},1,2} d_{\{2\},2}}{\sqrt{1 - Q_{\{2\},1,2}^2}} \right\}$$

$$\frac{\partial \mathbf{N}_2(d_{\{2\},1}, d_{\{2\},2}; [\mathcal{Q}_{\{2\},g,h}]_{2 \times 2})}{\partial G_\ell} = \sum_{j=1}^2 f(d_{\{2\},j}) \frac{\partial d_{\{2\},j}}{\partial G_\ell} \mathbf{N}_1 \left\{ \left(\left[\frac{d_{\{2\},g} - d_{\{2\},j} \mathcal{Q}_{\{2\},j,g}}{\sqrt{1 - \mathcal{Q}_{\{2\},j,g}^2}} \right]_{2 \times 1} \right)^{(-j,-j)} \right\}$$

Hence Eq. (2) stands for $k=2$.

Assuming that Eq. (2) is true for k , the following proves that it is also true for $k+1$. By Leibnitz’s rule,

$$\begin{aligned} & \frac{\partial \mathbf{N}_{k+1} \{ [d_{\{k+1\},g}]_{k+1 \times 1}; [\mathcal{Q}_{\{k+1\},g,h}]_{(k+1) \times (k+1)} \}}{\partial G_\ell} \\ &= f(d_{\{k+1\},1}) \frac{\partial d_{\{k+1\},1}}{\partial G_\ell} \\ & \times \mathbf{N}_k \left\{ \left[\frac{d_{\{k+1\},g+1} - \mathcal{Q}_{\{k+1\},1,g+1} d_{\{k+1\},1}}{\sqrt{1 - \mathcal{Q}_{\{k+1\},1,g+1}^2}} \right]_{k \times 1}; \left(\left[\frac{\mathcal{Q}_{\{k+1\},g,h} - \mathcal{Q}_{\{k+1\},1,g} \mathcal{Q}_{\{k+1\},1,h}}{\sqrt{(1 - \mathcal{Q}_{\{k+1\},1,g}^2)(1 - \mathcal{Q}_{\{k+1\},1,h}^2)}} \right]_{(k+1) \times (k+1)} \right)^{(-1,-1)} \right\} + \tilde{\mathbf{N}}_{k+1,1} \end{aligned} \tag{C.1}$$

$$\tilde{\mathbf{N}}_{k+1,1} \equiv \int_{-d_{\{k+1\},1}}^{\infty} f(z) \frac{\partial}{\partial G_\ell} \left(\mathbf{N}_k \left\{ \left[\frac{d_{\{k+1\},g+1} + \mathcal{Q}_{\{k+1\},1,g+1} z}{\sqrt{1 - \mathcal{Q}_{\{k+1\},1,g+1}^2}} \right]_{k \times 1}; \left[\frac{\mathcal{Q}_{\{k+1\},g+1,h+1} - \mathcal{Q}_{\{k+1\},1,g+1} \mathcal{Q}_{\{k+1\},1,h+1}}{\sqrt{(1 - \mathcal{Q}_{\{k+1\},1,g+1}^2)(1 - \mathcal{Q}_{\{k+1\},1,h+1}^2)}} \right]_{k \times k} \right\} \right) dz$$

Using the corresponding result for $\frac{\partial \mathbf{N}_k([d_{\{k\},g}]_{k \times 1}; [\mathcal{Q}_{\{k\},g,h}]_{k \times k})}{\partial G_\ell}$, by substituting $\frac{d_{\{k+1\},g+1} + \mathcal{Q}_{\{k+1\},1,g+1} z}{\sqrt{1 - \mathcal{Q}_{\{k+1\},1,g+1}^2}}$ and $\frac{\mathcal{Q}_{\{k+1\},g+1,h+1} - \mathcal{Q}_{\{k+1\},1,g+1} \mathcal{Q}_{\{k+1\},1,h+1}}{\sqrt{(1 - \mathcal{Q}_{\{k+1\},1,g+1}^2)(1 - \mathcal{Q}_{\{k+1\},1,h+1}^2)}}$ as $d_{k,g}$, $\mathcal{Q}_{k,g,h}$ in Eq. (2) respectively and setting $Z_{k+1,j+1} = z + \frac{d_{\{k+1\},j+1} \mathcal{Q}_{\{k+1\},1,j+1}}{\sqrt{1 - \mathcal{Q}_{\{k+1\},1,j+1}^2}}$, $\tilde{\mathbf{N}}_{k+1,1}$ can be derived as

$$\begin{aligned} \tilde{\mathbf{N}}_{k+1,1} &= \sum_{j=1}^k f(d_{\{k+1\},j+1}) \frac{\partial d_{\{k+1\},j+1}}{\partial G_\ell} \\ & \times \int_{\frac{-d_{\{k+1\},1} + d_{\{k+1\},j+1} \mathcal{Q}_{\{k+1\},1,j+1}}{\sqrt{1 - \mathcal{Q}_{\{k+1\},1,j+1}^2}}}^{\infty} f(Z_{k+1,j+1}) \mathbf{N}_{k-1} \{ \tilde{\mathbf{H}}_1; \tilde{\mathbf{H}}_2 \} dZ_{k+1,j+1} \end{aligned} \tag{C.2}$$

The numerator and the denominator of $\tilde{\mathbf{H}}_1$ are multiplied by $\frac{\sqrt{1 - \mathcal{Q}_{\{k+1\},1,g+1}^2}}{\sqrt{1 - \mathcal{Q}_{\{k+1\},j+1,g+1}^2}}$ in order to match the format of Theorem 1. Therefore

$$\tilde{\mathbf{H}}_1 = \left(\left[\frac{d_{\{k+1\},g+1} - d_{\{k+1\},j+1} \mathcal{Q}_{\{k+1\},j+1,g+1}}{\sqrt{1 - \mathcal{Q}_{\{k+1\},j+1,g+1}^2}} + Z_{k+1,j+1} H_{1,g}} \right]_{k \times 1} \right)^{(-j)} \quad \text{and} \quad \tilde{\mathbf{H}}_2 = \left([H_{2,g,h}]_{k \times k} \right)^{(-j,-j)},$$

where $H_{1,g} \equiv \frac{Q_{\{k+1\},1,g+1} - Q_{\{k+1\},1,j+1}Q_{\{k+1\},j+1,g+1}}{\sqrt{(1 - Q_{\{k+1\},1,j+1}^2)(1 - Q_{\{k+1\},j+1,g+1}^2)}}$, $\forall 1 \leq g \leq k$;

$$H_{2,g,h} = \frac{\frac{Q_{\{k+1\},g+1,h+1} - Q_{\{k+1\},1,g+1}Q_{\{k+1\},1,h+1}}{\sqrt{(1 - Q_{\{k+1\},1,g+1}^2)(1 - Q_{\{k+1\},1,h+1}^2)}} - \frac{Q_{\{k+1\},j+1,g+1} - Q_{\{k+1\},1,j+1}Q_{\{k+1\},1,g+1}}{\sqrt{(1 - Q_{\{k+1\},1,j+1}^2)(1 - Q_{\{k+1\},1,g+1}^2)}} \frac{Q_{\{k+1\},j+1,h+1} - Q_{\{k+1\},1,j+1}Q_{\{k+1\},1,h+1}}{\sqrt{(1 - Q_{\{k+1\},1,j+1}^2)(1 - Q_{\{k+1\},1,h+1}^2)}}}{\sqrt{\left[1 - \left(\frac{Q_{\{k+1\},j+1,g+1} - Q_{\{k+1\},1,j+1}Q_{\{k+1\},1,g+1}}{\sqrt{(1 - Q_{\{k+1\},1,j+1}^2)(1 - Q_{\{k+1\},1,g+1}^2)}}\right)^2\right] \left[1 - \left(\frac{Q_{\{k+1\},j+1,h+1} - Q_{\{k+1\},1,j+1}Q_{\{k+1\},1,h+1}}{\sqrt{(1 - Q_{\{k+1\},1,j+1}^2)(1 - Q_{\{k+1\},1,h+1}^2)}}\right)^2\right]}}$$

$\forall 1 \leq g, h \leq k$.

The integration of $\tilde{N}_{k+1,1}$ can be performed by applying Theorem 1. Hence,

$$\tilde{N}_{k+1,1} = \sum_{j=1}^k f(d_{\{k+1\},j+1}) \frac{\partial d_{\{k+1\},j+1}}{\partial G_{\ell}} \mathbf{N}_k \left\{ \left(\left[\frac{d_{\{k+1\},g} - d_{\{k+1\},j+1}Q_{\{k+1\},j+1,g}}{\sqrt{1 - Q_{\{k+1\},j+1,g}^2}} \right]_{(k+1) \times 1} \right)^{(-j-1,1)} ; \tilde{\mathbf{H}}_3 \right\} \tag{C.3}$$

where $\tilde{\mathbf{H}}_3 = \begin{bmatrix} 1 & \tilde{\mathbf{H}}_4^T \\ \tilde{\mathbf{H}}_4 & \tilde{\mathbf{H}}_5 \end{bmatrix}$, $\tilde{\mathbf{H}}_4 = \left(\left[\frac{Q_{\{k+1\},1,g+1} - Q_{\{k+1\},1,j+1}Q_{\{k+1\},j+1,g+1}}{\sqrt{(1 - Q_{\{k+1\},1,j+1}^2)(1 - Q_{\{k+1\},j+1,g+1}^2)}} \right]_{k \times 1} \right)^{(-j)}$,

By Theorem 1, $\tilde{\mathbf{H}}_3$ and $\tilde{\mathbf{H}}_5$ are symmetric with diagonal elements equal to 1. For $1 \leq g < h$, $H_{5,g,h} = H_{1,g}H_{1,h} + \sqrt{(1 - H_{1,g}^2)(1 - H_{1,h}^2)}H_{2,g,h}$. Thus

$$\tilde{\mathbf{H}}_5 = \left(\left[\frac{Q_{\{k+1\},g+1,h+1} - Q_{\{k+1\},j+1,g+1}Q_{\{k+1\},j+1,h+1}}{\sqrt{(1 - Q_{\{k+1\},j+1,g+1}^2)(1 - Q_{\{k+1\},j+1,h+1}^2)}} \right]_{k \times k} \right)^{(-j,-j)} \text{ and}$$

$$\tilde{\mathbf{H}}_3 = \left(\left[\frac{Q_{\{k+1\},g,h} - Q_{\{k+1\},j+1,g}Q_{\{k+1\},j+1,h}}{\sqrt{(1 - Q_{\{k+1\},j+1,g}^2)(1 - Q_{\{k+1\},j+1,h}^2)}} \right]_{(k+1) \times (k+1)} \right)^{(-j-1,-j-1)}$$

Substitute $\tilde{\mathbf{H}}_3$ into Eq. (C.3) and change the index j to obtain

$$\tilde{N}_{k+1,1} = \sum_{j=2}^{k+1} \frac{1}{\sqrt{2\pi}} f(d_{\{k+1\},j}) \frac{\partial d_{\{k+1\},j}}{\partial G_{\ell}} \times \mathbf{N}_k \left\{ \left(\left[\frac{d_{\{k+1\},g} - d_{\{k+1\},j}Q_{\{k+1\},j,g}}{\sqrt{1 - Q_{\{k+1\},j,g}^2}} \right]_{(k+1) \times 1} \right)^{(-j)} ; \left(\left[\frac{Q_{\{k+1\},g,h} - Q_{\{k+1\},j,g}Q_{\{k+1\},j,h}}{\sqrt{(1 - Q_{\{k+1\},j,g}^2)(1 - Q_{\{k+1\},j,h}^2)}} \right]_{(k+1) \times (k+1)} \right)^{(-j,-j)} \right\} \tag{C.4}$$

Substituting the above result into Eq. (C.1), the consequence is obtained: $\frac{\partial \mathbf{N}_{k+1} \{ [d_{\{k+1\},g}]_{(k+1) \times 1}; [Q_{\{k+1\},g,h}]_{(k+1) \times (k+1)} \}}{\partial G_{\ell}}$. \square

Appendix D. Sketch Proof of Lemma 2

The left-hand sides of Theorem 2 (a) and (b) are identical, hence the on right-hand sides are also the same. Lemma 2 can be proved according to the above result. Lemma 2 can also be proved

directly through a multivariate normal integration whose correlation matrix can be partitioned into “four quadrants”. The top-right and the bottom-left quadrants are zero matrices, so the integrals can be represented as the product of two uncorrelated normal integrals (Bickel and Doksum, 2001, Theorem B.6.4). □

Appendix E. Proof of Theorem 4

For part (a), $\frac{\partial \Psi_i(T_0)}{\partial S_0} = A_{i,1} e^{-\int_{T_0}^{T_i} q(u)du} \mathbf{N}_i \left\{ [A_{i,g} a_{i,g}]_{i \times i}; [\tilde{\rho}_{g,h}]_{i \times i} \right\} + \tilde{\Psi}_{\partial S,1} - \tilde{\Psi}_{\partial S,2}$, where $\tilde{\Psi}_{\partial S,1} \equiv A_{i,1} e^{-\int_{T_0}^{T_i} q(u)du} S_0 \sum_{v=1}^i \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} a_{i,v}^2} \left(A_{i,v} \frac{\partial a_{i,v}}{\partial S_0} \right) \mathbf{s}_{i-1,a,-v}$,

$$\tilde{\Psi}_{\partial S,2} \equiv \sum_{j=1}^i A_{j,1} e^{-\int_{T_0}^{T_j} r(u)du} K_j \sum_{v=1}^j \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} b_{i,v}^2} \left(A_{i,v} \frac{\partial b_{i,v}}{\partial S_0} \right) \mathbf{s}_{j-1,b,-v}$$

The sequential paragraphs demonstrate $\tilde{\Psi}_{\partial S,1} - \tilde{\Psi}_{\partial S,2} = 0$.

By definition,

$$e^{-\frac{1}{2} a_{i,v}^2} = \frac{S_{\#v,i}}{S_0} e^{-\frac{1}{2} b_{i,v}^2 - \int_{T_0}^{T_v} r(u)du}, \forall 1 \leq v \leq i. \tag{E.1}$$

The $\tilde{\Psi}_{\partial S,3}$ is denoted as $\tilde{\Psi}_{\partial S,3} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} b_{i,v}^2} \left(A_{i,v} \frac{\partial b_{i,v}}{\partial S_0} \right) \mathbf{s}_{v-1,b}$ for convenience.

According to Lemma 2, Eq. (E.1) and the fact that $\frac{\partial a_{i,v}}{\partial S_0} = \frac{\partial b_{i,v}}{\partial S_0}$, $\tilde{\Psi}_{\partial S,1}$ can be reformulated as

$$\tilde{\Psi}_{\partial S,1} = \sum_{v=1}^i \tilde{\Psi}_{\partial S,3} \left(\left[\mathbf{1}_{\{v < i\}} e^{-\int_{T_v}^{T_i} q(u)du} + \mathbf{1}_{\{v=i\}} \right] A_{i,1} S_{\#v,i} \mathbf{N}_{i-v} \left\{ [A_{i,v+g} a_{i,g,\#v}]_{(i-v) \times 1}; [\tilde{\rho}_{g,h,*v}]_{(i-v) \times (i-v)} \right\} \right).$$

$$\tilde{\Psi}_{\partial S,2} = \sum_{j=1}^i A_{j,1} e^{-\int_{T_0}^{T_j} r(u)du} K_j \sum_{v=1}^j \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} b_{i,v}^2} \left(A_{i,v} \frac{\partial b_{i,v}}{\partial S_0} \right) \mathbf{s}_{v-1,b}$$

$$\times \mathbf{N}_{j-v} \left\{ [A_{i,v+g} b_{i,g,\#v}]_{(j-v) \times 1}; [\tilde{\rho}_{g,h,*v}]_{(j-v) \times (j-v)} \right\}$$

$$= \sum_{v=1}^i \tilde{\Psi}_{\partial S,3} \left(A_{v,1} K_v + \mathbf{1}_{\{v < i\}} \sum_{j=1}^{i-v} e^{-\int_{T_v}^{T_{v+j}} r(u)du} \right.$$

$$\left. \times A_{v+j,1} K_{v+j} \mathbf{N}_j \left\{ [A_{i,v+g} b_{i,g,\#v}]_{(j-v) \times 1}; [\tilde{\rho}_{g,h,*v}]_{(j-v) \times (j-v)} \right\} \right)$$

The last equality is obtained by interchange of the two summations.

$$\tilde{\Psi}_{\partial S,1} - \tilde{\Psi}_{\partial S,2} = \tilde{\Psi}_{\partial S,3} (A_{i,1} S_{\#i,i} N_0 - A_{i,1} K_i) + \sum_{v=1}^{i-1} \tilde{\Psi}_{\partial S,3} (A_{v,1} \tilde{\Psi}_{\partial S,4} - A_{v,1} K_v),$$

where $\tilde{\Psi}_{\partial S,4} = e^{-\int_{T_v}^{T_i} q(u)du} A_{i,v+1} S_{\#v,i} \mathbf{N}_{i-v} \left\{ [A_{i,v+g} a_{i,g,\#v}]_{(i-v) \times 1}; [\tilde{\rho}_{g,h,*v}]_{(i-v) \times (i-v)} \right\} - \sum_{j=1}^{i-v} e^{-\int_{T_v}^{T_{v+j}} r(u)du} A_{v+j,v+1} K_{v+j} \mathbf{N}_j \left\{ [A_{i,v+g} b_{i,g,\#v}]_{(j-v) \times 1}; [\tilde{\rho}_{g,h,*v}]_{(j-v) \times (j-v)} \right\}$.

By definitions, $S_{\#i,i} = K_i$, hence $A_{i,1} S_{\#i,i} N_0 - A_{i,1} K_i = 0$. $\tilde{\Psi}_{\partial S,4}$ is the $(i-v)$ -fold compound option price with start time T_v (instead of T_0). In other words, $\tilde{\Psi}_{\partial S,4} = \Psi_{i-v}(T_v)$ with initial asset price $S_{\#v,1}$. Thus, by definitions, $\tilde{\Psi}_{\partial S,4} = K_v$, and $\tilde{\Psi}_{\partial S,1} - \tilde{\Psi}_{\partial S,2} = 0$. Part (b) and (c) can be proved by similar method to part (a). □

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