行政院國家科學委員會專題研究計畫 期末報告

正規勞倫級數之度量非齊次丟番圖逼近

計 畫 類 別 ： 個別型
計 畫 編 號 ： NSC 101-2115-M-009-010-
執 行 期 間 ： 101 年 08 月 01 日至 102 年 07 月 31 日
執 行 單 位 ： 國立交通大學應用數學系（所）

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報 告 附 件 ： 出席國際會議研究心得報告及發表論文

公 開 資 訊 ： 本計畫可公開查詢

中 華 民 國 102 年 08 月 27 日

中 文 摘 要 ： 在近期的一篇論文中，Kim 和 Nakada 證明了一個在有限體之下，正規勞倫級數之非齊次丟番圖逼近的 Kurzweil 定理。在這個計畫中，我們證明了一個 simultaneous 丟番圖逼近的推廣。此外，我們證明了其他歸納 Kurzweil 定理的結果。

中文關鍵詞： 非齊次丟番圖逼近、正規勞倫級數、Kurzweil 定理、強大數法則。

英 文 摘 要 ： This project was concerned with generalizations of Kim and Nakada ‘s recent analogue of Kurzweil ‘s theorem in the field of formal Laurent series. Kim and Nakada ‘s proof used continued fraction expansion which made a generalization to simultaneous Diophantine approximation complicated. We proposed a new approach which works for all dimensions. Moreover, we also considered other extensions of Kurzweil ‘s theorem in dimension one.

英文關鍵詞： Inhomogeneous Diophantine approximation, formal Laurent series, Kurzweil ‘s theorem, strong laws of large numbers.

Project: Metric Inhomogeneous Diophantine Approximation for Formal Laurent Series

by

Michael Fuchs

Abstract

This project was concerned with generalizations of Kim and Nakada's recent analogue of Kurzweil's theorem in the field of formal Laurent series. Kim and Nakada's proof used continued fraction expansion which made a generalization to simultaneous Diophantine approximation complicated. We proposed a new approach which works for all dimensions. Moreover, we also considered other extensions of Kurzweil's theorem in dimension one.

1 General

This is the final report on the National Science Council project "Metric inhomogeneous Diophantine approximation for formal Laurent series" with grant number NSC-101-2115-M-009-010- and running time from August 1st, 2012 to July 31st, 2013.

Before going into details, we shortly summarize the main achievements.

- The paper [1] contains the main findings of this project (a preprint is attached).
- Generalizations of Kurzweil's theorem in dimension one are work in progress.
- Two graduate students have worked on this project (one of them graduate in June 2012; the other is expected to graduate in June 2014).

2 Results

We start by fixing some notations. First, let $\mathbb{F}_q$ denote the finite field of q elements ($q = p^t, t \in \mathbb{N}, p \in \mathbb{P}$). Moreover, set

$$\mathbb{F}((T^{-1})) = \left\{ f = \sum_{-\infty}^{n_0} a_n T^n \ : \ a_{n_0} \neq 0, a_n \in \mathbb{F}_q \right\} \cup \{0\}$$

which with addition and multiplication defined similar as for polynomials is a field, the so-called *field of formal Laurent series*. We will also use the notation

$$\{f\} = a_{-1}T^{-1} + a_{-2}T^{-2} + \cdots, \qquad f = a_{n_0}T^{n_0} + \cdots \in \mathbb{F}_q((T^{-1})), a_{n_0} \neq 0.$$

Now, equip the field of formal Laurent series with a norm as follows: $|f| = q^{n_0}$ for $f = a_{n_0}T^{n_0} + \cdots, a_{n_0} \neq 0$ and $|0| = 0$. Moreover, set

$$\mathbb{L} = \{f \in \mathbb{F}_q((T^{-1})) \ : \ |f| < 1\}.$$

Note that the norm restricted to $\mathbb{L}$ gives a compact topological group. Thus, there exists a unique, translation-invariant probability measure denoted by m.

We consider on the above probability space the Diophantine approximation problem

$$|\{Qf\} - g| < \frac{1}{q^{n+l_n}}, \quad \deg(Q) = n, \tag{1}$$

where $f, g \in \mathbb{L}, Q \in \mathbb{F}_q[T]$ and l_n is a fixed sequence of positive integers. We are interested in the number of solutions in Q for a randomly chosen g (note that the corresponding cases where g if fixed and f random and both f, g random have been discussed in [2]; see also [6] for higher-dimensional generalizations).

It follows immediately from the lemma of Borel-Cantelli that (1) has finitely many solutions for almost all g whenever $\sum_n q^{-l_n}$ converges. Moreover, it is well-known, that the number of solutions of (1) obeys a 0-1 law, i.e., the number is either finite or infinite for almost all g. However, the divergence of $\sum_n q^{-l_n}$ does not necessarily imply that there are infinite many solutions for almost all g. Thus, it is interesting to consider the following set

$$\mathcal{W} = \left\{ f \in \mathbb{L} \ : \ \forall l_n \text{ with } \sum_n q^{-l_n} = \infty, \text{ (1) has infinitely many solutions for almost all } g \right\}.$$

Kim and Nakada gave in [3] the following surprisingly easy characterization of $\mathcal{W}$ (their result is an analogue of Kurzweil's theorem [7] from the real number case).

Theorem 1 (Kim and Nakada)**.** *We have,*

$$\mathcal{W} = \{f \in \mathbb{L} \ : \ f \text{ is badly approximable}\}.$$

Kim and Nakada's proof of the above result made use of continued fraction expansion and thus is not easily extended to simultaneous Diophantine approximation.

In [1], we gave a new proof of Kim and Nakada's result which allowed us in addition to generalize their result to higher dimension. In order to describe our result, we have to fix some more notation. Let $r, s \in \mathbb{N}$. Consider vectors $\mathbf{f} = (f_1, \ldots, f_r) \in \mathbb{F}_q((T^{-1}))^r$ and let

$$\{\mathbf{f}\} = (\{f_1\}, \ldots, \{f_r\}).$$

Moreover, define a norm as $\|\mathbf{f}\| = \max_{1 \le i \le r} |f_i|$.

Consider now the following extension of (1)

$$\|\{\mathbf{q}A\} - \mathbf{g}\| < \frac{1}{q^{\lfloor nr/s \rfloor + l_n}}, \qquad \deg \mathbf{q} = n, \tag{2}$$

where $A \in \mathbb{L}^{r \times s}, \mathbf{g} \in \mathbb{L}^s, \mathbf{q} \in \mathbb{F}_q[T]^r$ and l_n is a again a fixed sequence of positive integers (here $\deg \mathbf{q} = \max_{1 \le i \le r} \deg q_i$ with $\mathbf{q} = (q_1, \ldots, q_r)$). As before, we are interested in the number of solutions of (2) in $\mathbf{q}$ for almost all $\mathbf{g}$ (note that $r = s = 1$ is the situation from above).

In [1], we proved that (2) has either finitely many or infinitely many solutions in $\mathbf{q}$ for almost all $\mathbf{g}$. A simple application of the lemma of Borel-Cantelli shows that the number of solutions is finite whenever $\sum_n q^{-sl_n}$ is convergent. Therefore, it is again interesting to consider the following set

$$\mathcal{W}_{r,s} = \left\{ A \in \mathbb{L}^{r \times s} \ : \ \forall l_n \text{ with } \sum_n q^{-sl_n} = \infty, \text{ (2) has infinitely many solutions for almost all } \mathbf{g} \right\}.$$

Our main result from [1] is the following theorem.

Theorem 2. *We have,*

$$\mathcal{W} = \{A \in \mathbb{L}^{r\times s} \ : \ A \text{ is badly approximable}\}.$$

Apart from the above generalization, other generalizations of Kim and Nakada's result have been considered as well. For example, in [5], the author considered the following extension of (1)

$$|\{Qf\} - g| < \frac{1}{q^{l_n}}, \quad \deg(Q) = n. \tag{3}$$

Set

$$\Omega(\tau_n) = \left\{ f \in \mathbb{L} \ : \ \exists c \in \mathbb{N} \text{ such that for all } Q \in \mathbb{F}_q[T], \text{ we have } |\{Qf\}| \geq \frac{1}{q^{c+\tau_n}} \right\},$$

where τ_n is a sequence of positive integers. Then, in [5], they proved the following result.

Theorem 3 (Kim, Tan, Wang, Xu)**.** *Let* $\rho \geq 1$. *Then,*

$$\left\{ f \in \mathbb{L} \ : \ \forall l_n \text{ with } \sum_n q^{n-\rho l_n} = \infty, \text{ (3) has infinitely many solutions for almost all } g \right\}$$
$$= \Omega(\lfloor \rho n \rfloor).$$

Note that for $\rho = 1$ the latter result is Kurzweil's theorem. The main question is now, whether the above theorem can be generalized to arbitrary sequence τ_n (with possible mild restrictions). This is work in progress with one of my graduate students.

3 Conclusion

In this project, we gave a new proof of a recent result of Kim and Nakada. Moreover, our proof could be used to extend Kim and and Nakada's result to simultaneous Diophantine approximation. Other extensions of Kim and Nakada's result were considered as well.

Another promising direction of future research comes from a recent paper of Kim, Nakada and Natsui [4]. In this paper, they gave a sufficient and necessary condition of (1) having an infinite number of solutions for almost all g for all l_n which are non-decreasing. A similar result for Diophantine approximation in the real number field has not been proved yet. Does such a result hold? Moreover, is it possible to give a similar result when the monotonicity restriction on l_n is removed? Can the result be generalized to higher dimension? All this might be further directions of future research.

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A Higher-dimensional Kurzweil Theorem for Formal Laurent Series over Finite Fields

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Abstract

In a recent paper, Kim and Nakada proved an analogue of Kurzweil's theorem for inhomogeneous Diophantine approximation of formal Laurent series over finite fields. Their proof used continued fraction theory and thus cannot be easily extended to simultaneous Diophantine approximation. In this note, we give another proof which works for simultaneous Diophantine approximation as well.

1 Introduction and Result

We start by fixing some notation which we are going to use throughout this work. First, let $\mathbb{F}_q$ denote the finite field with q elements. Moreover, denote by $\mathbb{F}_q[T]$ the polynomial ring and by

$$\mathbb{F}_q((T^{-1})) = \{f = a_nT^n + a_{n-1}T^{n-1} + \cdots \ : \ a_i \in \mathbb{F}_q, n \in \mathbb{Z}\}$$

the field of formal Laurent series.

For a formal Laurent series $f = a_nT^n + a_{n-1}T^{n-1} + \cdots$, we define its fractional part $\{f\}$ by

$$\{f\} = a_{-1}T^{-1} + a_{-2}T^{-2} + \cdots .$$

and its valuation by $|f| = q^{\deg f}$, where $\deg f$ is the generalized degree function. It is straightforward to prove that $|\cdot|$ satisfies the ultra-metric property, i.e., $|f-g| \leq \max\{|f|, |g\}$ for all $f, g \in \mathbb{F}_q((T^{-1}))$ with equality whenever $|f| \neq |g|$. This property implies that balls, which we denote by

$$B(f; q^{-d}) = \{g \in \mathbb{F}_q((T^{-1})) \ : \ |g - f| < q^{-d}\},$$

are either disjoint or contained in each other.

Next, let

$$\mathbb{L} = \{f \in \mathbb{F}_q((T^{-1})) \ : \ |f| < 1\}.$$

Restricting the valuation to this set gives a compact topological group. Hence, there exists a unique, translation-invariant probability measure (the Haar measure) which we are going to denote by m.

*Partially supported by NSC under the grant NSC 101-2115-M-009-010

In several recent papers, the following inhomogeneous Diophantine approximation problem was investigated: for $f, g \in \mathbb{L}$ consider

$$|\{Qf\} - g| < \frac{1}{q^{n+l_n}}, Q \in \mathbb{F}_q[T], \deg Q = n, \tag{1}$$

where l_n is a sequence of non-negative integers. One is interested in the number of solutions in Q of (1). Three situations have been studied: (D) f and g are both random; (S1) g is fixed; f is random; (S2) f is fixed; g is random. The first case is called the *double-metric* case and the other two cases are called *single-metric* cases.

We are going to recall some previous results concerning the number of solutions of (1). First, in all three cases, it follows immediately from the Borel-Cantelli lemma that the number of solutions is finite almost surely whenever $\sum_{n\geq 0} q^{-l_n}$ converges. Moreover, in the double-metric case and the single-metric case (S1) it was proved by Ma and Su [8] and Fuchs [3] that divergence of the latter series entails that the number of solutions is infinite almost surely. Interestingly, the same result does not hold for the single-metric case (S2). More precisely, for some functions f, the number of solutions remains finite almost surely even for sequences l_n for which $\sum_{n\geq 0} q^{-l_n} = \infty$. This then raises to question of characterizing those f where the convergence or divergence of $\sum_{n\geq 0} q^{-l_n}$ determines whether the number of solutions is finite or infinite almost surely.

To this end, we define the following set

$$W = \{f \in \mathbb{L} \ : \ \forall\, l_n \text{ with} \sum_{n=0}^{\infty} q^{-l_n} = \infty, \text{(1) has infinitely many solutions for almost all } g\}.$$

A characterization of this set was given in a recent paper by Kim and Nakada [5], their result being an analogue of Kurzweil's theorem from the real case. In order to state the result, we need a notation: $f \in \mathbb{L}$ is called *badly approximable* if there exists a $c \in \mathbb{N}$ such that for all $Q \in \mathbb{F}_q[T]$ with $\deg Q = n$, we have

$$|\{Qf\}| \geq \frac{1}{q^{n+c}}.$$

Then, Kim and Nakada proved the following result.

Theorem 1 (Kim and Nakada)**.** *We have,*

$$W = \{f \in \mathbb{L} \ : \ f \text{ is badly approximable}\}.$$

As for the proof of the above result, Kim and Nakada used continued fraction theory. Hence, their proof is not easily extended to simultaneous Diophantine approximation. It is the purpose of this note to give another proof which works for simultaneous Diophantine approximation as well. Our new approach combines ideas of Kurzweil's original proof [7] and Kim and Nakada's approach from [5] (for a more recent proof of Kurzweil's theorem see Fayad [2]).

In order to state our result, we need further notation. Therefore, fix non-negative integers r and s. Then, we denote by $\mathbb{F}_q[T]^r$ the r-th fold Cartesian product of $\mathbb{F}_q[T]$ and by $\mathbb{F}_q((T^{-1}))^r$ the r-th dimensional vector space over $\mathbb{F}_q((T^{-1}))$. Throughout this work, vectors will always be row vectors and will be denoted by bold, lower-case letters.

Let $\mathbf{f} = (f_1, \ldots, f_r) \in \mathbb{F}_q((T^{-1}))^r$ be a vector. Then, we define its fractional part by

$$\{\mathbf{f}\} = (\{f_1\}, \ldots, \{f_r\})$$

and its valuation $\|\mathbf{f}\| = q^{\deg \mathbf{f}} = \max_{1\leq i\leq r} |f_i|$, where $\deg \mathbf{f} = \max_{1\leq i\leq r} \deg f_i$. Note that $\|\cdot\|$ again satisfies the ultra-metric property and balls

$$B(\mathbf{f}; q^{-d}) = \{\mathbf{g} \in \mathbb{F}_q((T^{-1}))^r \ : \ \|\mathbf{g} - \mathbf{f}\| < q^{-d}\}$$

are again either disjoint or contained in each other.

Finally, we let $\mathbb{L}^r$ denote the r-th fold Cartesian product of $\mathbb{L}$ which we equip with the product measure of $\mathbb{L}$ (also denoted by m). Note that due to Tychonov's theorem, $\mathbb{L}^r$ is again a compact topological group and hence the product measure is the unique Haar measure.

Now, we consider the following extension of (1): for $A \in \mathbb{L}^{r\times s}$ and $\mathbf{g} \in \mathbb{L}^s$ consider

$$\Vert\{\mathbf{q}A\} - \mathbf{g}\Vert < \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor + l_n}},\ \mathbf{q} \in \mathbb{F}_q[T]^r,\ \deg \mathbf{q} = n, \tag{2}$$

where l_n is a sequences of non-negative integers. Again, one has three cases: (D) A and $\mathbf{g}$ are both random; (S1) $\mathbf{g}$ is fixed and A is random; (S2) A is fixed and $\mathbf{g}$ is random.

In this note, we are interested in case (S2). We mention in passing that similar results as in the one-dimensional case have been proved for the double-metric case and the single-metric case (S1) by Kristensen in [6]. So, the only case which has not been studied yet is (S2). In this case, we again have from the Borel-Cantelli lemma that if $\sum_{n\geq 0} q^{-l_n s}$ is convergent, then the number of solutions of (2) is finite almost surely. As for the other direction, we again define the set

$$W_{r,s} = \{A \in \mathbb{L}^{r\times s}\ :\ \forall\, l_n \text{ with} \sum_{n=0}^{\infty} q^{-l_n s} = \infty, \text{(2) has infinitely many solutions for almost all } \mathbf{g}\}.$$

We need the following notation: $A \in \mathbb{L}^{r\times s}$ is called *badly approximable* if there exists a $c \in \mathbb{N}$ such that for all $\mathbf{q} \in \mathbb{F}_q[T]^r$ with $\deg \mathbf{q} = n$, we have

$$\Vert\{\mathbf{q}A\}\Vert \geq \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor + c}}. \tag{3}$$

Then, our main result is the following extension of Theorem 1.

Theorem 2. *We have,*

$$W_{r,s} = \{A \in \mathbb{L}^{r\times s}\ :\ A \text{ is badly approximable}\}.$$

The structure of the paper is as follows: in the next section, we will collect a couple of results which are needed in the proof of Theorem 2. The proof of Theorem 2 is then presented in Section 3.

2 Some Preliminaries

Throughout this section, let $A \in \mathbb{L}^{r\times s}$ with

$$A = \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,s} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ f_{r,1} & f_{r,2} & \cdots & f_{r,s} \end{pmatrix}.$$

We first recall the higher-dimensional version of Dirichlet's theorem.

Theorem 3. *The following diophantine inequality*

$$\Vert\{\mathbf{q}A\}\Vert < \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor}}, \mathbf{q} \in \mathbb{F}_q[T]^r, \deg \mathbf{q} = n$$

has infinitely many solutions.

Proof. This is proved as in the real case. ▮

Next, we need the the following result.

Lemma 1. *If $A\mathbf{u}^\intercal \in \mathbb{F}_q[T]^r$ for some $\mathbf{u} \in \mathbb{F}_q[T]^s$ with $\mathbf{u} \neq \mathbf{0}$, then A is not badly approximable.*

Proof. Let $\mathbf{u} = (U_1, \dots, U_s)$ with $U_j \in \mathbb{F}_q[T]$ and assume w.l.o.g. that $U_s \neq 0$. From the assumption, we obtain that

$$A\mathbf{u}^\intercal = (V_1, \dots, V_r)^\intercal$$

with $V_i = \sum_{j=1}^{s} f_{i,j} U_j \in \mathbb{F}_q[T]$.

Next, denote by A' the matrix A with the last column removed. Then, by Dirichlet's theorem,

$$\|\{\mathbf{q}A'\}\| < q^{-\lfloor \frac{nr}{s-1} \rfloor}, \mathbf{q} \in \mathbb{F}_q[T]^r, \deg \mathbf{q} = n$$

has infinitely many solutions. The latter is equivalent to

$$\begin{aligned}
&|Q_1 f_{1,1} + Q_2 f_{2,1} + \cdots + Q_r f_{r,1} - P_1| < q^{-\lfloor \frac{nr}{s-1} \rfloor}, \\
&|Q_1 f_{1,2} + Q_2 f_{2,2} + \cdots + Q_r f_{r,2} - P_2| < q^{-\lfloor \frac{nr}{s-1} \rfloor}, \\
&\qquad\qquad\qquad\vdots \\
&|Q_1 f_{1,s-1} + Q_2 f_{2,s-1} + \cdots + Q_r f_{r,s-1} - P_{s-1}| < q^{-\lfloor \frac{nr}{s-1} \rfloor}
\end{aligned}$$

has infinitely many solutions in $Q_1, \dots, Q_r, P_1, \dots, P_{s-1}$ with $\max_{1 \le i \le r} \deg Q_i = n$. Multiplying by U_s and setting $Q_i' = U_s Q_i, 1 \le i \le r$ and $P_j' = U_s P_j, 1 \le j \le s-1$ implies that

$$\begin{aligned}
&|Q_1' f_{1,1} + Q_2' f_{2,1} + \cdots + Q_r' f_{r,1} - P_1'| < q^{-\lfloor \frac{n'r}{s-1} \rfloor - c_1}, \\
&|Q_1' f_{1,2} + Q_2' f_{2,2} + \cdots + Q_r' f_{r,2} - P_2'| < q^{-\lfloor \frac{n'r}{s-1} \rfloor - c_1}, \\
&\qquad\qquad\qquad\vdots \\
&|Q_1' f_{1,s-1} + Q_2' f_{2,s-1} + \cdots + Q_r' f_{r,s-1} - P_{s-1}'| < q^{-\lfloor \frac{n'r}{s-1} \rfloor - c_1}
\end{aligned} \tag{4}$$

has infinitely many solutions, where $\max_{1 \le i \le r} \deg Q_i' = n'$ and c_1 is a suitable constant.

Now, fix a solution of the latter system and observe that

$$\begin{aligned}
&U_s Q_1' f_{1,s} + U_s Q_2' f_{2,s} + \cdots + U_s Q_r' f_{r,s} \\
&\qquad = \sum_{i=1}^{r} (V_i - U_1 f_{i,1} - \cdots - U_{s-1} f_{i,s-1}) Q_i' \\
&\qquad = \sum_{i=1}^{r} V_i Q_i' - \sum_{j=1}^{s-1} U_j (Q_1' f_{1,j} + \cdots + Q_r' f_{r,j} - P_j') - \sum_{j=1}^{s-1} U_j P_j'.
\end{aligned}$$

Rearranging yields

$$U_s \sum_{i=1}^{r} Q_i' f_{i,s} + \sum_{j=1}^{s-1} U_j P_j' - \sum_{i=1}^{r} V_i Q_i' = -\sum_{j=1}^{s-1} U_j (Q_1' f_{1,j} + \cdots + Q_r' f_{r,j} - P_j').$$

Hence,

$$\left| U_s \sum_{i=1}^{r} Q_i' f_{i,s} + \sum_{j=1}^{s-1} U_j P_j' - \sum_{i=1}^{r} V_i Q_i' \right| \le \max_{1 \le j \le s-1} |U_j| |Q_1' f_{1,j} + \cdots + Q_r' f_{r,j} - P_j'| < q^{-\lfloor \frac{n'r}{s-1} \rfloor - c_2},$$

where the last line follows from (4) and c_2 is a suitable constant. Dividing both sides by $|U_s|$ gives

$$\left|\sum_{i=1}^{r} Q'_i f_{i,s} + \frac{\sum_{j=1}^{s-1} U_j P'_j - \sum_{i=1}^{r} V_i Q'_i}{U_s}\right| < q^{-\lfloor \frac{n'r}{s-1} \rfloor - c_3},$$

where c_3 is a suitable constant. Note that $U_s | Q'_i, 1 \le i \le r$ and $U_s | P'_j, 1 \le j \le s-1$ and hence

$$R = \frac{\sum_{j=1}^{s-1} U_j P'_j - \sum_{i=1}^{r} V_i Q'_i}{U_s}$$

is a polynomial. Overall, we have proved that

$$|Q'_1 f_{1,s} + \cdots + Q'_r f_{r,s} + R| < q^{-\lfloor \frac{n'r}{s-1} \rfloor - c_3}.$$

So, we can add this equation to (4) and the resulting system still has infinitely many solutions. This in turn yields that if we set $\mathbf{q}' = (Q'_1, \ldots, Q'_r)$ and $c_4 = \min\{c_1, c_3\}$, then

$$\|\{\mathbf{q}'A\}\| < q^{-\lfloor \frac{n'r}{s-1} \rfloor - c_4}, \mathbf{q}' \in \mathbb{F}_q[T]^r, \deg \mathbf{q}' = n' \tag{5}$$

has infinitely many solutions.

The latter, however, implies that A is not badly approximable because otherwise (3) would hold which clearly contradicts (5). Hence, the proof is finished. ▮

Remark 1. In the real case, a matrix A is badly approximable if and only if $A^{\intercal}$ is badly approximable (see Theorem VIII in [1]). If the same is true for formal Laurent series as well (which we expect), then Lemma 1 would follow from this as a simple consequence.

For the final two results of this section, assume that A is badly approximable, i.e., (3) holds.

Lemma 2. *The set* $\{\{\mathbf{q}A\} : \mathbf{q} \in \mathbb{F}_q[T]^r\}$ *is dense in* $\mathbb{L}^s$.

Proof. Fix $n \in \mathbb{N}$ and $\mathbf{g} = (g_1, \ldots, g_s) \in \mathbb{L}^s$ with

$$g_j = g_1^{(j)} T^{-1} + g_2^{(j)} T^{-2} + \cdots .$$

We have to show that there exists a $\mathbf{q} \in \mathbb{F}_q[T]^r$ with

$$\|\{\mathbf{q}A\} - \mathbf{g}\| < q^{-n}. \tag{6}$$

In order to do so, we reformulate (6) as a solvability problem for a system of linear equations. Therefore, let $\mathbf{q} = (Q_1, \ldots, Q_r)$ with

$$Q_i = a_0^{(i)} + a_1^{(i)} T + \cdots + a_N^{(i)} T^N$$

and for $1 \le i \le r$ and $1 \le j \le s$

$$f_{i,j} = f_1^{(i,j)} T^{-1} + f_2^{(i,j)} T^{-2} + \cdots .$$

Moreover,

$$\mathbf{u}_i = \begin{pmatrix} a_0^{(i)} \\ a_1^{(i)} \\ \vdots \\ a_N^{(i)} \end{pmatrix}^{\intercal}, \quad A_{i,j} = \begin{pmatrix} f_1^{(i,j)} & f_2^{(i,j)} & \cdots & f_n^{(i,j)} \\ f_2^{(i,j)} & f_3^{(i,j)} & \cdots & f_{n+1}^{(i,j)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{N+1}^{(i,j)} & f_{N+2}^{(i,j)} & \cdots & f_{N+n}^{(i,j)} \end{pmatrix}, \quad \mathbf{v}_j = \begin{pmatrix} g_1^{(j)} \\ g_2^{(j)} \\ \vdots \\ g_n^{(i,j)} \end{pmatrix}^{\intercal}$$

for $1 \leq i \leq r$ and $1 \leq j \leq s$. Finally, set

$$\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_r \end{pmatrix}^{\intercal}, \quad A' = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,s} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r,1} & A_{r,2} & \cdots & A_{r,s} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_s \end{pmatrix}^{\intercal}.$$

Then, (6) has a solution if and only if the system of linear equations $\mathbf{u}A' = \mathbf{v}$ has a solution $\mathbf{u}$.

In order to show that the latter system is solvable, it suffices to show that $\mathrm{rank}(A') = ns$ for N large enough. Assume that this is wrong. Then, there exist $\alpha_1, \ldots, \alpha_{ns}$ not all 0 with

$$\alpha_1\left(f_1^{(1,1)}, \ldots, f_{N+1}^{(1,1)}, f_1^{(2,1)}, \ldots, f_{N+1}^{(2,1)}, \ldots, f_1^{(r,1)}, \ldots, f_{N+1}^{(r,1)}\right) \\ + \cdots + \alpha_{ns}\left(f_n^{(1,s)}, \ldots, f_{N+n}^{(1,s)}, f_n^{(2,s)}, \ldots, f_{N+n}^{(2,s)}, \ldots, f_n^{(r,s)}, \ldots, f_{N+n}^{(r,s)}\right) = \mathbf{0}. \quad (7)$$

If we now set $\mathbf{u} = (U_1, \ldots, U_s)$ with

$$\begin{aligned} U_1 &= \alpha_1 + \alpha_2 T + \cdots + \alpha_n T^{n-1}, \\ U_2 &= \alpha_{n+1} + \alpha_{n+2} T + \cdots + \alpha_{2n} T^{n-1}, \\ &\vdots \\ U_s &= \alpha_{n(s-1)+1} + \alpha_{n(s-1)+2} T + \cdots + \alpha_{ns} T^{n-1}, \end{aligned}$$

then (7) can be reformulated as

$$|\{f_{i,1}U_1 + \cdots + f_{i,s}U_s\}| < q^{-N-1}$$

for $1 \leq i \leq r$. This in turn gives that

$$\|A\mathbf{u}^{\intercal}\| < q^{-N-1}. \quad (8)$$

Now, since A is badly approximable, Lemma 1 implies that $\|A\mathbf{u}^{\intercal}\| > 0$. Consequently, since there are only finitely many possible choices of $\mathbf{u}$ (since n is fixed), (8) becomes wrong if N is large enough. This gives a contradiction and hence our result is proved. ∎

Lemma 3. *Let $E \subseteq \mathbb{L}^s$ and assume that E is invariant under the action $\cdot + \{\mathbf{q}A\}$ for all $\mathbf{q} \in \mathbb{F}_q[T]^r$. Then, $m(E) = 0$ or $m(E) = 1$.*

Proof. First, recall from the introduction that $\mathbb{L}^s$ is a compact topological group and m is its Haar measure.

Now, assume that $m(E) > 0$. We have to show that $m(E) = 1$. In order to do so, we use Lebesgues's density theorem for compact topological groups (see Remark 5 on page 268 in [4]): for all $\epsilon > 0$, there exists a $d \in \mathbb{Z}$ with

$$\int \left| \chi_E(\mathbf{g}) - \frac{m\left(E \cap B\left(\mathbf{g}; q^{-d}\right)\right)}{m\left(B\left(\mathbf{g}; q^{-d}\right)\right)} \right| \mathrm{d}m < \epsilon m(E),$$

where χ_E denotes the indicator function of E. The latter implies that

$$\int_E \left(1 - \frac{m\left(E \cap B\left(\mathbf{g}; q^{-d}\right)\right)}{m\left(B\left(\mathbf{g}; q^{-d}\right)\right)}\right) \mathrm{d}m < \epsilon m(E).$$

Hence, there exists a $\mathbf{g} \in \mathbb{L}^s$ with

$$1 - \frac{m\left(E \cap B\left(\mathbf{g}; q^{-d}\right)\right)}{m\left(B\left(\mathbf{g}; q^{-d}\right)\right)} < \epsilon$$

and consequently,

$$m\left(E\cap B\left(\mathbf{g};q^{-d}\right)\right) > (1-\epsilon)m\left(B\left(\mathbf{g};q^{-d}\right)\right).$$

Since E is invariant under the action $\cdot + \{\mathbf{q}A\}$ and m is translation-invariant, we obtain

$$m\left(E\cap\left(B\left(\mathbf{g};q^{-d}\right)+\{\mathbf{q}A\}\right)\right) > (1-\epsilon)m\left(B\left(\mathbf{g};q^{-d}\right)+\{\mathbf{q}A\}\right)$$

for all $\mathbf{q}\in\mathbb{F}_q[T]^r$. This together with Lemma 2 clearly implies that $m(E) > 1-\epsilon$ and since this holds for all $\epsilon > 0$, we have $m(E) = 1$ as desired. ∎

3 Proof of the Main Result

In this section, we will prove Theorem 2. We will start with the case where A is badly approximable. For the next two results again assume that A satisfies (3).

Lemma 4. *Let $\mathbf{g}\in\mathbb{L}^s$ and $d > 0$. Then, the number of $\mathbf{q}\in\mathbb{F}_q[T]^r$ with $\deg\mathbf{q}\le N$ such that $\{\mathbf{q}A\}\in B(\mathbf{g};q^{-d})$ is at most $\max\{q^{Nr+cs-ds},1\}$.*

Proof. First, fix $\mathbf{q},\mathbf{q}'\in\mathbb{F}_q[T]^r$ with $\deg\mathbf{q},\deg\mathbf{q}'\le N$. Then, since A is badly approximable, we have

$$\|\{\mathbf{q}A\}-\{\mathbf{q}'A\}\| = \|\{(\mathbf{q}-\mathbf{q}')A\}\| \ge q^{-\lfloor\frac{\deg(\mathbf{q}-\mathbf{q}')r}{s}\rfloor-c} \ge q^{-\lfloor\frac{Nr}{s}\rfloor-c}.$$

This means that the distance between any two points $\{\mathbf{q}A\}$ and $\{\mathbf{q}'A\}$ is at least $q^{-\lfloor\frac{Nr}{s}\rfloor-c}$.

Now, we consider two cases.

Case 1. If $q^{-\lfloor\frac{Nr}{s}\rfloor-c}\ge q^{-d}$, then there is at most one point in $B(\mathbf{g};q^{-d})$.

Case 2. If $q^{-\lfloor\frac{Nr}{s}\rfloor-c} < q^{-d}$, then the number of points in $B(\mathbf{g};q^{-d})$ is at most

$$\frac{\left(q^{-d}\right)^s}{\left(q^{-\lfloor\frac{Nr}{s}\rfloor-c}\right)^s}\le q^{Nr+cs-ds}.$$

Hence, our claimed result is proved. ∎

Lemma 5. *Let l_n be a sequence with $\sum_{n\ge0}q^{-l_ns}=\infty$. Then, for all $k\ge0$*

$$m\left(\bigcup_{n=k}^{\infty}\bigcup_{\deg\mathbf{q}=n}B\left(\{\mathbf{q}A\};q^{-\lfloor\frac{nr}{s}\rfloor-l_n}\right)\right) > \frac{1}{q^{cs+1}}.$$

Proof. We first exclude the case $q=2$ and $r=1$.

Let $l'_n=\max\{l_n,c+1\}$. Then, $\sum_{n\ge0}q^{-l'_ns}=\infty$. We will use proof by contradiction. Therefore, assume that the claim is wrong. Hence, there exists a $k_0\ge0$ such that for all $N\ge k_0$, we have

$$m\left(\bigcup_{n=k_0}^{N}\bigcup_{\deg\mathbf{q}=n}B\left(\{\mathbf{q}A\};q^{-\lfloor\frac{nr}{s}\rfloor-l'_n}\right)\right)\le q^{-cs-1}. \tag{9}$$

Next, define the following set

$$L_N=\left\{\deg\mathbf{q}=N\ :\ \{\mathbf{q}A\}\in\bigcup_{n=k_0}^{N}\bigcup_{\deg\mathbf{q}'=n}B\left(\{\mathbf{q}'A\};q^{-\lfloor\frac{nr}{s}\rfloor-l'_n}\right)\right.$$
$$\left.\setminus\bigcup_{n=k_0}^{N-1}\bigcup_{\deg\mathbf{q}'=n}B\left(\{\mathbf{q}'A\};q^{-\lfloor\frac{nr}{s}\rfloor-l'_n}\right)\right\}.$$

Our first goal is to estimate the cardinality of L_N. Therefore, set

$$\bigcup_{n=k_0}^{N-1} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right) = \bigcup_i B\left(\{\mathbf{q}'_i A\}; q^{-d_i}\right),$$

where the $B(\{\mathbf{q}'_i A\}; q^{-d_i})$ are disjoint for all i. Then, from (9),

$$q^{-cs-1} \geq m\left(\bigcup_{n=k_0}^{N-1} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right)\right) = m\left(\bigcup_i B\left(\{\mathbf{q}'_i A\}; q^{-d_i}\right)\right) = \sum_i q^{-d_i s}.$$

Using Lemma 4 gives that the number of $\mathbf{q}$ with $\deg \mathbf{q} \leq N$ such that $\{\mathbf{q}A\} \in \bigcup_i B(\{\mathbf{q}'_i A\}; q^{-d_i})$ is at most

$$\sum_i \max\left\{q^{Nr+cs-d_i s}, 1\right\} = \max\left\{q^{Nr+cs}\sum_i q^{-d_i s}, q^{Nr}\right\} = q^{Nr}.$$

Hence, the number of elements in L_N is at least

$$q^{(N+1)r} - q^{Nr} - q^{Nr} = q^{Nr}(q^r - 2) = dq^{Nr},$$

where $d > 0$ is a constant.

Next, we claim that

$$\bigcup_{\mathbf{q} \in L_N} B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{Nr}{s} \rfloor - l'_N}\right) \subseteq \bigcup_{n=k_0}^{N} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right) \setminus \bigcup_{n=k_0}^{N-1} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right). \quad (10)$$

In order to show this, fix a $\mathbf{q} \in L_N$ and assume that there exists a $\mathbf{q}'$ with $\deg \mathbf{q}' = n < N$ such that

$$B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{Nr}{s} \rfloor - l'_N}\right) \cap B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right) \neq \emptyset.$$

Since we know that $\{\mathbf{q}A\} \notin B(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n})$, we obtain that

$$B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right) \subseteq B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{Nr}{s} \rfloor - l'_N}\right)$$

and hence $\{\mathbf{q}'A\} \in B(\{\mathbf{q}A\}; q^{-\lfloor \frac{Nr}{s} \rfloor - l'_N})$. The number of $\mathbf{q}$ with $\deg \mathbf{q} \leq N$ and $\{\mathbf{q}A\}$ belonging to the latter set is, however, at most

$$\max\left\{q^{Nr+cs-(\lfloor \frac{Nr}{s} \rfloor + l'_N)s}, 1\right\} \leq \max\left\{q^{(c+1)s - l'_N s}, 1\right\} = 1.$$

This gives a contradiction and hence (10) is established.

Finally, we claim that the balls appearing on the left-hand side of (10) are pairwise disjoint. Therefore, consider $\mathbf{q}_1, \mathbf{q}_2 \in L_N$ with

$$B\left(\{\mathbf{q}_1 A\}; q^{-\lfloor \frac{Nr}{s} \rfloor - l'_N}\right) \cap B\left(\{\mathbf{q}_2 A\}; q^{-\lfloor \frac{Nr}{s} \rfloor - l'_N}\right) \neq \emptyset.$$

Thus, these two balls are equal and hence

$$\|\{\mathbf{q}_1 A\} - \{\mathbf{q}_2 A\}\| = \|\{(\mathbf{q}_1 - \mathbf{q}_2)A\}\| < q^{-\lfloor \frac{Nr}{s} \rfloor - l'_N}.$$

Now, as above, the ball $B(\mathbf{0}; q^{-\lfloor \frac{Nr}{s} \rfloor - l'_N})$ contains at most one point $\{\mathbf{q}A\}$ with $\deg \mathbf{q} \leq N$. Consequently, $\mathbf{q}_1 = \mathbf{q}_2$ and our claim is proved.

Now, from (10) and the latter claim, we obtain

$$\begin{aligned} m&\left(\bigcup_{n=k_0}^{N} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right) \right) \\ &\geq m\left(\bigcup_{n=k_0}^{N-1} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right) \right) + m\left(\bigcup_{\mathbf{q} \in L_N} B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{Nr}{s} \rfloor - l'_N}\right) \right) \\ &\geq m\left(\bigcup_{n=k_0}^{N-1} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right) \right) + dq^{Nr}\left(q^{-\lfloor \frac{Nr}{s} \rfloor - l'_N}\right)^s \\ &\geq m\left(\bigcup_{n=k_0}^{N-1} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right) \right) + dq^{-l'_N s}. \end{aligned}$$

Iterating yields

$$m\left(\bigcup_{n=k_0}^{N} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right) \right) \geq d \sum_{n=k_0}^{N} q^{-l'_n s}.$$

Since $\sum_{n\geq 0} q^{-l'_n s} = \infty$ this gives a contradiction when N is large enough.

Now, what is left is to consider the case $q = 2$ and $r = 1$. Here, we note that since $\sum_{n\geq 0} q^{-l'_n s} = \infty$, we have either $\sum_{n\geq 0} q^{-l'_{2n} s} = \infty$ or $\sum_{n\geq 0} q^{-l'_{2n+1} s} = \infty$. W.l.o.g. assume that the first case holds. Then, the same proof as above can be used with the only difference that instead of L_N, we consider

$$\tilde{L}_N = \left\{ \deg \mathbf{q} = 2N \ : \ \{\mathbf{q}A\} \in \bigcup_{n=k_0}^{2N} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right) \right. \\ \left. \setminus \bigcup_{n=k_0}^{2N-2} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right) \right\}.$$

Details are straightforward and we leave them to the reader. ▮

Now, we can prove one half of Theorem 2.

Proposition 1. *Let $A \in \mathbb{L}^{r\times s}$ be badly approximable. Then, for all sequences l_n with $\sum_{n\geq 0} q^{-l_n s} = \infty$, we have that (2) has infinitely many solutions for almost all $\mathbf{g} \in \mathbb{L}^s$.*

Proof. Consider

$$E = \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} \bigcup_{\deg \mathbf{q}=n} B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l_n}\right).$$

Then, we have for all $\mathbf{g} \in \mathbb{L}^s$ that $\mathbf{g} \in E$ if and only if (2) has infinitely many solutions. Moreover, Lemma 5 implies that $m(E) > 0$. Since E is invariant under the action $\cdot + \{\mathbf{q}A\}$ for all $\mathbf{q} \in \mathbb{F}_q[T]^r$, the latter and Lemma 3 yields $m(E) = 1$ which is the desired result. ▮

In order to conclude the proof of Theorem 2 what is left is to consider the case where A is not badly approximable.

Proposition 2. *Let $A \in \mathbb{L}^{r\times s}$ be not badly approximable. Then, there exists a sequence l_n with $\sum_{n\geq 0} q^{-l_n s} = \infty$ but (2) has only finitely many solutions for almost all $\mathbf{g} \in \mathbb{L}^s$.*

Proof. First, since A is not badly approximable, there exists a sequence $\mathbf{q}_i = (Q_1^{(i)}, \dots, Q_r^{(i)}) \in \mathbb{F}_q[T]^r$ with $\deg \mathbf{q}_i = n_i$ and n_i increasing such that

$$\|\{\mathbf{q}_i A\}\| < q^{-\lfloor \frac{(n_i+i)r}{s} \rfloor - i}.$$

Now, define $t_0 = 0$ and $t_i = n_i + i$ for all i. Moreover, for n with $t_{i-1} \le n < t_i$ set

$$l_n = \left\lfloor \frac{(t_i - n)r}{s} \right\rfloor.$$

Note that l_n is a sequence with

$$\sum_{n \ge 0} q^{-l_n s} \ge \sum_{i=1}^{\infty} q^{-l_{t_i-1} s} \ge \sum_{i=1}^{\infty} q^{-\lfloor \frac{r}{s} \rfloor s} \ge \sum_{i=1}^{\infty} q^{-r} = \infty.$$

Next, assume w.l.o.g. that $q^{n_i} = \|\mathbf{q}_i\| = |Q_1^{(i)}|$. We claim that

$$\bigcup_{t_{i-1} \le n < t_i} \bigcup_{\deg \mathbf{q} = n} B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l_n}\right) \subseteq \bigcup B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{t_i r}{s} \rfloor + 2}\right),$$

where the second union runs over all $\mathbf{q}' = (Q_1', \dots, Q_r')$ with

$$|Q_1'| \le q^{n_i - 1}, |Q_2'| \le q^{t_i - 1}, \dots, |Q_r'| \le q^{t_i - 1}.$$

In order to show this, fix $\mathbf{q} = (Q_1, \dots, Q_r)$ with $t_{i-1} \le \deg \mathbf{q} = n < t_i$. Using division with remainder gives a $P \in \mathbb{F}_q[T]$ with $|Q_1 + PQ_1^{(i)}| \le q^{n_i - 1}$. Note that $|P| \le q^{t_i - 1 - n_i}$. Now set

$$\mathbf{q}' = (Q_1 + PQ_1^{(i)}, \dots, Q_r + PQ_r^{(i)}).$$

Then,

$$\|\{\mathbf{q}A\} - \{\mathbf{q}'A\}\| \le |P| \|\{\mathbf{q}_i A\}\| < q^{t_i - 1 - n_i - \lfloor \frac{t_i r}{s} \rfloor - i} = q^{-\lfloor \frac{t_i r}{s} \rfloor - 1}.$$

Also, note that

$$q^{-\lfloor \frac{nr}{s} \rfloor - l_n} = q^{-\lfloor \frac{nr}{s} \rfloor - \lfloor \frac{(t_i - n)r}{s} \rfloor} < q^{-\frac{nr}{s} - \frac{(t_i - n)r}{s} + 2} \le q^{-\lfloor \frac{t_i r}{s} \rfloor + 2}.$$

Consequently,

$$B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l_n}\right) \subseteq B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{t_i r}{s} \rfloor + 2}\right)$$

which proves the claim.

In order to conclude the proof, observe that the claim implies

$$m\left(\bigcup_{t_{i-1} \le n < t_i} \bigcup_{\deg \mathbf{q} = n} B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l_n}\right)\right) \le q^{(-\lfloor \frac{t_i r}{s} \rfloor + 2)s} q^{n_i + t_i(r-1)} < q^{3s - i}.$$

Hence,

$$\sum_{i=1}^{\infty} m\left(\bigcup_{t_{i-1} \le n < t_i} \bigcup_{\deg \mathbf{q} = n} B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l_n}\right)\right) \le \sum_{i=1}^{\infty} q^{3s - i} < \infty.$$

The Borel-Cantelli lemma now implies that for almost all $\mathbf{g} \in \mathbb{L}^s$

$$\mathbf{g} \in \bigcup_{t_{i-1} \le n < t_i} \bigcup_{\deg \mathbf{q} = n} B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l_n}\right)$$

for only finitely many n which proves the desired result. ∎

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Participation in conferences within NSC-99-2115-M-009-007-MY2

by

Michael Fuchs

This is a short report concerning participation in international conferences within my national science council project NSC 101-2115-M-009-010 in 2013.

I participated in the Seventh Cross-strait Conference on Graph Theory and Combinatorics which took place in Changsha, Hunan, China from June 27 to June 30. I was one of roughly 30 invited speakers out of roughly 350 participants. My Ph.D. student (who was employed in this project) was also giving a contributed talk at the conference. This conference is a biyearly conference and serves as an important platform to foster academic exchange between people working in graph theory and combinatorics in China and Taiwan. The conference is conducted on a rotational basis with every second meeting taking place in China.

This conference was different from the conference I applied for in the project proposal because of two reasons: (i) I was an invited speaker; (ii) I wanted to give my Ph.D. student a chance to give a talk at an international meeting (this was the first international meeting he gave a talk) and this was a conference which was both relevant to our field and fitted in our budget.

As mentioned above, I delivered an invited talk on June 28 entitled “On Set Partitions, Words, Approximate Counting and Digital Search Trees” (slides are attached); the talk of my Ph.D. student was scheduled just after my talk and was entitled “Limit Laws for Wiener Index of Random Digital Trees” (slides are attached as well).

On Set Partitions, Words, Approximate Counting and Digital Search Trees

(joint with Chung-Kuei Lee and Helmut Prodinger)

Michael Fuchs

Institute of Applied Mathematics
National Chiao Tung University

Hsinchu, Taiwan

Changsha, June 28, 2013

Set Partitions

Example.

$$\{\{2,7\},\{1,3,4\},\{5,6\}\}.$$

Set partition of $\{1,2,3,4,5,6,7\}$ with three blocks.

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$\#$ of set partitions of $\{1,\ldots,n\}$: **Bell number** B_n.

Set Partitions

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$$\{\{2,7\},\{1,3,4\},\{5,6\}\}.$$

Set partition of $\{1,2,3,4,5,6,7\}$ with three blocks.

$\#$ of set partitions of $\{1,\dots,n\}$: **Bell number** B_n.

We have,

$$B_n \sim n!\frac{e^{e^r-1}}{r^n\sqrt{2\pi r(r+1)e^r}},$$

where $re^r = n+1$, i.e., asymptotically

$$r = \log n - \log\log n + o(1)$$

Number of Blocks

$\#$ of set partitions with k blocks: **Stirling partition number** $S(n,k)$.

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Theorem (Harper; 1967)

We have,

$$\mathbb{E}(X_n) \sim \frac{n}{\log n}, \qquad \mathrm{Var}(X_n) \sim \frac{n}{\log^2 n}.$$

Moreover,

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathrm{Var}(X_n)}} \xrightarrow{d} N(0,1).$$

Set Partitions as Words

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$\omega_1 \cdots \omega_7$ with $w_i = \#$ of the block, i.e.,

$$\omega_1 \cdots \omega_7 = 1211332.$$

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Arrange blocks such that block leaders are increasing.

$$\{\overbrace{\{1,3,4\}}^{\#1},\overbrace{\{2,7\}}^{\#2},\overbrace{\{5,6\}}^{\#3}\}$$

$\omega_1 \cdots \omega_7$ with $w_i = \#$ of the block, i.e.,

$$\omega_1 \cdots \omega_7 = 1211332.$$

This gives a 1-1 correspondence between set partitions and certain words.

Geometric Words

$\omega = \omega_1 \cdots \omega_n$: word corresponding to a set partition.

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ω_i independent, geometric random variables with success probability p.

Words generated by this random model are called **geometric words**.

p_n: probability that a geometric word satisfies RGP.

Results for p_n (i)

$q = 1 - p.$

$(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}).$

$(x;q)_\infty = \lim_{n\to\infty}(x;q)_n.$

Theorem (Oliver, Prodinger; 2011; Mansour, Shattuck; 2012)

We have,

$$p_n = p\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}q^j(p;q)_j$$
$$= \sum_{j=0}^{n}(-1)^j\binom{n}{j}(p;q)_j.$$

Results for p_n (ii)

$Q = 1/q$.

$L = \log Q$.

$\chi_k = 2\pi i k/L$.

Theorem (Oliver, Prodinger; 2011)

We have,

$$p_n \sim \frac{(p;q)_\infty}{L(q;q)_\infty}\Gamma(-\log_Q p)n^{\log_Q p} + n^{\log_Q p}\Psi(\log_Q n),$$

where $\Psi(z)$ *is the* 1*-periodic function with average value* 0 *and*

$$\Psi(z) = \frac{(p;q)_\infty}{L(q;q)_\infty}\sum_{k\neq 0}\Gamma(-\log_Q p + \chi_k)e^{-2\pi i k z}.$$

Largest Letter

$p_{n,k}$: probability that a geometric word has largest letter k and satisfies RGP.

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$$n_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}.$$

$$n_q! = 1_q 2_q \cdots n_q.$$

$$\binom{n}{k}_q = \frac{n_q}{k_q(n-k)_q}.$$

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Average Value of Largest Letter

X_n: largest letter of geometric word subject to RGP. Then,

$$P(X_n = k) = \frac{p_{n,k}}{p_n}.$$

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Theorem (Prodinger; 2012)

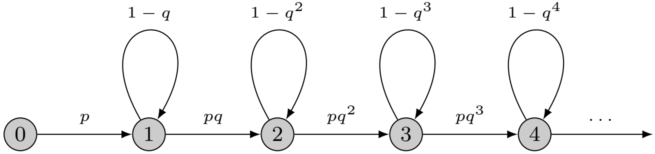
We have,

$$\mathbb{E}(X_n) \sim \log_Q n - \alpha_p - \frac{\psi(-\log_Q p)}{L} + \Phi(\log_Q n),$$

where $\Phi(z)$ is a 1-periodic function with average value 0, $\psi = \Gamma'/\Gamma$ and

$$\alpha_p = \sum_{l \geq 0} \frac{pq^l}{1 - pq^l}.$$

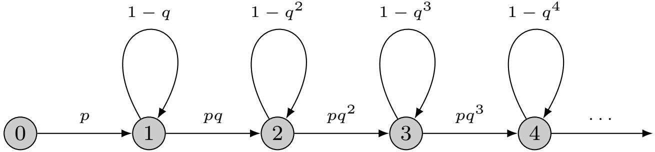
State diagram:



In every state there is a positive probability of violating RGP.

Approximate Counting with Black Holes

State diagram:



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Above diagram implies

$$p_{n,k} = pq^{k-1}p_{n-1,k-1} + (1 - q^k)p_{n-1,k}.$$

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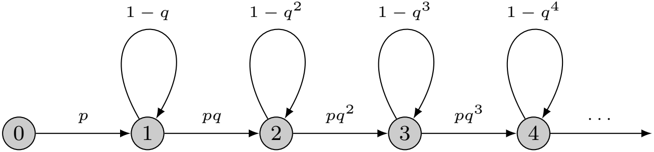

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Prodinger used this as starting point for his analysis.

Approximate Counting

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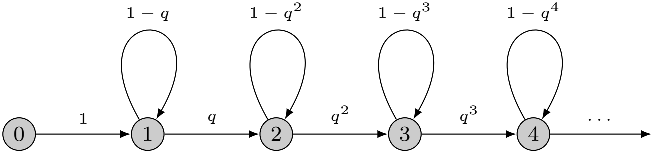

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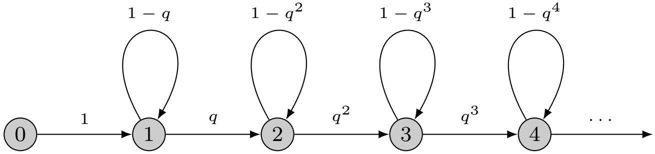

Approximate Counting (Morris 1978):

Counter C_n with $C_0 = 0$ and

$$C_{n+1} = \begin{cases} C_n + 1, & \text{with probability } q^{C_n}; \\ C_n, & \text{with probability } 1 - q^{C_n}. \end{cases}$$

Approximate Counting

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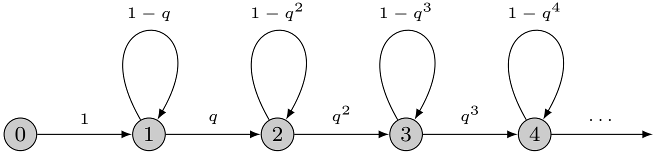

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Only $\Theta(\log \log n)$ space is needed for counting n objects.

Applications

Approximate counting has found many applications:

- Analysis of the Webgraph.
- Monitoring network traffic.
- Finding patterns in protein and DNA sequencing.
- Computing frequency moments of data streams.
- Data storage in flash memory.
- Etc.

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Many refinements have been proposed.

Analysis of Approximate Counting

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$$\text{Var}(C_n) \sim C_{\text{var}} + G(\log_Q n),$$

where $G(z)$ is a 1-periodic function and

$$C_{\text{var}} = \frac{\pi^2}{6L^2} - \alpha - \beta + \frac{1}{12} - \frac{1}{L} \sum_{l \geq 1} \frac{1}{l \sinh(2l\pi^2/L)}$$

with $\alpha = \sum_{l \geq 1} q^l/(1-q^l)$ and $\beta = \sum_{l \geq 1} q^{2l}/(1-q^l)^2$.

Methods

Many different methods have been used:

- **Mellin Transform**: Flajolet (1985); Prodinger (1992)
- **Rice Method**: Kirschenhofer & Prodinger (1991)
- **Euler Transform**: Prodinger (1994)
- **Analysis of Extreme Value Distributions**: Louchard & Prodinger (2006)
- **Martingale Theory**: Rosenkrantz (1987)
- **Probability Theory**: Robert (2005)
- **Poisson-Laplace-Mellin Method**: F. & Lee & Prodinger (2012).

Digital Search Tree (DST)

Introduced by Coffman and Eve (1970).

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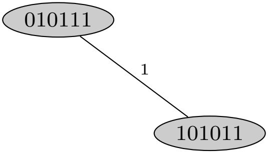

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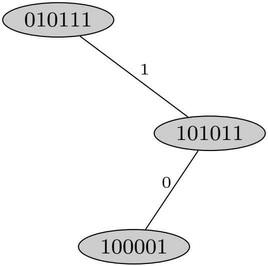

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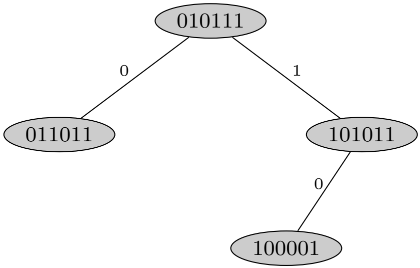

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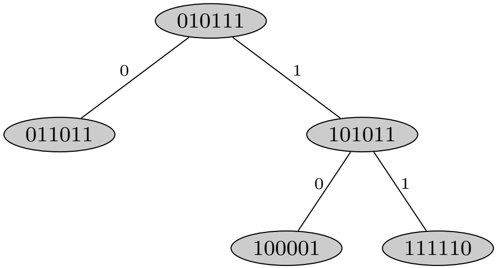

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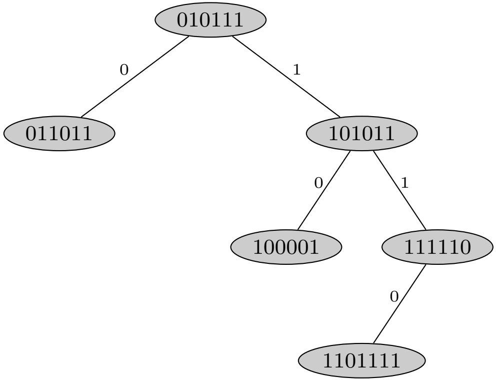

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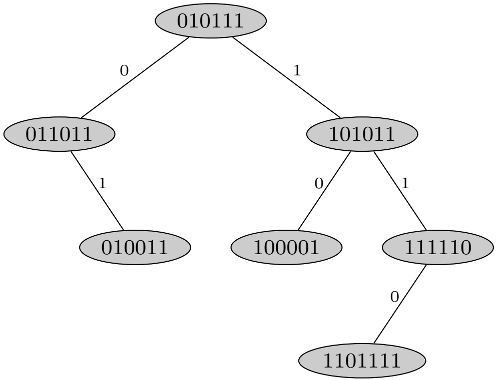

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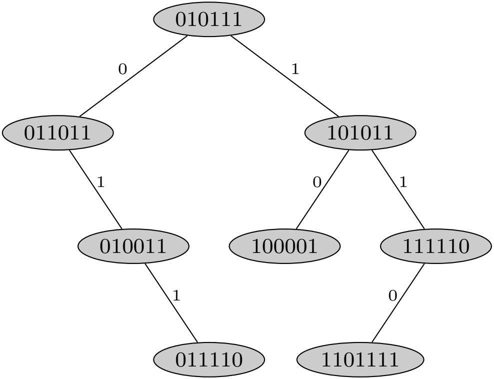

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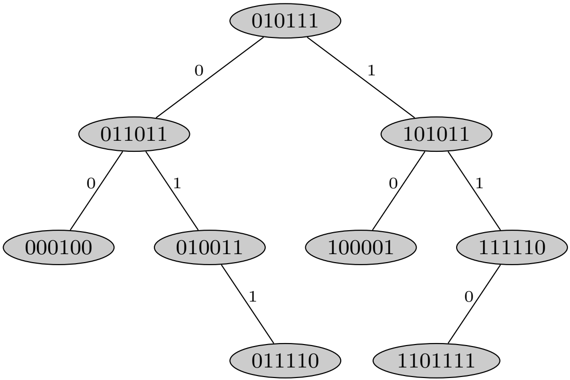

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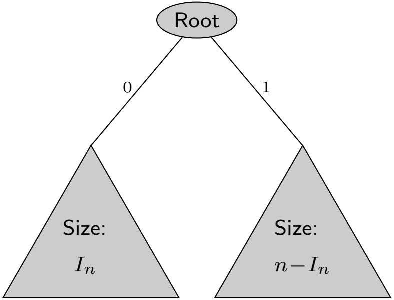
Note that:

$$X_n \stackrel{d}{=} C_n.$$

Distributional Recurrence of X_n

$$X_{n+1} \stackrel{d}{=} X_{I_n} + 1$$

- $I_n \stackrel{d}{=} \text{Binomial}(n, q)$;
- X_n, I_n independent.



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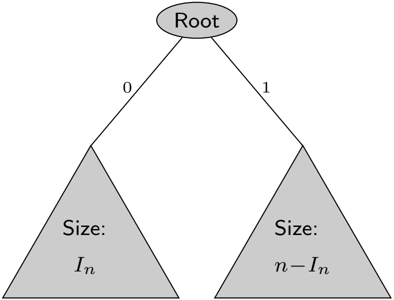

Recurrence of moments:

$$f_{n+1} = \sum_{j=0}^{n} \binom{n}{j} q^j p^{n-j} f_j + g_n.$$

Analytic Methods for DSTs

- **Rice Method**:

  Introduced by Flajolet and Sedgewick.

- **Approach of Flajolet and Richmond**:

  Based on Euler transform, Mellin transform, and singularity analysis.

- **Approach via Analytic Depoissonization**:

  Introduced by Jacquet & Regnier and Jacquet & Szpankowski. Based on saddle point method and Mellin transform.

- **Poisson-Laplace-Mellin Approach**:

  Introduced by F. & Hwang & Zacharovas. Based on analytic depoissonization and a combination of Laplace and Mellin transform.

Variance of Approximate Counting

$Q_n = (q;q)_\infty/(q^{n+1};q)_\infty$; $Q_\infty = \lim_{n\to\infty} Q_n$.

Theorem (F., Lee, Prodinger; 2012)

We have,

$$\mathrm{Var}(C_n) \sim \sum_k g_k e^{2k\pi i \log_Q n},$$

where

$$g_k = \frac{Q_\infty}{L\Gamma(1+\chi_k)} \sum_{h,l,j\ge 0} \frac{(-1)^j q^{h+l+\binom{j+1}{2}}}{Q_h Q_l Q_j} \varphi(\chi_k; q^{h+j} + q^{l+j}).$$

Here,

$$\varphi(\chi;x) = \begin{cases} \pi(x^\chi - 1)/(\sin(\pi\chi)(x-1)), & \textit{if } x \neq 1; \\ \pi\chi/\sin(\pi\chi), & \textit{if } x = 1. \end{cases}$$

An Identity

Corollary (F., Lee, Prodinger; 2012)

We have,

$$\frac{Q_\infty}{L}\sum_{h,l,j\ge 0}\frac{(-1)^j q^{h+l+\binom{j+1}{2}}}{Q_hQ_lQ_j}\psi(q^{h+j}+q^{l+j})$$
$$=\frac{\pi^2}{6L^2}-\alpha-\beta+\frac{1}{12}-\frac{1}{L}\sum_{l\ge 1}\frac{1}{l\sinh(2l\pi^2/L)},$$

where

$$\psi(x)=\begin{cases}\log x/(x-1), & \textit{if } x\neq 1;\\ 1, & \textit{if } x=1.\end{cases}$$

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p_n: probability that a geometric word satisfies GRGP.

$p_{n,k}$: probability that a geometric word with largest letter k satisfies GRGP.

X_n: largest letter of geometric word subject to GRGP. Again,

$$P(X_n = k) = \frac{p_{n,k}}{p_n}.$$

Analysis of p_n (i)

Conditioning on first letter and # of letters $\leq$ first letter:

$$p_{n+1} = \sum_{l=1}^{d} pq^{l-1} \sum_{j=0}^{n} \binom{n}{j} (1-q^l)^{n-j} q^{lj} p_j.$$

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Then,

$$\tilde{f}(z) + \tilde{f}'(z) = \sum_{l=1}^{d} pq^{l-1} \tilde{f}(q^l z).$$

This is the probability in the **Poisson model**.

Poisson Heuristic

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$$p_n \text{ sufficiently smooth} \quad \Longrightarrow \quad p_n \approx \tilde{f}(n) = e^{-n} \sum_{j \geq 0} p_j \frac{n^j}{j!}.$$

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More precisely: if p_n is smooth enough,

$$p_n \sim \sum_{j \geq 0} \frac{\tilde{f}^{(j)}(n)}{n!} \tau_j(n) = \tilde{f}(n) - \frac{n}{2} \tilde{f}''(n) + \dots,$$

where $\tau_j(n) := n![z^n](z-n)^j e^z$.

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This is called *Poisson-Charlier expansion* (can be already found in Ramanujan's notebooks).

Jacquet-Szpankowski-admissibility (JS-admissibility)

$\tilde{f}(z)$ is called JS-admissible if

$(\mathbf{I})$ Uniformly for $|\arg(z)| \le \epsilon$,

$$\tilde{f}(z) = \mathcal{O}\left(|z|^{\alpha} \log^{\beta} |z|\right),$$

$(\mathbf{O})$ Uniformly for $\epsilon < |\arg(z)| \le \pi$,

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Theorem (Jacquet, Szpankowski; 1998)

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Depoissonization

JS-admissibility satisfies closure properties:

(i) $\tilde{f}, \tilde{g}$ JS-admissible, then $\tilde{f} + \tilde{g}$ JS-admissible.

(ii) $\tilde{f}$ JS-admissible, then $\tilde{f}'$ JS-admissible. Etc.

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Proposition

Consider

$$\tilde{f}(z) + \tilde{f}'(z) = \sum_{l=1}^{d} pq^{l-1} \tilde{f}(q^l z) + \tilde{g}(z).$$

We have,

$$\tilde{g}(z) \text{ JS-admissible} \iff \tilde{f}(z) \text{ JS-admissible.}$$

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Recall

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We only have to find an asymptotic of $\tilde{f}(z)$.

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$$p_n \sim \tilde{f}(n).$$

We only have to find an asymptotic of $\tilde{f}(z)$.

This can be done via Mellin transform.

$$\mathscr{M}[\tilde{f}(z); s] = \int_0^\infty \tilde{f}(z) z^{s-1} \mathrm{d}z.$$

Analysis of p_n (iii)

We have,

$$\mathscr{M}[\tilde{f}(z);s]=\frac{q^d\Omega(1)\Gamma(s)}{P(q^{-s})\Omega(q^{-s})},$$

where

$$P(z)=1-p\sum_{l=1}^{d}q^{l-1}z^l$$

and

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Lemma

Let ρ be the smallest positive root of $P(z)$. Then, ρ is simple and the only root with $|z|\leq\rho$.

Converse Mapping Theorem

Theorem (Flajolet, Gourdon, Dumas; 1995)

Let the Mellin transform of $\tilde{f}(z)$ *exist in the strip* $\langle \alpha, \beta \rangle$*.*

Assume that $\mathscr{M}[\tilde{f}(z); s]$ *can be continued to a meromorphic function on* $\langle \alpha, \gamma \rangle$ *with* $\beta < \gamma$ *with simple poles at* $s_1, \cdots, s_k$*.*

Then, under some technical conditions,

$$\tilde{f}(z) = -\sum_{j=1}^{k} \mathrm{Res}(\mathscr{M}[\tilde{f}(z); s], s = s_j) z^{-s_j} + \mathcal{O}\left(z^{-\gamma}\right)$$

as $z \to \infty$*.*

Analysis of p_n (iv)

$\mathscr{M}[\tilde{f}(z);s]$ has simple poles at $\log_Q \rho + \chi_k$ with

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Thus,

$$\tilde{f}(z) \sim -\frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)} z^{-\log_Q \rho} \sum_k \Gamma(\log_Q \rho + \chi_k) z^{-\chi_k}$$

and

$$p_n \sim -\frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)} n^{-\log_Q \rho} \sum_k \Gamma(\log_Q \rho + \chi_k) n^{-\chi_k}.$$

Result for p_n

Theorem (F., Prodinger; 2013)

We have,

$$p_n \sim -\frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)}\Gamma(\log_Q \rho)n^{-\log_Q \rho} + n^{-\log_Q \rho}\Psi(\log_Q n),$$

where $\Psi(z)$ is the 1-periodic function with average value 0 and

$$\Psi(z) = -\frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)}\sum_{k\neq 0}\Gamma(\log_Q \rho + \chi_k)e^{-2\pi ikz}.$$

For $d = 1$: $\rho = 1/p$ and result coincides with Oliver and Prodinger's result.

Average Value of X_n

X_n: largest letter of geometric word subject to GRGP.

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Similar (but more involved) analysis gives:

Theorem (F., Prodinger; 2013)

We have,

$$\mathbb{E}(X_n) \sim \log_Q n - \alpha_p - \frac{\psi(\log_Q \rho)}{L} + \Phi(\log_Q n),$$

where $\Phi(z)$ *is a* 1*-periodic function with average value* 0, $\psi = \Gamma'/\Gamma$ *and*

$$\alpha_p = -\sum_{l\geq 0} \frac{q^l P'(q^l)}{P(q^l)}.$$

Further Extensions

- **Further Restrictions on Geometric Words**:

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- **More properties of X_n**:

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- **Generality of our method**:

  The method seems to be applicable to asymmetric DSTs with $\log p/\log q \in \mathbb{Q}$. This might yield simplifications of expressions in asymptotics of total path length, peripheral path length, profile, number of leaves, patterns, etc.

Limit Laws for Wiener Index of Random Digital Trees

(joint work with M. Fuchs)

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Department of Applied Mathematics
National Chiao Tung University

2013 Conference on Graph Theory and Combinatorics and
Seventh Cross-strait Conference on Graph Theory and Combinatorics

Changsha, June 28, 2013

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- Studied for all kinds of trees since tree arises as molecular graphs of acyclic organic molecules.

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- Many others.

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Wagner (2012) Same resides of simple generated tree hold for non-plane unlabeled tree.

Motivation of Our Research

Random split tree is a huge class of random trees which includes many important subclasses such as binary search trees, m-ary search trees, median-of-(2k+1) search trees, quadtrees, simplex trees, digital trees, etc.

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Neininger's questions:

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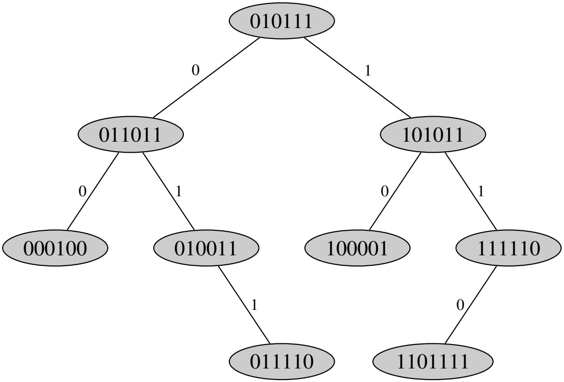
2. Is Wiener index of digital trees asymptotically normal?

Digital Search Tree (DST)

First proposed by Coffman and Eve in 1970.

Example: A DST built by 9 keys



$010111\cdots$
$101011\cdots$
$100001\cdots$
$011011\cdots$
$111110\cdots$
$110111\cdots$
$010011\cdots$
$011110\cdots$
$000100\cdots$

Random model and Recurrence

Random Model: Bernoulli model.
Bits of keys: i.i.d. Bernoulli random variables
Probability: $\mathbb{P}(I_n = 0) = q^n$

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Recurrence Relation:

1. T_n: Random variable of total path length of DST on n nodes.
2. W_n: Random variable of the Wiener index of DST on n nodes.
3. B_n=Binomial$(n, \frac{1}{2})$.

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1. T_n: Random variable of total path length of DST on n nodes.
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3. B_n=Binomial$(n, \frac{1}{2})$.

$$\begin{cases} T_{n+1} = & T_{B_n} + T^*_{n-B_n} + n \\ W_{n+1} = & W_{B_n} + W^*_{n-B_n} + n + T_{B_n} + T^*_{n-B_n} + (n-B_n)T_{B_n} \\ & +B_n T^*_{n-B_n} + 2B_n(n-B_n). \end{cases}$$

Relation for the Mean:

$$\begin{cases} \tilde{f}_{1,0}(z) + \tilde{f}'_{1,0}(z) = 2\tilde{f}_{1,0}(\frac{z}{2}) + z \\ \tilde{f}_{0,1}(z) + \tilde{f}'_{0,1}(z) = 2\tilde{f}_{0,1}(\frac{z}{2}) + (z+2)\tilde{f}_{1,0}(\frac{z}{2}) + \frac{z^2}{2} + z \end{cases}$$

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Relation for the Variance and Covariance:

$$\begin{cases} \tilde{V}(z) + \tilde{V}'(z) = & 2\tilde{V}(\frac{z}{2}) + z\tilde{f}''_{1,0}(z)^2 \\ \tilde{C}(z) + \tilde{C}'(z) = & 2\tilde{C}(\frac{z}{2}) + (z+2)\tilde{V}(\frac{z}{2}) + z\tilde{f}''_{1,0}(z)\tilde{f}''_{0,1}(z), \\ \tilde{W}(z) + \tilde{W}'(z) = & 2\tilde{W}(\frac{z}{2}) + \left(\frac{z^2}{2} + 3z + 2\right)\tilde{V}(\frac{z}{2}) + (2z+4)\,\tilde{U}(\frac{z}{2}) \\ & +z^2\tilde{f}'_{1,0}(\frac{z}{2})^2 + 2z^2\tilde{f}'_{1,0}(\frac{z}{2}) + z\tilde{f}''_{0,1}(z)^2 + z^2. \end{cases}$$

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We use the Poisson-Laplace-Mellin method to handle above equations and generate asymptotic expressions.

Results of DST

Apply inverse Mellin transform and inverse Laplace transform, we would get the desired result. For the others, we apply the same method and it yields

$$\begin{aligned}
\mathbb{E}(T_n) &= n\log_2 n + nP_1(\log_2 n) + \mathcal{O}(\log n) \\
\mathbb{E}(W_n) &= n^2\log_2 n + n^2P_1(\log_2 n) - n^2 + \mathcal{O}(|n\log n|) \\
\mathrm{Var}(T_n) &= nP_2(\log_2 n) + \mathcal{O}(1) \\
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$$P_1(z) = \frac{\gamma - 1}{\log 2} + \frac{1}{2} - \sum_{k\geq 1}\frac{1}{2^k - 1} + \frac{1}{\log 2}\sum_{k\neq 0}\Gamma(-1+\chi_k)e^{2k\pi i}$$

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$$P_2(z) = \frac{1}{\log 2}\sum_{k\in\mathbb{Z}}\sum_{j,h,l\geq 0}\frac{Q_\infty(-1)^j 2^{-\binom{j+1}{2}}}{Q_jQ_hQ_l}\frac{e^{2k\pi i}}{\Gamma(2+\chi_k)}\varphi(2+\chi_k, 2^{-j-h}+2^{-j-l})$$

Limiting Distribution

Theorem (Fuchs, Lee)

$$\left(\frac{T_n - \mathbb{E}(T_n)}{\sqrt{\mathrm{Var}(T_n)}}, \frac{W_n - \mathbb{E}(W_n)}{\sqrt{\mathrm{Var}(W_n)}}\right) \xrightarrow{d} (X, X)$$

where X is a standard normal distributed random variable and $\xrightarrow{d}$ denote weak convergence.

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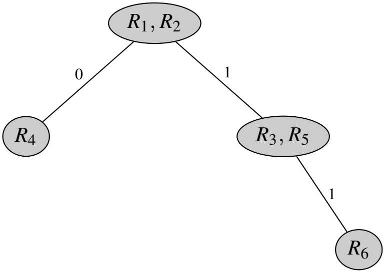
Proof (Sketch):

It is well-known that T_n is Gaussian. We let

$$Y_n = \frac{W_n - \mathbb{E}(W_n)}{\sqrt{\mathrm{Var}(W_n)}} - \frac{T_n - \mathbb{E}(T_n)}{\sqrt{\mathrm{Var}(T_n)}}.$$

Compute $\mathbb{E}(Y_n^2)$, use Markov's inequality to show that $Y_n \xrightarrow{P} 0$ and the rest follows by Slutsky's Theorem.

Bucket Digital Search Trees



$$
\begin{aligned}
R_1 &= 000001\cdots\\
R_2 &= 000110\cdots\\
R_3 &= 110111\cdots\\
R_4 &= 011011\cdots\\
R_5 &= 100001\cdots\\
R_6 &= 111110\cdots
\end{aligned}
$$

Figure : A bucket digital search tree with $b = 2$ built from 6 keys with key-wise path length $= 5$, key-wise Wiener index $= 19$, node-wise path length $= 4$ and node-wise Wiener index $= 10$.

Key-wise Wiener Index of Bucket DST

Key-wise Wiener index is the sum of distance between unordered pairs of data (keys). The same method as DST yields

$$
\begin{aligned}
\mathbb{E}(T_n) &= n \log_2 n P_1(\log_2 n) + \mathcal{O}(n) \\
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$$P_1(z) = \frac{\gamma - 1 + \lim_{\omega\to 2}\frac{G_1(\omega)-1}{(\omega-2)}}{\log 2} + \frac{1}{2} + \frac{1}{\log 2}\sum_{k\neq 0}\frac{G_1(2+\chi_k)}{\Gamma(2+\chi_k)}e^{2k\pi i}$$

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$$

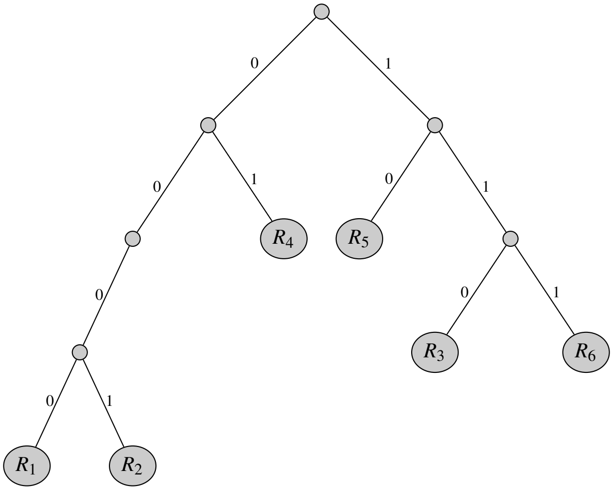
$P_2(z)$ is another periodic function with complicated expression.

Node-wise Wiener Index of Bucket DST

Let N_n be the random variable of number of nodes in a Bucket DST, the same method gave us

$$
\begin{aligned}
\mathbb{E}(N_n) =& nP_1(\log_2 n) + \mathcal{O}(1) \\
\mathbb{E}(T_n) =& n\log_2 nP_1(\log_2 n) + \mathcal{O}(n) \\
\mathbb{E}(W_n) =& n^2\log_2 nP_1(\log_2 n)^2 + \mathcal{O}(n^2) \\
\mathrm{Var}(N_n) =& nP_2(\log_2 n) + \mathcal{O}(1) \\
\mathrm{Cov}(N_n, T_n) =& n\log_2 nP_2(\log_2 n) + \mathcal{O}(n) \\
\mathrm{Cov}(N_n, W_n) =& 2n^2\log_2 nP_1(\log_2 n)P_2(\log_2 n) + \mathcal{O}(n) \\
\mathrm{Var}(T_n) =& n(\log_2 n)^2P_2(\log_2 n) + \mathcal{O}(n\log n) \\
\mathrm{Cov}(T_n, W_n) =& 2n^2(\log_2 n)^2P_1(\log_2 n)P_2(\log_2 n) + \mathcal{O}(n^2\log n) \\
\mathrm{Var}(W_n) =& 4n^3(\log_2 n)^2P_1(\log_2 n)^2P_2(\log_2 n) + \mathcal{O}(n^3\log n)
\end{aligned}
$$

Tries



$$R_1 = 000001\cdots$$
$$R_2 = 000110\cdots$$
$$R_3 = 110111\cdots$$
$$R_4 = 011011\cdots$$
$$R_5 = 100001\cdots$$
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PATRICIA Tries

A space-optimized trie data structure where each node with only one child is merged with its child. Some researchers call it radix tree, radix trie or compact prefix tree.

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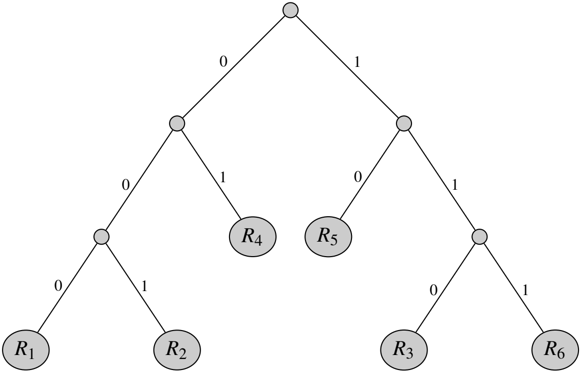

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Internal and External Wiener Index of Tries and PATRICIA Tries

External Wiener index of tries and PATRICIA Tries followed the same pattern of key-wise Wiener index of Bucket-DST while internal Wiener index being similar to the node-wise Wiener index of Bucket DST. **The only difference is that the periodic functions are different.**

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Theorem (Fuchs, Lee)

$$\left(\frac{N_n - \mathbb{E}(N_n)}{\sqrt{\mathrm{Var}(N_n)}}, \frac{T_n - \mathbb{E}(T_n)}{\sqrt{\mathrm{Var}(T_n)}}, \frac{W_n - \mathbb{E}(W_n)}{\sqrt{\mathrm{Var}(W_n)}}\right) \xrightarrow{d} (X, X, X)$$

where X is a standard normal distributed random variable and $\xrightarrow{d}$ denote weak convergence.

Summary and Perspective

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- Asymmetric case can also be solved by the same method.
- Wiener index is in fact a special case of **Steiner** k**-distance**, which is also called Steiner index, with $k = 2$.

國科會補助計畫衍生研發成果推廣資料表

日期:2013/08/26

<table>
<tr><td rowspan="3">國科會補助計畫</td><td>計畫名稱: 正規勞倫級數之度量非齊次丟番圖逼近</td></tr>
<tr><td>計畫主持人: 符麥克</td></tr>
<tr><td>計畫編號: 101-2115-M-009-010- 學門領域: 代數和解析數論</td></tr>
<tr><td colspan="2">無研發成果推廣資料</td></tr>
</table>

101 年度專題研究計畫研究成果彙整表

<table>
<tr><td colspan="3">計畫主持人：符麥克</td><td colspan="5">計畫編號：101-2115-M-009-010-</td></tr>
<tr><td colspan="8">計畫名稱：正規勞倫級數之度量非齊次丟番圖逼近</td></tr>
<tr><td colspan="3" rowspan="2">成果項目</td><td colspan="3">量化</td><td rowspan="2">單位</td><td rowspan="2">備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）</td></tr>
<tr><td>實際已達成數（被接受或已發表）</td><td>預期總達成數(含實際已達成數)</td><td>本計畫實際貢獻百分比</td></tr>
<tr><td rowspan="12">國內</td><td rowspan="4">論文著作</td><td>期刊論文</td><td>0</td><td>0</td><td>100%</td><td rowspan="3">篇</td><td></td></tr>
<tr><td>研究報告/技術報告</td><td>0</td><td>0</td><td>100%</td><td></td></tr>
<tr><td>研討會論文</td><td>0</td><td>0</td><td>100%</td><td></td></tr>
<tr><td>專書</td><td>0</td><td>0</td><td>100%</td><td></td><td></td></tr>
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<tr><td>已獲得件數</td><td>0</td><td>0</td><td>100%</td><td></td></tr>
<tr><td rowspan="2">技術移轉</td><td>件數</td><td>0</td><td>0</td><td>100%</td><td>件</td><td></td></tr>
<tr><td>權利金</td><td>0</td><td>0</td><td>100%</td><td>千元</td><td></td></tr>
<tr><td rowspan="4">參與計畫人力（本國籍）</td><td>碩士生</td><td>1</td><td>0</td><td>100%</td><td rowspan="4">人次</td><td></td></tr>
<tr><td>博士生</td><td>1</td><td>0</td><td>100%</td><td></td></tr>
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</table>

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	None

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估 ■達成目標 □未達成目標（請說明，以 100 字為限） 　　□實驗失敗 　　□因故實驗中斷 　　□其他原因 說明：
2. 研究成果在學術期刊發表或申請專利等情形： 論文：■已發表 □未發表之文稿 □撰寫中 □無 專利：□已獲得 □申請中 ■無 技轉：□已技轉 □洽談中 ■無 其他：（以 100 字為限）
3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限） This project was concerned with generalizations of Kim and Nakada's recent analogue of Kurzweil's theorem in the field of formal Laurent series. Kim and Nakada's proof used continued fraction expansion which made a generalization to simultaneous Diophantine approximation complicated. We proposed a new approach which works for all dimensions. Moreover, we also considered other extensions of Kurzweil's theorem in dimension one.