

MINIMIZING SONET ADMs IN UNIDIRECTIONAL WDM RINGS WITH GROOMING RATIO SEVEN*

CHARLES J. COLBOURN[†], HUNG-LIN FU[‡], GENNIAN GE[§], ALAN C. H. LING[¶], AND
HUI-CHUAN LU^{||}

Abstract. In order to reduce the number of add-drop multiplexers (ADM) in SONET/WDM networks using wavelength add-drop multiplexing, certain graph decompositions can be used to form a “grooming” that specifies the assignment of traffic to wavelengths. When traffic among nodes is all-to-all and uniform, the drop cost of such a decomposition is the sum, over all graphs in the decomposition, of the number of vertices of nonzero degree in the graph. The number of ADMs required is this drop cost. The existence of such decompositions with minimum cost, when every pair of sites employs no more than $\frac{1}{7}$ of the wavelength capacity, is determined within an additive constant. Indeed when the number n of sites satisfies $n \equiv 1 \pmod{3}$ and $n \neq 19$, the determination is exact; when $n \equiv 0 \pmod{3}$, $n \not\equiv 18 \pmod{24}$, and n is large enough, the determination is also exact; and when $n \equiv 2 \pmod{3}$ and n is large enough, the gap between the cost of the best construction and the cost of the lower bound is independent of n and does not exceed 4.

Key words. traffic grooming, combinatorial designs, block designs, group-divisible designs, optical networks, wavelength-division multiplexing

AMS subject classifications. 68M10, 68R05

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1. Introduction. *Traffic grooming* in optical (SONET) rings arises from amalgamating C low rate signals onto a higher capacity wavelength [15, 25, 26]; C is the *grooming ratio*. Nodes initiate or terminate traffic on a wavelength using an *add-drop multiplexer* (ADM). Finding the minimum number of ADMs, $A(C, n)$, required in an n -node SONET ring with grooming ratio C , is equivalent to the following problem in graphs [4]: Given a number of nodes n and a grooming ratio C , find a partition of the edges of K_n into subgraphs B_ℓ , $\ell = 1, \dots, s$, with $|E(B_\ell)| \leq C$ such that $\sum_{1 \leq \ell \leq s} |V(B_\ell)|$ is minimum.

Optimal constructions for given grooming ratio C have been obtained using tools of graph and design theory [9]. Results are known for grooming ratio $C = 3$ [1], $C = 4$ [5, 23], $C = 5$ [3], $C = 6$ [2], $C \leq \frac{1}{6}n(n-1)$ [5], and for large values of C [5]. Related problems have been studied for variable traffic requirements [8, 14, 22, 27, 29], for fixed traffic requirements [1, 3, 4, 5, 15, 21, 23, 24, 25, 28, 30], and in the case of

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[†]Department of Computer Science and Engineering, Arizona State University, P. O. Box 878809, Tempe, AZ 85287-8809 (colbourn@asu.edu).

[‡]Department of Applied Mathematics, National Chiao Tung University, Hsin Chu, Taiwan, Republic of China (hlfu@math.nctu.edu.tw). This author’s work has been partially funded by NSC-94-2115-M009-017.

[§]Corresponding author. Department of Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, People’s Republic of China (gnge@zju.edu.cn). This author’s work has been partially funded by the National Outstanding Youth Science Foundation of China under grant 10825103, the National Natural Science Foundation of China under grant 10771193, the Zhejiang Provincial Natural Science Foundation of China, and the Program for New Century Excellent Talents in University.

[¶]Department of Computer Science, University of Vermont, Burlington, VT 05405 (aling@cems.uvm.edu).

^{||}Center of General Education, National United University, Miaoli, Taiwan, Republic of China (hht0936@seed.net.tw). This author’s work has been partially funded by NSC-96-2115-M239-002.

bidirectional rings [10, 13]. The explicit correspondence between grooming and graph decomposition is developed in detail in [1, 11].

In this paper we consider grooming with grooming ratio 7. In section 2 we employ linear programming duality to establish a general lower bound on $A(7, n)$. In section 3 we present a complete solution to the existence problem of 4-GDDs of types $24^u m^1$ and $84^u m^1$, which will be used for recursion in subsequent sections. In section 4 we determine $A(7, n)$ with the possible exception of $n = 19$ when $n \equiv 1 \pmod{3}$. When $n \equiv 0 \pmod{3}$ (section 5) we determine $A(7, n)$ with finitely many possible exceptions except when $n \equiv 18 \pmod{24}$; in the latter case we establish a construction whose cost exceeds the lower bound by 1. When $n \equiv 2 \pmod{3}$ (section 6) we develop a set of constructions to establish that, with finitely many possible exceptions, the cost does not exceed the lower bound by more than 4, independent of n .

It is natural to ask why the case when $C = 7$ is of independent interest. Unlike all cases when $C \leq 6$, the graph with the lowest ratio of number of vertices to number of edges does not have C edges; rather it is K_4 , a 6-edge graph. This necessitates consideration of decompositions that do not use the minimum number of graphs, and hence determining the minimum number of wavelengths required is quite different than determining the minimum drop cost.

2. The lower bounds. We adapt a strategy using linear programming from [12] that was used in [11] to determine both the cost and the structure of certain optimal groomings. A grooming with ratio 7 is a decomposition of K_n into subgraphs each having at most 7 edges. Its *drop cost*, or just *cost*, is the sum of the numbers of vertices of nonzero degree over all graphs in the decomposition. $A(7, n)$ is the minimum drop cost of a grooming of K_n with grooming ratio 7. Figure 1 displays all of the connected graphs having at most 7 edges. The naming convention is as follows. For each number q of edges and p of vertices, suppose that there are $\gamma_{q,p}$ nonisomorphic graphs. These are named $G_{\ell,q,p}$ for $1 \leq \ell \leq \gamma_{q,p}$.

In a decomposition, let $\alpha_{\ell,q,p}$ be the number of occurrences of $G_{\ell,q,p}$, and let $\alpha_{q,p} = \sum_{\ell=1}^{\gamma_{q,p}} \alpha_{\ell,q,p}$. Then because every edge appears in exactly one of the chosen subgraphs,

$$(1) \quad \sum_{q=1}^7 \sum_{p=1}^8 \sum_{\ell=1}^{\gamma_{q,p}} q \cdot \alpha_{\ell,q,p} = \binom{n}{2}.$$

In order to minimize drop cost, we must compute

$$(2) \quad \min \sum_{q=1}^7 \sum_{p=1}^8 \sum_{\ell=1}^{\gamma_{q,p}} p \cdot \alpha_{\ell,q,p}.$$

Figure 1 does not list disconnected graphs, but the cost of a disconnected graph is the sum of the costs of its components, so all feasible decompositions are accounted for. For every graph $G_{\ell,q,p}$, we find that $\frac{p}{q} \geq \frac{2}{3}$. Subtract $\frac{2}{3} \times (1)$ from (2) to restate the minimum drop cost $A(7, n)$ as

$$(3) \quad \frac{n(n-1)}{3} + \min \sum_{q=1}^7 \sum_{p=1}^8 \sum_{\ell=1}^{\gamma_{q,p}} \left(p - \frac{2}{3}q \right) \cdot \alpha_{\ell,q,p}.$$

In (3) the triple summation is always nonnegative; it can be zero only when all graphs are isomorphic to K_4 . However, structural restrictions can prohibit such a

Graph	$G_{l,q,p}$	deg. seq.	$\phi_{l,q,p}$	$\psi_{l,q,p}$	Graph	$G_{l,q,p}$	deg. seq.	$\phi_{l,q,p}$	$\psi_{l,q,p}$
$G_{1,7,5}$		44222	4	3.5	$G_{4,6,5}$		33321	1.5	1.5
$G_{2,7,5}$		43322	2.5	2	$G_{1,6,6}$		522111	4.5	4.5
$G_{3,7,5}$		43331	1	2	$G_{2,6,6}$		422211	4.5	4.5
$G_{4,7,5}$		33332	1	0.5	$G_{3,6,6}$		432111	3	4.5
$G_{1,7,6}$		442211	4	5	$G_{4,6,6}$		222222	6	3
$G_{2,7,6}$		522221	5.5	3.5	$G_{5,6,6}$		322221	4.5	3
$G_{3,7,6}$		422222	5.5	3.5	$G_{6,6,6}$		332211	3	3
$G_{4,7,6}$		532211	4	3.5	$G_{7,6,6}$		333111	1.5	3
$G_{5,7,6}$		432221	4	3.5	$G_{1,6,7}$		5211111	4.5	6
$G_{6,7,6}$		433211	2.5	3.5	$G_{2,6,7}$		4221111	4.5	6
$G_{7,7,6}$		332222	4	2	$G_{3,6,7}$		4311111	3	6
$G_{1,7,7}$		4421111	4	6.5	$G_{4,6,7}$		6111111	3	6
$G_{2,7,7}$		5222111	5.5	5	$G_{5,6,7}$		2222211	6	4.5
$G_{3,7,7}$		4222211	5.5	5	$G_{6,6,7}$		3222111	4.5	4.5
$G_{4,7,7}$		5321111	4	5	$G_{7,6,7}$		3321111	3	4.5
$G_{5,7,7}$		4322111	4	5	$G_{1,5,4}$		3322	2	1
$G_{6,7,7}$		4331111	2.5	5	$G_{1,5,5}$		42211	3.5	4
$G_{7,7,7}$		2222222	7	3.5	$G_{2,5,5}$		22222	5	2.5
$G_{8,7,7}$		3222221	5.5	3.5	$G_{3,5,5}$		32221	3.5	2.5
$G_{9,7,7}$		3322211	4	3.5	$G_{4,5,5}$		33211	2	2.5
$G_{10,7,7}$		3332111	2.5	3.5	$G_{1,5,6}$		421111	3.5	5.5
$G_{1,7,8}$		71111111	4	8	$G_{2,5,6}$		511111	3.5	5.5
$G_{2,7,8}$		44111111	4	8	$G_{3,5,6}$		322111	3.5	4
$G_{3,7,8}$		52211111	5.5	6.5	$G_{4,5,6}$		331111	2	4
$G_{4,7,8}$		42221111	5.5	6.5	$G_{5,5,6}$		222211	5	3
$G_{5,7,8}$		62111111	4	6.5	$G_{1,4,4}$		2222	4	2
$G_{6,7,8}$		53111111	4	6.5	$G_{2,4,4}$		3221	2.5	2
$G_{7,7,8}$		43211111	4	6.5	$G_{1,4,5}$		41111	2.5	5
$G_{8,7,8}$		22222211	7	5	$G_{2,4,5}$		22211	4	3.5
$G_{9,7,8}$		32222111	5.5	5	$G_{3,4,5}$		32111	2.5	3.5
$G_{10,7,8}$		33221111	4	5	$G_{1,3,3}$		222	3	1.5
$G_{11,7,8}$		33311111	2.5	5	$G_{1,3,4}$		2211	3	3
$G_{1,6,4}$		3333	0	0	$G_{2,3,4}$		3111	1.5	3
$G_{1,6,5}$		42222	4.5	3	$G_{1,2,3}$		211	2	2.5
$G_{2,6,5}$		43221	3	3	$G_{1,1,2}$		11	1	2
$G_{3,6,5}$		33222	3	1.5					

FIG. 1. The graphs.

selection. In particular, considering the number $\binom{n}{2}$ of edges modulo 6,

$$(4) \quad \sum_{q=1}^7 \sum_{p=1}^8 \sum_{\ell=1}^{\gamma_{q,p}} (q \bmod 6) \cdot \alpha_{\ell,q,p} \equiv \begin{cases} 0 & (\bmod 6) \text{ if } n \equiv 0, 1, 4, 9 \pmod{12}, \\ 1 & (\bmod 6) \text{ if } n \equiv 2, 11 \pmod{12}, \\ 3 & (\bmod 6) \text{ if } n \equiv 3, 6, 7, 10 \pmod{12}, \\ 4 & (\bmod 6) \text{ if } n \equiv 5, 8 \pmod{12}. \end{cases}$$

We can relax this congruence to linear inequalities. For example, if $n \equiv 3, 6, 7, 10 \pmod{12}$, then

$$(5) \quad \sum_{p=1}^8 \left[\sum_{\ell=1}^{\gamma_{3,p}} \alpha_{\ell,3,p} + \frac{1}{3} \left(\sum_{q \in \{1,4,7\}} \sum_{\ell=1}^{\gamma_{q,p}} \alpha_{\ell,q,p} \right) + \frac{2}{3} \left(\sum_{q \in \{2,5\}} \sum_{\ell=1}^{\gamma_{q,p}} \alpha_{\ell,q,p} \right) \right] \geq 1,$$

because if there is no graph on three edges, there must be at least three graphs having 1 (mod 3) edges, or one having 1 (mod 3) edges and one having 2 (mod 3) edges.

Every vertex of K_n has degree congruent to $n - 1 \pmod{3}$; placing a K_4 in the decomposition does not change this congruence class at any vertex, and hence subgraphs other than K_4 may be needed to accommodate these vertex degrees. Let $\omega_{\ell,q,p}$ be the number of vertices whose degree is congruent to 1 modulo 3 in $G_{\ell,q,p}$, and let $\tau_{\ell,q,p}$ be the number of vertices whose degree is congruent to 2 modulo 3. Now if $n \equiv 0 \pmod{3}$, then every vertex has degree 2 modulo 3, and hence at every vertex there must either be a graph itself having degree 2 modulo 3, or two graphs each having degree 1 modulo 3 (there may be more). And if $n \equiv 2 \pmod{3}$, then every vertex has degree 1 modulo 3, and hence at every vertex there must either be a graph itself having degree 1 modulo 3, or two graphs each having degree 2 modulo 3. For convenience we write $\phi_{\ell,q,p} = \frac{1}{2}\omega_{\ell,q,p} + \tau_{\ell,q,p}$ and $\psi_{\ell,q,p} = \omega_{\ell,q,p} + \frac{1}{2}\tau_{\ell,q,p}$. These are tabulated for each graph in Figure 1. We conclude that

$$(6) \quad \begin{aligned} \sum_{q=1}^7 \sum_{p=1}^8 \sum_{\ell=1}^{\gamma_{q,p}} \phi_{\ell,q,p} \cdot \alpha_{\ell,q,p} &\geq n \quad \text{if } n \equiv 0 \pmod{3}, \\ \sum_{q=1}^7 \sum_{p=1}^8 \sum_{\ell=1}^{\gamma_{q,p}} \psi_{\ell,q,p} \cdot \alpha_{\ell,q,p} &\geq n \quad \text{if } n \equiv 2 \pmod{3}. \end{aligned}$$

THEOREM 2.1. *The cost of an optimal grooming of K_n with grooming ratio 7, $A(7, n)$, is at least*

$$\begin{aligned} \frac{2}{3} \binom{n}{2} & \quad \text{if } n \equiv 1, 4 \pmod{12}, \\ \frac{2}{3} \binom{n}{2} + 1 & \quad \text{if } n \equiv 7, 10 \pmod{12}, \\ \frac{2}{3} \binom{n}{2} + \lceil \frac{n}{12} \rceil & \quad \text{if } n \equiv 0, 3, 6, 15, 18, 21 \pmod{24}, \\ \frac{2}{3} \binom{n}{2} + \lceil \frac{n}{12} \rceil + 1 & \quad \text{if } n \equiv 9, 12 \pmod{24}, \\ \lceil \frac{2}{3} \binom{n}{2} \rceil + \frac{2n}{21} & \quad \text{if } n \equiv 5, 8, 17 \pmod{21} \\ & \quad \text{or } n \equiv 2, 23, 32, 53, 56, 77, 62, 83 \pmod{84}, \\ \lceil \frac{2}{3} \binom{n}{2} \rceil + \frac{2n}{21} + 1 & \quad \text{if } n \equiv 14, 35, 20, 41, 44, 65, 74, 11 \pmod{84}. \end{aligned}$$

Proof. We follow the strategy in [12]. Form a linear program whose variables are

the $\{\alpha_{\ell,q,p}\}$ s,

$$(7) \quad \min \sum_{q=1}^7 \sum_{p=1}^8 \sum_{\ell=1}^{\gamma_{q,p}} (p - \frac{2}{3}q) \cdot \alpha_{\ell,q,p}$$

subject to (4) suitably relaxed, (6), and nonnegativity of each variable.

If z^* is the minimum, then the cost of any grooming must be at least $\lceil \frac{2}{3} \binom{n}{2} + z^* \rceil$, since the cost is integral. By forming the dual of (7), any feasible solution to the dual gives a lower bound on all primal feasible solutions, and hence a lower bound on z^* .

Case 1. $n \equiv 1 \pmod{3}$: When $n \equiv 1, 4 \pmod{12}$, the linear program is constrained only by nonnegativity, and the dual optimum is 0. When $n \equiv 7, 10 \pmod{12}$, (5) holds. Call its dual variable y_1 . An assignment y_1^* is dual feasible if $y_1^* \leq p - 2$ for every graph $G_{\ell,3,p}$; $y_1^* \leq \frac{3}{2}(p - \frac{2}{3}q)$ for every graph $G_{\ell,q,p}$ with $q \in \{2, 5\}$; and $y_1^* \leq 3(p - \frac{2}{3}q)$ for every graph $G_{\ell,q,p}$ with $q \in \{1, 4, 7\}$. By considering the graphs in Figure 1 the dual optimum of 1 occurs when $y_1^* = 1$. This raises the lower bound by 1.

Case 2. $n \equiv 0 \pmod{3}$: Consider the inequality from (6), and let y_2 be its dual variable. Each graph $G_{\ell,q,p}$ leads to the dual inequality $\phi_{\ell,q,p}y_2 \leq p - \frac{2}{3}q$. The dual optimum of $\frac{n}{12}$ arises when $y_2^* = \frac{1}{12}$; the only graph whose dual inequality is binding is $G_{1,7,5}$ with $\phi_{1,7,5} = 4$ and $5 - \frac{2}{3}7 = \frac{1}{3}$. We can compute the slackness of each variable; for $\alpha_{\ell,q,p}$, the slackness is $p - \frac{2}{3}q - \frac{1}{12}\phi_{\ell,q,p}$. A unit increase in the variable $\alpha_{\ell,q,p}$ increases the dual objective function value by the slackness. The only variables with slackness at most $\frac{1}{2}$ are $\alpha_{2,7,5}$ with slackness $\frac{1}{8}$, $\alpha_{3,7,5}$ and $\alpha_{4,7,5}$ with slackness $\frac{1}{4}$, and $\alpha_{1,5,4}$ with slackness $\frac{1}{2}$. Hence any decomposition of cost less than $\frac{n}{12} + \frac{1}{2}$ consists solely of graphs in $\{G_{\ell,7,5}\}$. To satisfy (6), $\alpha_{7,5} \geq \lceil \frac{n}{4} \rceil$. If $\alpha_{7,5} \geq \frac{n}{4} + \delta$, then adjoining this inequality with dual variable y_3 yields a dual solution $\{y_2 = 0, y_3 = \frac{1}{3}\}$ of cost $\frac{n}{12} + \frac{\delta}{3}$, increasing the bound when $\delta \geq 3$. So $\lceil \frac{n}{4} \rceil \leq \alpha_{7,5} < \frac{n}{4} + 3$. Because all of the graphs in the decomposition have six or seven edges, $\alpha_{7,5} \equiv 0 \pmod{3}$. Thus when $n \equiv 9, 12 \pmod{24}$, $\alpha_{7,5} \equiv 3 \pmod{6}$, violating (4). This increases the bound by 1 when $n \equiv 9, 12 \pmod{24}$.

Case 3. $n \equiv 2 \pmod{3}$: Again consider the inequality from (6), and let y_2 be its dual variable. Each graph $G_{\ell,q,p}$ leads to the dual inequality $\psi_{\ell,q,p}y_2 \leq p - \frac{2}{3}q$. The dual optimum of $\frac{2n}{21}$ arises when $y_2^* = \frac{2}{21}$; the only graph whose dual inequality is binding is $G_{1,7,5}$ with $\psi_{1,7,5} = \frac{7}{2}$ and $5 - \frac{2}{3}7 = \frac{1}{3}$. We can compute the slackness of each variable; for $\alpha_{\ell,q,p}$, the slackness is $p - \frac{2}{3}q - \frac{2}{21}\psi_{\ell,q,p}$. The only variables with slackness at most $\frac{4}{7}$ are $\alpha_{2,7,5}$ and $\alpha_{3,7,5}$ with slackness $\frac{1}{7}$, $\alpha_{4,7,5}$ with slackness $\frac{2}{7}$, and $\alpha_{1,5,4}$ with slackness $\frac{4}{7}$. An increase of $\frac{4}{7}$ would result in an increase in the integer ceiling when $n \equiv 2, 11, 14, 20 \pmod{21}$, so in these cases we are restricted to K_4 s and graphs in $\{G_{\ell,7,5}\}$ to meet the bound. To satisfy (6), $\alpha_{7,5} \geq \lceil \frac{2n}{7} \rceil$. If $\alpha_{7,5} \geq \frac{2n}{7} + \delta$, then adjoining this inequality with dual variable y_3 yields a dual solution $\{y_2 = 0, y_3 = \frac{1}{3}\}$ of cost $\frac{2n}{21} + \frac{\delta}{3}$, increasing the bound when $\delta \geq 3$. So $\lceil \frac{2n}{7} \rceil \leq \alpha_{7,5} < \frac{2n}{7} + 3$. Because all of the graphs in the decomposition have six or seven edges, $\alpha_{7,5} \equiv 1 \pmod{3}$. Thus when $n = 21s + x$ for $x \in \{2, 11, 14, 20\}$, $\alpha_{7,5} = 6s + 1, 6s + 4, 6s + 4, 6s + 7$, respectively. This violates (4) precisely when $n \equiv 44, 65; 11, 74; 14, 35; 20, 41 \pmod{84}$, increasing the bound by 1 in these cases. \square

We denote by $\mathcal{L}(7, n)$ the lower bound prescribed by Theorem 2.1.

3. Group divisible designs with block size four. A *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$, where X is a set of points, \mathcal{G} is a partition of X into *groups*,

and \mathcal{B} is a collection of subsets of X called *blocks* such that any pair of distinct points from X occur together either in some group or in exactly one block, but not both. A K -GDD of type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ is a GDD in which every block has size from the set K and in which there are u_i groups of size g_i , $i = 1, 2, \dots, s$.

A group divisible design $(X, \mathcal{G}, \mathcal{B})$ is *resolvable* if its block set \mathcal{B} admits a partition into *parallel classes*, each parallel class being a partition of the point set X .

A *pairwise balanced design* (PBD) with parameters $(K; v)$ is a K -GDD of type 1^v .

The interested reader may refer to [6, 9] for the undefined terms as well as a general overview of design theory. The main recursive construction that we use is Wilson's fundamental construction (WFC) for GDDs (see, e.g., [9]).

CONSTRUCTION 3.1. *Let $(X, \mathcal{G}, \mathcal{B})$ be a GDD, and let $w : X \rightarrow Z^+ \cup \{0\}$ be a weight function on X . Suppose that for each block $B \in \mathcal{B}$, there exists a K -GDD of type $\{w(x) : x \in B\}$. Then there is a K -GDD of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.*

A *double group divisible design* (DGDD) is a quadruple $(X, \mathcal{H}, \mathcal{G}, \mathcal{B})$, where X is a set of points, \mathcal{H} and \mathcal{G} are partitions of X (into holes and groups, respectively), and \mathcal{B} is a collection of subsets of X (blocks) such that

- (i) for each block $B \in \mathcal{B}$ and each hole $H \in \mathcal{H}$, $|B \cap H| \leq 1$, and
- (ii) any pair of distinct points from X which are not in the same hole occur either in some group or in exactly one block, but not both.

A K -DGDD of type $(g_1, h_1^v)^{u_1} (g_2, h_2^v)^{u_2} \cdots (g_s, h_s^v)^{u_s}$ is a DGDD in which every block has size from the set K and in which there are u_i groups of size g_i , each of which intersects each of the v holes in h_i points. (Thus, $g_i = h_i v$ for $i = 1, 2, \dots, s$. Not every DGDD can be expressed this way, of course, but this is the most general type that we require.) Thus, for example, a *modified group divisible design* (MGDD) K -MGDD of type g^u is a K -DGDD of type $(g, 1^v)^u$. A k -DGDD of type $(g, h^v)^k$ is an incomplete transversal design (ITD) $\text{ITD}(k, g; h^v)$ and is equivalent to a set of $k - 2$ holey MOLS of type h^v (see, e.g., [9]). A DGDD is *resolvable* if its block set admits a partition into parallel classes. We use the following existence result.

THEOREM 3.2 (see [20]). *There exists a 4-DGDD of type $(mt, m^t)^n$ if and only if $t, n \geq 4$ and $(t-1)(n-1)m \equiv 0 \pmod{3}$ except for $(m, n, t) = (1, 4, 6)$ and except possibly for $m = 3$ and $(n, t) \in \{(6, 14), (6, 15), (6, 18), (6, 23)\}$.*

We also make use of the following simple construction for 4-GDDs.

CONSTRUCTION 3.3 (see [19]). *Suppose that there is a 4-DGDD of type $(g_1, h_1^v)^{u_1} (g_2, h_2^v)^{u_2} \cdots (g_s, h_s^v)^{u_s}$ and that for each $i = 1, 2, \dots, s$ there is a 4-GDD of type $h_i^v a^1$ where a is a fixed nonnegative integer. Then there is a 4-GDD of type $h^v a^1$, where $h = \sum_{i=1}^s u_i h_i$.*

The following results on transversal designs (TDs) are known.

THEOREM 3.4. *A $\text{TD}(k, m)$ exists if*

1. $k = 5$ and $m \geq 4$ and $m \notin \{6, 10\}$,
2. $k = 6$ and $m \geq 5$ and $m \notin \{6, 10, 14, 18, 22\}$,
3. $k = 7$ and $m \geq 7$ and $m \notin \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60, 62\}$.

Finally, we employ the following results on 4-GDDs.

THEOREM 3.5 (see [9, IV.4, Theorem 4.8]). *A 4-GDD of type $3^u m^1$ exists if and only if either $u \equiv 0 \pmod{4}$ and $m \equiv 0 \pmod{3}$, $0 \leq m \leq (3u - 6)/2$; or $u \equiv 1 \pmod{4}$ and $m \equiv 0 \pmod{6}$, $0 \leq m \leq (3u - 3)/2$; or $u \equiv 3 \pmod{4}$ and $m \equiv 3 \pmod{6}$, $0 < m \leq (3u - 3)/2$.*

THEOREM 3.6 (see [17, Theorem 1.7]). *There exists a 4-GDD of type $g^4 m^1$ with $m > 0$ if and only if $g \equiv m \equiv 0 \pmod{3}$ and $0 < m \leq \frac{3g}{2}$.*

THEOREM 3.7 (see [18, Theorem 1.6]). *There exists a 4-GDD of type $6^u m^1$ for every $u \geq 4$ and $m \equiv 0 \pmod{3}$ with $0 \leq m \leq 3u - 3$ except for $(u, m) = (4, 0)$*

and except possibly for $(u, m) \in \{(7, 15), (11, 21), (11, 24), (11, 27), (13, 27), (13, 33), (17, 39), (17, 42), (19, 45), (19, 48), (19, 51), (23, 60), (23, 63)\}$.

THEOREM 3.8 (see [16, Theorem 3.16]). *There exists a 4-GDD of type $12^u m^1$ for each $u \geq 4$ and $m \equiv 0 \pmod{3}$ with $0 \leq m \leq 6(u-1)$.*

THEOREM 3.9 (see [16, Theorem 5.21]). *There exists a 4-GDD of type $2^u m^1$ for each $u \geq 6$, $u \equiv 0 \pmod{3}$ and $m \equiv 2 \pmod{3}$ with $2 \leq m \leq u-1$ except for $(u, m) = (6, 5)$ and except possibly for $(u, m) \in \{(21, 17), (33, 23), (33, 29), (39, 35), (57, 44)\}$.*

3.1. $g \in \{24, 84\}$.

LEMMA 3.10. *For each $u \geq 4$, $u \notin \{7, 11, 13, 17, 19, 23\}$, there exists a 4-GDD of type $24^u m^1$ with $m \equiv 0 \pmod{3}$ and $0 \leq m \leq 12(u-1)$.*

Proof. For $u = 4$, see Theorem 3.6. For each $u \geq 5$, $u \notin \{7, 11, 13, 17, 19, 23\}$, take a 4-GDD of type $6^u v^1$ with $v \equiv 0 \pmod{3}$ and $0 \leq v \leq 3(u-1)$, and remove the points on the last group of size v ; apply weight 4, using 4-MGDDs of type 4^4 and resolvable $\{3\}$ -MGDDs of type 4^3 , to obtain a $\{3, 4\}$ -DGDD of type $(24, 6^4)^u$ whose triples fall into $3v$ parallel classes. Adjoin $3v$ infinite points to complete the parallel classes, and then fill in 4-GDDs of type $6^{ut} t^1$ with $t \equiv 0 \pmod{3}$ and $0 \leq t \leq 3(u-1)$ to obtain a 4-GDD of type $24^u(3v+t)^1$, as desired. \square

LEMMA 3.11. *For each $u \in \{7, 11, 13, 17, 19, 23\}$, there exists a 4-GDD of type $24^u m^1$ with $m \equiv 0 \pmod{3}$ and $3(u-1) \leq m \leq 12(u-1)$.*

Proof. For each u , start with a TD(5, u) and adjoin an infinite point ∞ to the groups; then delete a finite point in order to form a $\{5, u+1\}$ -GDD of type $4^u u^1$. Note that each block of size $u+1$ intersects the group of size u in the infinite point ∞ and each block of size 5 intersects the group of size u , but certainly not in ∞ . Now, in the group of size u , we give ∞ weight 0 or $3(u-1)$ and give the remaining points weight 3, 6, or 9. Give all other points in the $\{5, u+1\}$ -GDD weight 6. Replace the blocks in the $\{5, u+1\}$ -GDD by 4-GDDs of types 6^u , $6^u(3(u-1))^1$, $6^4 3^1$, $6^4 6^1$, or $6^4 9^1$ to obtain the 4-GDDs, as desired. \square

LEMMA 3.12. *For each $u \in \{7, 11, 13, 17, 19, 23\}$, there exists a 4-GDD of type $24^u m^1$ with $m \equiv 0 \pmod{3}$ and $0 \leq m \leq 3(u-2)$.*

Proof. Starting from a 4-DGDD of type $(24, 6^4)^u$ coming from Theorem 3.2 and applying Construction 3.3 with 4-GDDs of type $6^u m^1$ to fill in holes, we obtain most of the designs except for $(u, m) \in \{(7, 15), (11, 21), (11, 24), (11, 27), (13, 27), (13, 33), (17, 39), (17, 42), (19, 45), (19, 48), (19, 51), (23, 60), (23, 63)\}$.

For the remaining choices for (u, m) , take a 4-GDD of type $6^u 3^1$ and remove the points of the last group of size 3; apply weight 4, using 4-MGDDs of type 4^4 and resolvable $\{3\}$ -MGDDs of type 4^3 , to obtain a $\{3, 4\}$ -DGDD of type $(24, 6^4)^u$ whose triples fall into 9 parallel classes. Adjoin $m-9$ infinite points to complete the parallel classes and then fill in 4-GDDs of type $6^u(m-9)^1$. \square

Combining Lemmas 3.10–3.12, we have the following theorem.

THEOREM 3.13. *There exists a 4-GDD of type $24^u m^1$ for each $u \geq 4$ and $m \equiv 0 \pmod{3}$ with $0 \leq m \leq 12(u-1)$.*

THEOREM 3.14. *There exists a 4-GDD of type $84^u m^1$ for each $u \geq 4$ and $m \equiv 0 \pmod{3}$ with $0 \leq m \leq 42(u-1)$.*

Proof. The proof is similar to that of Lemma 3.10. For each u , take a 4-GDD of type $12^u v^1$ with $v \equiv 0 \pmod{3}$ and $0 \leq v \leq 6(u-1)$, and remove the points on the last group of size v ; apply weight 7, using 4-MGDDs of type 7^4 and resolvable $\{3\}$ -MGDDs of type 7^3 , to obtain a $\{3, 4\}$ -DGDD of type $(84, 12^7)^u$ whose triples fall into $6v$ parallel classes. Adjoin $6v$ infinite points to complete the parallel classes, and

then fill in 4-GDDs of type $12^u t^1$ with $t \equiv 0 \pmod 3$ and $0 \leq t \leq 6(u-1)$ to obtain a 4-GDD of type $84^u(6v+t)^1$, as desired. \square

4. Constructions: $n \equiv 1 \pmod 3$. We settle the small cases first.

LEMMA 4.1. $A(7, n) = \mathcal{L}(7, n)$ for $n \in \{4, 7\}$.

Proof. The lower bound is met for $n = 4$ by a single K_4 . The lower bound is realized when $n = 7$: Let $V = \{\infty\} \cup \{0, \dots, 5\}$, and form the three $G_{1,7,5}$ s $\{\{i, i+3\}, \{i, i+1\}, \{i, i+4\}, \{i+1, i+3\}, \{i+3, i+4\}, \{\infty, i\}, \{\infty, i+3\}\}$ for $i \in \{0, 1, 2\}$, arithmetic modulo 6. \square

LEMMA 4.2. $A(7, 10) = \mathcal{L}(7, 10) + 1 = 32$.

Proof. The lower bound of 31 is not met. To see this, the only primal variables with slackness at most $\frac{1}{3}$ are for $\{G_{\ell,7,5}\}$. But $6x + 7y = 45$ and $4x + 5y = 31$ admits only the solution $x = 4$ and $y = 3$, i.e., four K_4 s and three graphs from $\{G_{\ell,7,5}\}$. There is a unique way to place four K_4 s in a K_{10} , and its complement does not partition into three graphs from $\{G_{\ell,7,5}\}$. To produce a decomposition of cost 32, on the 10 points $\{0, \dots, 9\}$ form K_4 s on $\{0, 1, 2, 3\}$ and $\{0, 4, 5, 6\}$, and form the graphs

$$\begin{aligned} G_{2,7,5} & \quad \{\{2, 4\}, \{2, 5\}, \{2, 7\}, \{2, 9\}, \{4, 7\}, \{5, 7\}, \{4, 9\}\}, \\ G_{3,7,5} & \quad \{\{3, 9\}, \{5, 9\}, \{6, 9\}, \{7, 9\}, \{3, 6\}, \{3, 7\}, \{6, 7\}\}, \\ G_{4,7,5} & \quad \{\{3, 4\}, \{3, 5\}, \{3, 8\}, \{4, 8\}, \{5, 8\}, \{1, 4\}, \{1, 5\}\}, \\ G_{4,7,5} & \quad \{\{0, 7\}, \{0, 8\}, \{0, 9\}, \{7, 8\}, \{8, 9\}, \{1, 7\}, \{1, 9\}\}, \\ G_{1,5,4} & \quad \{\{1, 8\}, \{1, 6\}, \{2, 8\}, \{2, 6\}, \{6, 8\}\}. \quad \square \end{aligned}$$

LEMMA 4.3. $\mathcal{L}(7, 19) + 1 \leq A(7, 19) \leq \mathcal{L}(7, 19) + 2 = 117$.

Proof. The lower bound of 115 cannot be met. A maximum packing on 19 points has 25 K_4 s [7]. Consider the linear program using (5). Using slackness, the only way to achieve a dual objective value of 1 in such a way that at least $21 = \binom{19}{2} - 25 \cdot 6$ edges do not appear in K_4 s is to use three graphs in $\{G_{\ell,7,5}\}$. There are 249 nonisomorphic graphs that can be left by a maximum packing of 25 K_4 s in K_{19} [2]. $G_{3,7,5}$ cannot be used because it contains a K_4 , and the 25 K_4 s form a maximum packing. Of the 249 graphs, 79 have degree sequence 3^{14} , 122 have degree sequence $6^1 3^{12}$, and 48 have degree sequence $6^2 3^{10}$. In order to use a $G_{1,7,5}$ there must be at least five vertices of degree 6 or larger; and for $G_{2,7,5}$ there must be at least three. Hence both are ruled out and the only possibility is three $G_{4,7,5}$ s. This case can be eliminated by a simple computer search. Thus the drop cost cannot be 115. A solution with drop cost 117 follows:

$$\begin{aligned} 24 K_4 \text{'s:} & \quad \{0, 1, 2, 4\}, \{0, 3, 5, 6\}, \{0, 7, 8, 9\}, \{0, 10, 11, 12\}, \{0, 13, 14, 15\}, \\ & \quad \{0, 16, 17, 18\}, \{1, 3, 7, 10\}, \{1, 5, 8, 11\}, \{1, 6, 13, 16\}, \{1, 9, 14, 17\}, \\ & \quad \{1, 12, 15, 18\}, \{2, 3, 8, 15\}, \{2, 5, 9, 18\}, \{2, 6, 10, 17\}, \{2, 7, 12, 13\}, \\ & \quad \{2, 11, 14, 16\}, \{3, 4, 14, 18\}, \{3, 9, 12, 16\}, \{4, 5, 12, 17\}, \{4, 6, 9, 15\}, \\ & \quad \{5, 10, 15, 16\}, \{6, 7, 11, 18\}, \{6, 8, 12, 14\}, \{8, 10, 13, 18\} \\ \text{one } G_{2,7,5}: & \quad \{\{3, 11\}, \{3, 13\}, \{3, 17\}, \{11, 15\}, \{11, 17\}, \{13, 17\}, \{15, 17\}\} \\ \text{two } G_{4,7,5}: & \quad \{\{4, 7\}, \{4, 8\}, \{4, 16\}, \{7, 16\}, \{7, 17\}, \{8, 16\}, \{8, 17\}\} \text{ and} \\ & \quad \{\{4, 10\}, \{4, 11\}, \{4, 13\}, \{9, 10\}, \{9, 11\}, \{9, 13\}, \{11, 13\}\} \\ \text{one } G_{7,6,6}: & \quad \{\{5, 7\}, \{5, 13\}, \{5, 14\}, \{7, 14\}, \{7, 15\}, \{10, 14\}\}. \quad \square \end{aligned}$$

THEOREM 4.4. When $n \equiv 1 \pmod 3$ and $n \notin \{10, 19\}$, $A(7, n) = \mathcal{L}(7, n)$. Moreover, $A(7, 10) = \mathcal{L}(7, 10) + 1$ and $\mathcal{L}(7, 19) + 1 \leq A(7, 19) \leq \mathcal{L}(7, 19) + 2$.

Proof. When $n \equiv 1, 4 \pmod{12}$, there is a 4-GDD of type 1^n with drop cost $\mathcal{L}(7, n)$. When $n \equiv 7, 10 \pmod{12}$ and $n \notin \{10, 19\}$, there is a 4-GDD of type $1^{n-7} 7^1$ [7]; fill the hole with a solution from Lemma 4.1. \square

5. Constructions: $n \equiv 0 \pmod{3}$. The lower bound is met for $n = 3$ by a single K_3 .

LEMMA 5.1. $A(7, 6) = \mathcal{L}(7, 6) + 1 = 12$.

Proof. The lower bound of 11 is not met. A decomposition of cost 12 can be produced as follows:

$$\begin{aligned} G_{2,7,5} & \{\{0, 1\}, \{0, 2\}, \{0, 4\}, \{0, 5\}, \{1, 4\}, \{1, 5\}, \{2, 4\}\}, \\ G_{2,7,5} & \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}, \\ G_{1,1,2} & \{\{0, 3\}\}. \quad \square \end{aligned}$$

LEMMA 5.2. $A(7, 9) = \mathcal{L}(7, 9) + 1 = 27$.

Proof. The lower bound of 26 is not met for $n = 9$ as follows. There can be at most three K_4 s on nine points. If there are zero, then at least six graphs are needed, each having a slackness of at least $\frac{1}{3}$; because the total increase in the dual objective function is 2, all of the graphs must be from $\{G_{\ell,7,5}\}$ and cannot account for 36 edges. In the same manner, with one K_4 , 30 edges must be accounted for by graphs in $\{G_{\ell,7,5}\}$, each with a slackness of $\frac{1}{3}$ and $G_{1,5,4}$ with a slackness of $\frac{2}{3}$; again this is not possible as 25 is not a multiple of 7. There remain cases with two or three K_4 s; each can be eliminated by an exhaustive search.

A decomposition of cost 27 using graphs on at most six edges is given in [2]. We give a different solution here:

$$\begin{aligned} G_{1,7,5} & \{\{0, 7\}, \{0, 8\}, \{1, 7\}, \{1, 8\}, \{2, 7\}, \{2, 8\}, \{7, 8\}\}, \\ G_{4,7,5} & \{\{0, 4\}, \{0, 5\}, \{0, 6\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{4, 5\}\}, \\ G_{4,7,5} & \{\{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\}, \{4, 6\}\}, \\ G_{4,7,5} & \{\{4, 7\}, \{4, 8\}, \{5, 6\}, \{5, 7\}, \{5, 8\}, \{6, 7\}, \{6, 8\}\}, \\ G_{1,6,4} & \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}, \\ G_{1,2,3} & \{\{3, 7\}, \{3, 8\}\}. \quad \square \end{aligned}$$

LEMMA 5.3. $A(7, 15) = \mathcal{L}(7, 15) = 72$.

Proof. Start with a Kirkman triple system of order 9 on $\{0, \dots, 8\}$, in which the first parallel class is $\{B_0, B_1, B_2\}$. Then adjoin points $\{x_0, x_1, x_2, y_0, y_1, y_2\}$. Form nine K_4 s by adding y_i to each block of the $(i+2)$ nd parallel class. For $i \in \{0, 1, 2\}$, form a K_4 on $\{x_{i+2}\} \cup B_i$ and a $G_{1,7,5}$ in which the degree 4 vertices are x_i and x_{i+1} and the degree 2 vertices are the elements of B_i . Form a K_4 on $\{x_2, y_0, y_1, y_2\}$. What remains is a $G_{3,6,5}$. \square

LEMMA 5.4. $A(7, 18) \leq \mathcal{L}(7, 18) + 1 = 105$.

Proof. Form a 4-GDD of type 3^5 with groups $\{B_j : j = 0, 1, 2, 3, 4\}$. Then adjoin points $\{x_0, x_1, x_2\}$. For $i \in \{0, 1, 2\}$, form a $G_{1,7,5}$ by using the edge $\{x_i, x_{i+1 \bmod 3}\}$, join these vertices to each vertex in B_i , and form a K_4 by adding $x_{i+2 \bmod 3}$ to B_i . For $i \in \{3, 4\}$, form a $G_{3,6,5}$ by joining the vertices x_0 and x_1 to vertices in B_i , and form a K_4 by adding x_2 to B_i . This decomposition is of cost 105. \square

LEMMA 5.5. $A(7, 24) = \mathcal{L}(7, 24) = 186$.

Proof. We give the solution on $\{0, 1, 2, 3, 4, 5, 6, 7\} \times \mathbb{Z}_3$, writing element (i, j) as i_j .

$$\begin{aligned} (0_0, 0_1, 1_0, 4_2), & \quad (0_0, 1_1, 5_0, 6_1), & \quad (0_0, 2_0, 3_1, 3_2), & \quad (0_0, 2_1, 5_1, 5_2), \\ (0_0, 2_2, 7_0, 7_2), & \quad (0_0, 6_0, 6_2, 7_1), & \quad (1_0, 1_1, 2_1, 7_0), & \quad (1_0, 2_2, 5_1, 6_1), \\ (1_0, 3_1, 5_0, 7_1), & \quad (1_0, 3_2, 4_1, 6_2), & \quad (2_0, 2_1, 4_2, 6_1), & \quad (3_0, 5_0, 6_2, 7_2), \\ (4_0, 4_1, 5_2, 7_2), & \quad (3_0, 4_0 : 0_0, 1_0, 2_0), & \quad (3_0, 4_1 : 5_1, 6_1, 7_1). \end{aligned}$$

The latter two orbits are graphs isomorphic to $G_{1,7,5}$. \square

THEOREM 5.6. $A(7, n) = \mathcal{L}(7, n)$ when $n \equiv 0 \pmod{3}$, $n \not\equiv 18 \pmod{24}$, and

1. $n \geq 96$ when $n \equiv 0, 3, 6, 9, 15 \pmod{24}$;
2. $n \geq 276$ when $n \equiv 12 \pmod{24}$;
3. $n \geq 309$ when $n \equiv 21 \pmod{24}$.

$\mathcal{L}(7, n) \leq A(7, n) \leq \mathcal{L}(7, n) + 1$ when $n \equiv 18 \pmod{24}$ and $n \geq 114$.

Proof. If $m = n \pmod{24} \in \{0, 3, 6, 9, 15, 18\}$ and $n \geq 96$, form a 4-GDD of type $24^{(n-m)/24}m^1$ from Theorem 3.13; place optimal groomings from Lemma 5.5 on each group of size 24 and an optimal grooming of size m on the exceptional group (from Lemmas 5.1, 5.2, or 5.3 when $m = 6, 9, 15$, respectively). When $m = 18$, use the grooming from Lemma 5.4, missing the lower bound by 1. When $m = 6$, reduce the drop cost by 1 by amalgamating the single edge from this grooming with a K_4 of the 4-GDD to form a $G_{3,7,5}$. When $m = 9$, reduce the drop cost by 1 by amalgamating both edges of the $G_{1,3,2}$ of this grooming with K_4 s of the 4-GDD to form $G_{3,7,5}$.

When $m = n \pmod{24} = 12$, form a 4-GDD of type 20^4 and add four infinite points. On each group, together with the four infinite points, place an optimal grooming from Lemma 5.5 aligning a K_4 on the four infinite points. Suppress the duplicate K_4 s so produced. This establishes that $\mathcal{L}(7, 84) = A(7, 84)$. Then filling groups in a 4-GDD of type $24^t 84^1$ establishes that $A(7, 24t + 84) = \mathcal{L}(7, 24t + 84)$ when $t \geq 8$, i.e., for all $n \geq 276$.

When $m = n \pmod{24} = 21$, form a 4-GDD of type 23^4 and add one infinite point. On each group, together with the infinite point, place an optimal grooming from Lemma 5.5. This establishes that $\mathcal{L}(7, 93) = A(7, 93)$. Then filling groups in a 4-GDD of type $24^t 93^1$ establishes that $A(7, 24t + 93) = \mathcal{L}(7, 24t + 93)$ when $t \geq 9$, i.e., for all $n \geq 309$. \square

6. Constructions: $n \equiv 2 \pmod{3}$.

LEMMA 6.1. $A(7, n) = \mathcal{L}(7, n)$ for $n \in \{5, 8\}$.

Proof. For K_5 , note that $G_{1,7,5} \equiv K_5 \setminus K_3$. Partition K_8 is as follows:

$$\begin{aligned} G_{1,7,5} & \quad \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}\}, \\ G_{1,7,5} & \quad \{\{6, 7\}, \{6, 2\}, \{6, 3\}, \{6, 4\}, \{7, 2\}, \{7, 3\}, \{7, 4\}\}, \\ G_{3,7,5} & \quad \{\{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}, \\ G_{4,7,5} & \quad \{\{1, 6\}, \{1, 7\}, \{0, 5\}, \{0, 6\}, \{0, 7\}, \{5, 6\}, \{5, 7\}\}. \quad \square \end{aligned}$$

LEMMA 6.2. $A(7, 11) = \mathcal{L}(7, 11) = 39$.

Proof. Partition K_{11} on $\{\infty_1, \infty_2\} \cup (\mathbb{Z}_3 \times \mathbb{Z}_3)$ as follows. Include the K_4 $\{\infty_2, 0_2, 1_2, 2_2\}$. Form three $G_{2,7,5}$ s as $\{\{i_0, (i+1)_1\}, \{i_0, (i+2)_1\}, \{i_0, (i+1)_2\}, \{i_0, (i+2)_2\}, \{(i+1)_1, (i+2)_1\}, \{(i+1)_1, (i+2)_2\}, \{(i+2)_1, (i+1)_2\}\}$ for $i \in \{0, 1, 2\}$. Then include three $G_{3,7,5}$ s as $\{\{\infty_1, i_0\}, \{\infty_1, i_1\}, \{\infty_1, i_2\}, \{i_0, i_1\}, \{i_0, i_2\}, \{i_1, i_2\}, \{\infty_2, i_1\}\}$ for $i \in \{0, 1, 2\}$. Include one last $G_{3,7,5}$: $\{\{\infty_1, \infty_2\}, \{\infty_2, 0_0\}, \{\infty_2, 1_0\}, \{\infty_2, 2_0\}, \{0_0, 1_0\}, \{0_0, 2_0\}, \{1_0, 2_0\}\}$. \square

LEMMA 6.3. $A(7, 17) \leq \mathcal{L}(7, 17) + 1 = 94$.

Proof. Start with an $S(2, 4, 16)$ on $\mathbb{Z}_{15} \cup \{\infty\}$ with blocks $\{i, i+1, i+3, i+7\}$ for $i \in \mathbb{Z}_{15}$ and $\{\infty, i, i+5, i+10\}$ for $i \in \{0, 1, 2, 3, 4\}$. We adjoin a new point α and modify six of the blocks in the first orbit as follows:

Block	Remove	Add
$\{5, 6, 8, 12\}$	$\{8, 12\}$	$\{\alpha, 5\}, \{\alpha, 8\}$
$\{7, 8, 10, 14\}$	$\{8, 14\}$	$\{\alpha, 7\}, \{\alpha, 10\}$
$\{0, 8, 9, 11\}$	$\{0, 8\}$	$\{\alpha, 0\}, \{\alpha, 9\}$
$\{3, 11, 12, 14\}$	$\{12, 14\}$	$\{\alpha, 3\}, \{\alpha, 12\}$
$\{0, 4, 12, 13\}$	$\{0, 12\}$	$\{\alpha, 4\}, \{\alpha, 13\}$
$\{0, 2, 6, 14\}$	$\{0, 14\}$	$\{\alpha, 2\}, \{\alpha, 14\}$

Now add the K_4 on $\{0, 8, 12, 14\}$. Then delete the K_4 on $\{\infty, 1, 6, 11\}$; on $\{\alpha, \infty, 1, 6, 11\}$, place a K_3 and a $G_{1,7,5}$. The result has 14 K_4 s, one K_3 , and seven graphs in $\{G_{\ell,7,5}\}$. \square

LEMMA 6.4. When $n \equiv 2 \pmod{6}$ and $n \geq 14$, $A(7, n) \leq \frac{2}{3}\binom{n}{2} + \frac{n}{6} = \frac{2}{3}\binom{n}{2} + \frac{2n}{21} + \frac{n}{14}$.

Proof. Write $h = \frac{n}{2}$. When $h \equiv 1 \pmod{3}$ and $h \geq 7$, a 4-GDD of type 2^h exists by Theorem 3.9. It has h groups and $\frac{h(h-1)}{3}$ blocks. For each group, choose a distinct block containing one point of the group (this is an easy exercise using systems of distinct representatives). Then adjoin the pair of each group to its corresponding block to obtain a $G_{3,7,5}$. \square

LEMMA 6.5. When $n \equiv 5 \pmod{6}$ and $n \geq 23$, $A(7, n) \leq \frac{2}{3}\binom{n}{2} + \frac{2n}{21} + \frac{n+7}{14}$.

Proof. Write $h = \frac{n-5}{2}$. When $h \equiv 0 \pmod{3}$ and $h \geq 9$, a 4-GDD of type $2^h 5^1$ exists by Theorem 3.9. For each group of size 2, choose a distinct block containing one point of the group and adjoin the pair of each group to its corresponding block to obtain a $G_{3,7,5}$. Then fill the group of size 5 using a solution from Lemma 6.1. \square

In order to treat larger cases, we now develop a recursion.

LEMMA 6.6. There exists a decomposition of K_{21} into nine partial parallel classes of K_3 s and six $G_{1,7,5}$ s.

Proof. We present a solution on $\{0, 1, \dots, 20\}$ with rows as partial parallel classes:

0 2 13	1 12 15	9 14 17	3 10 20	4 5 19	7 11 16	6 8 18
0 18 20	1 2 16	11 17 19	3 12 13	4 7 8	6 9 10	5 14 15
0 1 11	13 17 18	3 9 16	4 12 14	7 10 19	2 5 6	
0 3 5	1 8 17	4 13 16	7 9 20	6 11 15	2 10 14	
0 8 14	1 5 20	2 3 17	4 10 15	6 13 19	11 12 18	
0 9 15	1 13 14	3 18 19	4 6 20	2 7 12	5 8 16	
0 10 16	1 9 19	12 17 20	3 8 15	2 4 11	5 7 18	
0 12 19	1 10 18	15 16 17	6 7 14	2 8 9	11 13 20	
5 10 17	3 11 14	4 9 18	7 13 15	6 12 16	8 19 20.	

The remaining edges partition into six $G_{1,7,5}$ s: $\{\{7i+j, 7i+j+2\}, \{7i+j, 7i+4\}, \{7i+j, 7i+5\}, \{7i+j, 7i+6\}, \{7i+j+2, 7i+4\}, \{7i+j+2, 7i+5\}, \{7i+j+2, 7i+6\}\}$ for $j \in \{0, 1\}$ and $i \in \{0, 1, 2\}$. \square

We denote by $X(n)$ the excess over the lower bound, i.e., $X(n) = A(7, n) - \mathcal{L}(7, n)$.

THEOREM 6.7. Let $(V, \mathcal{G}, \mathcal{B})$ be a resolvable group-divisible design of type 7^n , in which the blocks of \mathcal{B} are partitioned into parallel classes $\mathcal{P}_1, \dots, \mathcal{P}_s$, and for $1 \leq i \leq s$ every block of \mathcal{P}_i has size k_i . Suppose that, for $1 \leq i \leq s$, a 4-GDD of type $3^{k_i} \sigma_i^1$ exists, and that $\sum_{i=1}^s \sigma_i > 0$. Then

$$A\left(7, 21n + 8 + \sum_{i=1}^s \sigma_i\right) \leq \mathcal{L}\left(7, 21n + 8 + \sum_{i=1}^s \sigma_i\right) + X\left(8 + \sum_{i=1}^s \sigma_i\right).$$

Proof. Suppose, without loss of generality, that $\sigma_1 > 0$. Give weight three to each point of the GDD $(V, \mathcal{G}, \mathcal{B})$. For $2 \leq i \leq s$, adjoin σ_i new infinite points, and place a 4-GDD of type $3^{k_i} \sigma_i^1$ on the inflation of each block of \mathcal{P}_i together with these infinite points. Then proceed similarly for \mathcal{P}_1 , but adding only $\sigma_1 - 1$ infinite points; in the 4-GDD, delete one point in the group of size σ_1 to form a $\{3, 4\}$ -GDD of type $3^{k_1}(\sigma_1 - 1)^1$ in which the blocks of size three form a (frame) parallel class on the $3k_i$ points. On each inflation of a group form a copy of the 21-point design from Lemma 6.6. The nine partial parallel classes of blocks of size 3 formed can be

completed to nine parallel classes on the $21n$ points using the triples from the $\{3, 4\}$ -GDDs. Finally, add nine further infinite points and extend each of the nine parallel classes to K_4 s using these infinite points. The resulting design has a hole on the $8 + \sum_{i=1}^s \sigma_i$ infinite points added in total, which can be filled with a solution of cost $A(7, 8 + \sum_{i=1}^s \sigma_i)$. \square

COROLLARY 6.8.

1. $X(92) \leq X(29)$.
2. For $n \in \{11, 14, 17, 20, 23, 26, 29\}$, $X(84 + n) \leq X(n)$.
3. For $n \in \{14, 20, 26, 32, 38, 44, 50\}$, $X(105 + n) \leq X(n)$.
4. For $29 \leq n \leq 71$ and $n \equiv 2 \pmod{3}$, $X(147 + n) \leq X(n)$.

Proof. Apply Theorem 6.7 using an $\text{RTD}(k, 7)$ with $k = 3, 4, 5, 7$ as a resolvable GDD of type 7^k with $s = 7$ and $k_1 = \dots = k_7 = k$. \square

COROLLARY 6.9.

1. For $29 \leq n \leq 80$ and $n \equiv 2 \pmod{3}$, $X(168 + n) \leq X(n)$.
2. For $32 \leq n \leq 92$ and $n \equiv 2 \pmod{6}$, $X(189 + n) \leq X(n)$.
3. For $41 \leq n \leq 107$ and $n \equiv 5 \pmod{6}$, $X(231 + n) \leq X(n)$.
4. For $44 \leq n \leq 134$ and $n \equiv 2 \pmod{6}$, $X(273 + n) \leq X(n)$.
5. For $53 \leq n \leq 164$ and $n \equiv 2 \pmod{3}$, $X(336 + n) \leq X(n)$.

Proof. Apply Theorem 6.7 using an $\text{RTD}(7, n)$ with $n = 8, 9, 11, 13, 16$ as a resolvable GDD of type 7^n with $s = n$ and $k_1 = \dots = k_{n-1} = 7$ and $k_n = n$. \square

THEOREM 6.10. For $x \geq 4$, $0 \leq m \leq 42(x - 1)$, $m \equiv 0 \pmod{3}$, and $r \in \{11, 14, 17, 20, 23, 26, 29\}$,

$$A(7, 84x + m + r) \leq \mathcal{L}(7, 84x + m + r) + X(m + r).$$

Equivalently, $X(84x + m + r) \leq X(m + r)$.

Proof. Form a 4-GDD of type $84^x m^1$ from Theorem 3.14. Adjoin r infinite points, and place a solution on each group of size 84 together with the r points, leaving a hole on the r points (from Corollary 6.8(2.)). On the $m + r$ points, place a solution with excess $X(m + r)$. \square

THEOREM 6.11. For $m \equiv 2 \pmod{3}$ and $2 \leq m \leq 83$, $\mathcal{L}(7, 84x + m) \leq A(7, 84x + m) \leq \mathcal{L}(7, 84x + m) + X(84x + m)$, where $X(84x + m)$ is given in Table 1 (using the final bold entry for $X(84x + m)$ in the row for m when the table does not provide a value). In particular, $A(7, 84x + m) \leq \mathcal{L}(7, 84x + m) + 4$ when $84x + m > 1094$.

Proof. Apply Lemmas 6.1, 6.2, and 6.3 for $x = 0$ and $m \in \{5, 8, 11, 17\}$; then apply Lemmas 6.4 and 6.5 to provide an upper bound on $X(84x + m)$ in general. Now apply Corollaries 6.8 and 6.9 to improve these upper bounds. Finally, apply Theorem 6.10. \square

7. Conclusions. Grooming with ratio 7 corresponds to the smallest ratio C for which optimal groomings do not consist primarily of C -edge graphs. Consequently, optimal grooming focuses on packings with K_4 s in this case. Despite this, the structures of the edges not appearing in K_4 s appear to exhibit patterns that repeat modulo 12, 24, and 84 when $n \equiv 1, 0, 2 \pmod{3}$, respectively. In the latter case, techniques for constructing optimal groomings in all cases would necessitate the direct construction of many “small” groomings. Therefore in this paper, we have instead found near-optimal groomings in which the construction deviates from the lower bound by a fixed constant independent of n . When $n \equiv 0, 1 \pmod{3}$, much more complete characterizations are given. Our conjecture is that, with few small exceptions, the lower bound proved here provides the correct cost of an optimal grooming.

TABLE 1
Least excesses for $84x + m$.

m	x													
	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2	1	6	12	18	2	6	6	6	6	6	6	6	6	2
5	0	6	12	4	4	6	6	6	6	6	4			
8	0	2	2	18	24	2								
11	0	0	12	4	0									
14	0	0	2	18	0									
17	1	1	13	5	1									
20	0	0	2	2	0									
23	2	2	14	6	2									
26	1	1	3	3	1									
29	2													
32	2	8	2	4	2									
35	2	0	2	20	0									
38	2	8	2	4	2									
41	2	0	2	20	0									
44	2	8	2	4	2									
47	3	1	3	21	1									
50	3	9	3	5	3									
53	4	2	2	22	4	2								
56	4	10	4	6	4									
59	4	2	2	22	4	2								
62	4	10	4	6	4									
65	4	2	2	2	4	2								
68	4	10	4	6	4									
71	5	3	3	3	5	3								
74	4	10	4	0	4	4	4	4	4	4	4	4	0	
77	6	12	4	4	6	6	6	6	6	4				
80	5	11	5	1	5	5	5	5	5	5	5	5	1	
83	6	12	4	4	6	6	6	6	6	4				

REFERENCES

- [1] J.-C. BERMOND AND S. CEROI, *Minimizing SONET ADMs in unidirectional WDM ring with grooming ratio 3*, Networks, 41 (2003), pp. 83–86.
- [2] J.-C. BERMOND, C. J. COLBOURN, D. COUDERT, G. GE, A. C. H. LING, AND X. MUÑOZ, *Traffic grooming in unidirectional wavelength-division multiplexed rings with grooming ratio $C = 6$* , SIAM J. Discrete Math., 19 (2005), pp. 523–542.
- [3] J.-C. BERMOND, C. J. COLBOURN, A. C. H. LING, AND M.-L. YU, *Grooming in unidirectional rings: $K_4 - e$ designs*, Discrete Math., 284 (2004), pp. 57–62.
- [4] J.-C. BERMOND AND D. COUDERT, *Traffic grooming in unidirectional WDM ring networks using design theory*, in Proceedings of the IEEE ICC, Anchorage, AK, 2003, pp. 1402–1406.
- [5] J.-C. BERMOND, D. COUDERT, AND X. MUÑOZ, *Traffic grooming in unidirectional WDM ring networks: The all-to-all unitary case*, in Proceedings of the 7th IFIP Working Conference on Optical Network Design & Modelling – ONDM’03, 2003, pp. 1135–1153.
- [6] T. BETH, D. JUNGnickel, AND H. LENZ, *Design Theory*, Bibliographisches Institut, Mannheim, Zurich, 1985.
- [7] A. E. BROUWER, *Optimal packings of K_4 ’s into a K_n* , J. Combin. Theory Ser. A, 26 (1979), pp. 278–297.
- [8] A. L. CHIU AND E. H. MODIANO, *Traffic grooming algorithms for reducing electronic multiplexing costs in WDM ring networks*, IEEE/OSA Journal of Lightwave Technology, 18 (2000), pp. 2–12.
- [9] C. J. COLBOURN AND J. H. DINITZ, EDs., *Handbook of Combinatorial Designs*, 2nd ed., CRC/Chapman and Hall, London, 2007.
- [10] C. J. COLBOURN AND A. C. H. LING, *Wavelength add-drop multiplexing and minimizing SONET ADMs*, Discrete Math., 261 (2003), pp. 141–156.
- [11] C. J. COLBOURN, G. QUATTROCCHI, AND V. R. SYROTIUK, *Grooming for two-period optical networks*, Networks, to appear.

- [12] C. J. COLBOURN, G. QUATTROCCHI, AND V. R. SYROTIUK, *Lower bounds for graph decompositions via linear programming duality*, Networks, to appear.
- [13] C. J. COLBOURN AND P.-J. WAN, *Minimizing drop cost for SONET/WDM networks with $\frac{1}{8}$ wavelength requirements*, Networks, 37 (2001), pp. 107–116.
- [14] R. DUTTA AND N. ROUSKAS, *On optimal traffic grooming in WDM rings*, IEEE J. Sel. Areas Commun., 20 (2002), pp. 110–121.
- [15] R. DUTTA AND N. ROUSKAS, *Traffic grooming in WDM networks: Past and future*, IEEE Network, 16 (2002), pp. 46–56.
- [16] G. GE AND A. C. H. LING, *Group divisible designs with block size four and group type $g^u m^1$ for small g* , Discrete Math., 285 (2004), pp. 97–120.
- [17] G. GE AND R. S. REES, *On group-divisible designs with block size four and group-type $g^u m^1$* , Des. Codes Cryptogr., 27 (2002), pp. 5–24.
- [18] G. GE AND R. S. REES, *On group-divisible designs with block size four and group-type $6^u m^1$* , Discrete Math., 279 (2004), pp. 247–265.
- [19] G. GE, R. S. REES, AND L. ZHU, *Group-divisible designs with block size four and group-type $g^u m^1$ with m as large or as small as possible*, J. Combin. Theory Ser. A, 98 (2002), pp. 357–376.
- [20] G. GE AND R. WEI, *HGDDs with block size four*, Discrete Math., 279 (2004), pp. 267–276.
- [21] O. GERSTEL, R. RAMASWANI, AND G. SASAKI, *Cost-effective traffic grooming in WDM rings*, IEEE/ACM Trans. Net., 8 (2000), pp. 618–630.
- [22] O. GOLDSCHMIDT, D. HOCHBAUM, A. LEVIN, AND E. OLINICK, *The SONET edge-partition problem*, Networks, 41 (2003), pp. 13–23.
- [23] J. Q. HU, *Optimal traffic grooming for wavelength-division-multiplexing rings with all-to-all uniform traffic*, J. Opt. Netw., 1 (2002), pp. 32–42.
- [24] J. Q. HU, *Traffic grooming in WDM ring networks: A linear programming solution*, J. Opt. Netw., 1 (2002), pp. 397–408.
- [25] E. MODIANO AND P. LIN, *Traffic grooming in WDM networks*, IEEE Commun. Mag., 39 (2001), pp. 124–129.
- [26] A. SOMANI, *Survivable traffic grooming in WDM networks*, in Broad Band Optical Fiber Communications Technology—BBOFCT, D. K. Gautam, ed., Jalgaon, India, 2001, pp. 17–45.
- [27] P.-J. WAN, G. CALINESCU, L. LIU, AND O. FRIEDER, *Grooming of arbitrary traffic in SONET/WDM BLSRs*, IEEE J. Sel. Areas Commun., 18 (2000), pp. 1995–2003.
- [28] J. WANG, W. CHO, V. VEMURI, AND B. MUKHERJEE, *Improved approaches for cost-effective traffic grooming in WDM ring networks: ILP formulations and single-hop and multihop connections*, IEEE/OSA J. Lightwave Tech., 19 (2001), pp. 1645–1653.
- [29] X. YUAN AND A. FULAY, *Wavelength assignment to minimize the number of SONET ADMs in WDM rings*, in Proceedings of the IEEE ICC, New York, 2002.
- [30] X. ZHANG AND C. QIAO, *An effective and comprehensive approach for traffic grooming and wavelength assignment in SONET/WDM rings*, IEEE/ACM Trans. Net., 8 (2000), pp. 608–617.