（二）背景及目的

有些化學工廠的廢水經由水井排放到地底，地面的有毒廢棄物也會溶入雨水而進入地底，核能電廠儲存在地底下的核廢棄物因為時間過長而容器腐蝕、或因為地層變動造成容器破裂等因素，核廢棄物也因此進入地下水。這些都會造成日常飲用水的不安全。這些問題不只發生在臺灣，世界其它國家也有同樣的情形。歐美有些國家甚至已設立專責機構負責這些污染源的清除工作。此外身體內的血液在血管中的流動，地底下的碳氫化合物的抽取等。以上這些都屬於多孔介質中的多相流問題[4,8,10,34]。因此了解多孔介質中的多相流的變化對解決很多實際問題是很有幫助的。

一般討論多孔介質中多相流的變化的數學模式都是由 Darcy law 與 transport equation 所組成[5,6,7,9,12,13,14,15]。前者為橢圓方程式而後者為拋物線方程式，在高度非均勻多孔介質中以上兩個方程式都是屬於非均勻的情形。換句話說，一個為非均勻橢圓方程式另一個為 非均勻拋物線方程式。理論分析的方法很難得到有實質義意的結果[1,2,3,24,28,30,31,35,37,38,42,43,44,45]，因此我們計劃探討數值計算方法來了解非均勻橢圓方程式與 非均勻拋物線方程式的特性。

這是一個二年期的計劃。我們希望探討計算非均勻橢圓方程式與 非均勻拋物線方程式的方法及這計算方法的誤差估計問題。假設 ε 代表一個很小的數字，Y_m 是一個以原點為中心半徑小於 1/2 的球，$D_m^\varepsilon = \sum \varepsilon(Y_m + j)$, $j \in Z^n$ ，$D_f^\varepsilon = R^n - D_m^\varepsilon$，$\Omega_f^\varepsilon = \Omega \bigcap D_f^\varepsilon$，$\Omega_m^\varepsilon = \Omega \bigcap D_m^\varepsilon$。 Denote absolute permeability by K_ε in Ω_f^ε and k_ε in Ω_m^ε, phase pressure by P_ε in Ω_f^ε and p_ε in Ω_m^ε, and external source by $Q_\varepsilon, F_\varepsilon$ in Ω_f^ε and $q_\varepsilon, f_\varepsilon$ in Ω_m^ε. 我們希望考慮的橢圓方程式為[11,16,21,22]

$$-\nabla\cdot(K_\varepsilon \nabla P_\varepsilon + Q_\varepsilon) = F_\varepsilon \qquad \text{in } \Omega_f^\varepsilon,$$

$$-\varepsilon\nabla\cdot(k_\varepsilon \varepsilon\nabla p_\varepsilon + q_\varepsilon) = f_\varepsilon \qquad \text{in } \Omega_m^\varepsilon$$

$$(K_\varepsilon \nabla P_\varepsilon + Q_\varepsilon)\cdot\vec{n} = \varepsilon k_\varepsilon(\varepsilon\nabla p_\varepsilon + q_\varepsilon)\cdot\vec{n} \qquad \text{on } \partial\Omega_m^\varepsilon,$$

$$P_\varepsilon = p_\varepsilon \qquad \text{on } \partial\Omega_m^\varepsilon$$

顯然的，當 ε 很小時，計算 p_ε 的代價會非常的高。如何使用很小的資源，穫得滿意的結果，這是我們有興趣討論的。我們希望考慮的拋物線方程式為

$$\partial_t P_\varepsilon - \nabla\cdot(K_\varepsilon \nabla P_\varepsilon + Q_\varepsilon) = F_\varepsilon \qquad \text{in } \Omega_f^\varepsilon \times (0,T),$$

$$\partial_t p_\varepsilon - \varepsilon\nabla\cdot(k_\varepsilon \varepsilon\nabla p_\varepsilon + q_\varepsilon) = f_\varepsilon \qquad \text{in } \Omega_m^\varepsilon \times (0,T)$$

$$(K_\varepsilon \nabla P_\varepsilon + Q_\varepsilon)\cdot\vec{n} = \varepsilon k_\varepsilon(\varepsilon\nabla p_\varepsilon + q_\varepsilon)\cdot\vec{n} \qquad \text{on } \partial\Omega_m^\varepsilon \times (0,T),$$

$$P_\varepsilon = p_\varepsilon \qquad \text{on } \partial\Omega_m^\varepsilon \times (0,T)$$

$P_\varepsilon = P_0 \qquad \text{in } \Omega_f^\varepsilon$

$p_\varepsilon = p_0 \qquad \text{in } \Omega_m^\varepsilon$

同樣的，我們也要找出，當ε很小時，如何計算p_ε的方法。

此計劃是之前計劃的沿續。在之前的計劃中，我們了解一些描述流體在破裂多孔介質中的微觀模式與宏觀模式的關係，同時也了解一些微觀模式的解的均勻估計的問題，這對此次計劃會有很大的幫助。

（三）研究方法、進行步驟及執行進度。

基本上，均勻橢圓方程式與均勻拋物線方程式的計算方法已經被研究了五、六十年，甚至更久[18,19,20,23,25,26,27,29,32,34,36,39,40,41,46]。這部份的知識是垂手可得也很完備。不過非均勻的情形仍十分欠缺，若只是直接把計算均勻問題的方法套用在非均勻的問題上，可以想像出這是十分沒有效率，非常糟糕的想法。我們則是想將計算均勻問題的方法與之前微觀模式的解的均勻估計做一結合，發展出一套有效率的計算方法。同時也討論計算方法的誤差估計問題。

（四）成果與自評。

1.得到不均勻拋物線微分方程式的解的導數的L^p norm的均勻估計結果[51]。此結果對我們以後要發展出有效率的、可處理複雜情形的計算方法有很重要的理論依據。理論推導的過程也是可做為進一步研究不均勻介質問題的有效工具。

2. 找出一個計算非均勻橢圓方程式的解的數值方法，並且得到數值解與正確解之間的誤差估計，此結果有很大的實用價值，它可以用來計算多重尺度的問題[52]。

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L^p gradient estimate for non-uniform elliptic equations with discontinuous coefficients

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Many elliptic equations defined in heterogeneous media are non-uniform elliptic equations with discontinuous coefficients. The heterogeneous media considered are periodic and consist of a connected high permeability sub-region and a disconnected matrix block sub-region with low permeability. Let ϵ denote the size ratio of matrix blocks to the whole domain and assume the permeability ratio of the matrix block sub-region to the connected high permeability sub-region is of the order ϵ^2. Elliptic equations with diffusion depending on the permeability of the media have fast diffusion in high permeability sub-region and slow diffusion in low permeability sub-region, and they are non-uniform elliptic equations. It is proved that the L^p norm of the gradient of the elliptic solutions in the high permeability sub-region are bounded uniformly in ϵ. One example also shows that the L^p norm of the second order derivatives of the elliptic solutions in the high permeability sub-region in general are not bounded uniformly in ϵ.

Keywords: permeability, perforated domain, fractured media, heterogeneous media

AMS Subject Classification: 35J15, 35J25, 35J67

1. Introduction

A priori L^p estimate for the gradient of the solutions of non-uniform elliptic equations with discontinuous coefficients is presented. Let $Y(\equiv [0,1]^n, n \geq 3)$ be a unit cube, $B_{1/24}(\frac{\vec{1}}{2})$ denote a ball centered at $\frac{\vec{1}}{2} \equiv (\frac{1}{2}, \cdots, \frac{1}{2})$ with radius $1/24$, $Y_m \subset B_{1/24}(\frac{\vec{1}}{2})$ be a smooth domain, $Y_f \equiv Y \setminus Y_m$, $\epsilon \in (0,1)$, $\Omega_m^\epsilon \equiv \{x | x \in \epsilon(Y_m - j) \text{ for } j \in \mathbb{Z}^n\}$ be a disconnected subset of $\mathbb{R}^n$, Ω_f^ϵ $(\equiv \mathbb{R}^n \setminus \Omega_m^\epsilon)$ denote a connected subset of $\mathbb{R}^n$, and $\partial\Omega_m^\epsilon$ represent the boundary of Ω_m^ϵ. The equations that we considered are

$$\begin{cases} -\nabla \cdot (\mathbf{K}_\epsilon \nabla \Psi_\epsilon + G_\epsilon) = V_\epsilon & \text{in } \Omega_f^\epsilon, \\ -\epsilon \nabla \cdot (\epsilon \mathbf{K}_\epsilon \nabla \psi_\epsilon + g_\epsilon) = v_\epsilon & \text{in } \Omega_m^\epsilon, \\ (\mathbf{K}_\epsilon \nabla \Psi_\epsilon + G_\epsilon) \cdot \vec{\mathbf{n}}^\epsilon = \epsilon(\epsilon \mathbf{K}_\epsilon \nabla \psi_\epsilon + g_\epsilon) \cdot \vec{\mathbf{n}}^\epsilon & \text{on } \partial\Omega_m^\epsilon, \\ \Psi_\epsilon = \psi_\epsilon & \text{on } \partial\Omega_m^\epsilon, \end{cases} \tag{1.1}$$

where $\vec{\mathbf{n}}^\epsilon$ is the unit normal vector on $\partial\Omega_m^\epsilon$ and $\mathbf{K}_\epsilon, G_\epsilon, g_\epsilon, V_\epsilon, v_\epsilon$ are given functions. System (1.1) has applications in flows in highly heterogeneous media, the stress in composite materials, and so on (see [3, 10, 15] and references therein).

Since $\epsilon \in (0,1)$, they are non-uniform elliptic equations with discontinuous coefficients. If $\mathbf{K}_\epsilon$ is positive and smooth as well as if $G_\epsilon, g_\epsilon, V_\epsilon, v_\epsilon$ are smooth and have compact supports, the regular solution of (1.1) in Hilbert space H^k ($k \geq 1$) exists uniquely for each ϵ [14]. By energy method, it is easy to see that the H^1 norm of the solutions of (1.1) in the connected region Ω_f^ϵ are bounded uniformly in ϵ if $G_\epsilon, g_\epsilon, V_\epsilon, v_\epsilon$ are bounded uniformly in ϵ in $L^2(\mathbb{R}^n)$. There are some literatures related to this problem. Lipschitz estimate and $W^{2,p}$ estimate for uniform elliptic equations with discontinuous coefficients could be found in [15, 18]. Uniform Hölder, L^p, and Lipschitz estimates in ϵ for uniform elliptic equations in periodic domains were proved in [4, 5]. Uniform L^p estimate in ϵ for Laplace equation in perforated domains was considered in [17] and the same problem in Lipschitz estimate was considered in [23]. For non-uniform elliptic equations with smooth coefficients, existence of $C^{2,\alpha}$ solution was studied in [11]. Uniform Hölder and Lipschitz estimates in ϵ for non-uniform elliptic equations with discontinuous coefficients were shown in [26]. Here we shall consider the non-uniform elliptic equations with discontinuous coefficients in L^p space case. It is proved that L^p estimate for the gradient of the solutions of (1.1) in the connected sub-region Ω_f^ϵ are bounded uniformly in ϵ but the L^p gradient estimate in the disconnected sub-region Ω_m^ϵ may not be. This is different from uniform elliptic equation case, in which uniform bound holds in the whole domain [4, 5, 11, 15, 18]. We also note that the L^p estimate of the second derivatives of the solutions of (1.1) may not be bounded uniformly in ϵ [26]. In [26], Lipschitz estimate for the solutions of (1.1) was derived under some regularity requirements on $G_\epsilon, g_\epsilon, V_\epsilon, v_\epsilon$ and under smallness of v_ϵ. In this work, no regularity requirements and no smallness of v_ϵ are needed. The assumption that the diameter of Y_m is less than $1/12$ is only for convenience of presentation. Indeed the results still hold if the diameter of Y_m is less than 1.

2. Notation and main result

L^p (resp. $H^s, W^{s,p}$) denotes a Sobolev space with norm $\|\cdot\|_{L^p}$ (resp. $\|\cdot\|_{H^s}, \|\cdot\|_{W^{s,p}}$), $C^{k,\alpha}$ denotes a Hölder space with norm $\|\cdot\|_{C^{k,\alpha}}$, $[\zeta]_{C^{0,\alpha}}$ is the Hölder semi-norm of ζ, $W^{s,p}_{loc}(D) \equiv \{\zeta | \zeta \in W^{s,p}(\mathbb{D})$ for any compact subset $\mathbb{D}$ in $D\}$, and $H^1_{loc}(D) \equiv W^{1,2}_{loc}(D)$, where $s \geq -1, p \in [1,\infty], k \geq 0, \alpha \in (0,1)$. $H^1_{per}(Y)$ contains functions in $H^1(Y)$ satisfying periodic boundary conditions on ∂Y, $C(\mathbb{R}^n)$ is a space of continuous functions in $\mathbb{R}^n$, and $C_0^\infty(D)$ is a space of infinitely differentiable functions with compact support in D. Define $\|\zeta_1, \cdots, \zeta_k\|_{\mathbf{B}} \equiv \|\zeta_1\|_{\mathbf{B}} + \cdots + \|\zeta_k\|_{\mathbf{B}}$ for any Banach space $\mathbf{B}$, $B(x)$ is a ball centered at x, and $B_r(x)$ is a ball centered at x with radius $r > 0$. For any set D, $|D|$ is the volume of D, $\overline{D}$ is the closure of D, $\mathcal{X}_D$ is the characteristic function on D, $dist(x, D)$ is the distance from x to D, $D/r \equiv \{x | rx \in D\}$ for $r > 0$, and

$$\fint_D \zeta(y)dy \equiv \frac{1}{|D|}\int_D \zeta(y)dy \qquad \text{if } \zeta \in L^1(D).$$

If $\zeta \in L^1(\mathbb{R}^n)$, $(\zeta)_{x,r} \equiv \fint_{B_r(x)} \zeta(y)dy$. Define $\Omega_m \equiv \{x|x \in Y_m - j \text{ for } j \in \mathbb{Z}^n\}$ and $\Omega_f \equiv \mathbb{R}^n \setminus \Omega_m$. So $\Omega_m = \Omega_m^\epsilon/\epsilon$ and $\Omega_f = \Omega_f^\epsilon/\epsilon$. For any $\epsilon, \omega > 0$ and $j \in \mathbb{Z}^n$,

$$\mathbf{E}^\epsilon \equiv \begin{cases} 1 & \text{in } \Omega_f, \\ \epsilon^2 & \text{in } \Omega_m, \end{cases} \qquad \widetilde{\mathbf{E}}^{\epsilon,j} \equiv \begin{cases} 1 & \text{in } \mathbb{R}^n \setminus (Y_m - j), \\ \epsilon^2 & \text{in } Y_m - j, \end{cases}$$
$$\mathbf{E}^\epsilon_\omega(x) \equiv \mathbf{E}^\epsilon(\frac{x}{\omega}), \qquad \widetilde{\mathbf{E}}^{\epsilon,j}_\omega(x) \equiv \widetilde{\mathbf{E}}^{\epsilon,j}(\frac{x}{\omega}).$$

Define $\|\zeta\|_{C^{0,\alpha}(D\cap\Omega_f^\omega)} \equiv \|\eta\|_{C^{0,\alpha}(D/\omega\cap\Omega_f^\omega/\omega)}$ and $\|\zeta\|_{C^{0,\alpha}(D\cap\Omega_m^\omega)} \equiv \|\eta\|_{C^{0,\alpha}(D/\omega\cap\Omega_m^\omega/\omega)}$ where $\eta(x) = \zeta(\omega x)$, $\omega > 0$, and $\alpha \in (0,1)$. Define the left and the right limits of ζ (denoted by $\zeta_{,-}$ and $\zeta_{,+}$) on ∂Y_m as

$$\zeta_{,-}(x) \equiv \lim_{\substack{x'\to 0\\ x+x'\in Y_m}} \zeta(x+x'), \quad \zeta_{,+}(x) \equiv \lim_{\substack{x'\to 0\\ x+x'\in Y_f}} \zeta(x+x') \quad \text{for } x \in \partial Y_m.$$

Denote by $x_i (i = 1, \cdots, n)$ the i-th component of $x \in \mathbb{R}^n$, $\mathbb{R}^n_+ \equiv \{x|x_n > 0\}$, and $\partial\mathbb{R}^n_+ \equiv \{x|x_n = 0\}$. Let $\mathcal{B} \equiv [0, \mathbf{M}]^n$ for some $\mathbf{M} \in \mathbb{N}$, $\mathcal{B}^\epsilon_m \equiv \{x|x \in \epsilon(Y_m + j) \subset \mathcal{B} \text{ for } j \in \mathbb{Z}^n\}$, and $\mathcal{B}^\epsilon_f \equiv \mathcal{B} \setminus \mathcal{B}^\epsilon_m$. For any function ζ on $\mathcal{B}$, $\Pi^e_{\mathcal{B}}(\zeta)$ and $\Pi^o_{\mathcal{B}}(\zeta)$ defined in $\mathbb{R}^n$ are extensions of ζ, are periodic with period $[0, 2\mathbf{M}]^n$, and satisfy

$$\begin{cases} \Pi^e_{\mathcal{B}}(\zeta)(x_1, \cdot\cdot, x_{i-1}, x_i, x_{i+1}, \cdot\cdot, x_n) = \Pi^e_{\mathcal{B}}(\zeta)(x_1, \cdot\cdot, x_{i-1}, -x_i, x_{i+1}, \cdot\cdot, x_n), \\ \Pi^o_{\mathcal{B}}(\zeta)(x_1, \cdot\cdot, x_{i-1}, x_i, x_{i+1}, \cdot\cdot, x_n) = -\Pi^o_{\mathcal{B}}(\zeta)(x_1, \cdot\cdot, x_{i-1}, -x_i, x_{i+1}, \cdot\cdot, x_n), \end{cases}$$

for all $i = 1, \cdots, n$. $\Pi^e_{\mathcal{B}}(\zeta)$ (resp. $\Pi^o_{\mathcal{B}}(\zeta)$) is symmetric (resp. antisymmetric) with respect to all coordinate planes (that is, $x_i = 0, i = 1, \cdots, n$) and is called even (resp. odd) extension of ζ in $\mathbb{R}^n$. For any function ζ on $\mathcal{B}$, $\widehat{\Pi}^{e,i}_{\mathcal{B}}(\zeta)$ (resp. $\widehat{\Pi}^{o,i}_{\mathcal{B}}(\zeta)$) in $\mathbb{R}^n$ is extensions of ζ, is periodic with period $[0, 2\mathbf{M}]^n$, is symmetric (resp. antisymmetric) in $x_i = 0$ plane, and is antisymmetric (resp. symmetric) in other coordinate planes, that is,

$$\begin{cases} \widehat{\Pi}^{e,i}_{\mathcal{B}}(\zeta)(x_1, \cdot\cdot, x_{j-1}, x_j, x_{j+1}, \cdot\cdot, x_n) = \widehat{\delta}^e_{i,j}\widehat{\Pi}^{e,i}_{\mathcal{B}}(\zeta)(x_1, \cdot\cdot, x_{j-1}, -x_j, x_{j+1}, \cdot\cdot, x_n), \\ \widehat{\Pi}^{o,i}_{\mathcal{B}}(\zeta)(x_1, \cdot\cdot, x_{j-1}, x_j, x_{j+1}, \cdot\cdot, x_n) = \widehat{\delta}^o_{i,j}\widehat{\Pi}^{o,i}_{\mathcal{B}}(\zeta)(x_1, \cdot\cdot, x_{j-1}, -x_j, x_{j+1}, \cdot\cdot, x_n), \end{cases}$$

where $\widehat{\delta}^e_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i \neq j, \end{cases}$ $\widehat{\delta}^o_{i,j} = \begin{cases} -1 & \text{if } i = j, \\ 1 & \text{if } i \neq j, \end{cases}$ and $i, j = 1, \cdots, n$. For any vector function $\zeta = (\zeta_1, \cdots, \zeta_n)$, we define $\widehat{\Pi}^e_{\mathcal{B}}\zeta \equiv (\widehat{\Pi}^{e,1}_{\mathcal{B}}\zeta_1, \cdots, \widehat{\Pi}^{e,n}_{\mathcal{B}}\zeta_n)$ and define $\widehat{\Pi}^o_{\mathcal{B}}\zeta \equiv (\widehat{\Pi}^{o,1}_{\mathcal{B}}\zeta_1, \cdots, \widehat{\Pi}^{o,n}_{\mathcal{B}}\zeta_n)$.

The following statements are assumed throughout this work:

A1. $\mathbf{K}_\epsilon \in [d_1, d_2]$ for some $d_1, d_2 > 0$ and $\|\mathbf{K}_\epsilon\|_{C^{0,\alpha}(\mathbb{R}^n)}$ for some $\alpha \in (0,1)$ is bounded independent of $\epsilon (< 1)$,

A2. $Y_m \subset B_{1/24}(\frac{\vec{1}}{2})$ is a smooth domain, where $\frac{\vec{1}}{2} \equiv (\frac{1}{2}, \cdots, \frac{1}{2})$.

Our main results are:

Theorem 2.1. *Any solution of (1.1) satisfies*

$$\|\nabla\Psi_\epsilon\|_{L^p(\Omega_f^\epsilon)}+\epsilon\|\nabla\psi_\epsilon\|_{L^p(\Omega_m^\epsilon)}\le c(\|\Psi_\epsilon\mathcal{X}_{\Omega_f^\epsilon}+\epsilon\psi_\epsilon\mathcal{X}_{\Omega_m^\epsilon},G_\epsilon\mathcal{X}_{\Omega_f^\epsilon}+g_\epsilon\mathcal{X}_{\Omega_m^\epsilon}\|_{L^p(\mathbb{R}^n)}$$
$$+\|V_\epsilon\mathcal{X}_{\Omega_f^\epsilon}+v_\epsilon\mathcal{X}_{\Omega_m^\epsilon}\|_{W^{-1,p}(\mathbb{R}^n)}+\epsilon^{-1}\|v_\epsilon\|_{W^{-1,p}(\Omega_m^\epsilon)}), \tag{2.1}$$

where $p\in(1,\infty)$ and c is a constant independent of $\epsilon(<1)$.

Theorem 2.2. *(1) In addition to*

A3. $Y_m-\vec{1}/2$ is symmetric with respect to all coordinate planes $x_i=0, i=1,\cdots,n$,

the solution of

$$\begin{cases}-\nabla\cdot(\mathbf{K}_\epsilon\nabla\Psi_\epsilon+G_\epsilon)=V_\epsilon & \text{in } \mathcal{B}_f^\epsilon,\\ -\epsilon\nabla\cdot(\epsilon\mathbf{K}_\epsilon\nabla\psi_\epsilon+g_\epsilon)=v_\epsilon & \text{in } \mathcal{B}_m^\epsilon,\\ (\mathbf{K}_\epsilon\nabla\Psi_\epsilon+G_\epsilon)\cdot\vec{\mathbf{n}}^\epsilon=\epsilon(\epsilon\mathbf{K}_\epsilon\nabla\psi_\epsilon+g_\epsilon)\cdot\vec{\mathbf{n}}^\epsilon & \text{on } \partial\mathcal{B}_m^\epsilon,\\ \Psi_\epsilon=\psi_\epsilon & \text{on } \partial\mathcal{B}_m^\epsilon,\\ \Psi_\epsilon=0 & \text{on } \partial\mathcal{B},\end{cases} \tag{2.2}$$

satisfies

$$\|\nabla\Psi_\epsilon\|_{L^p(\mathcal{B}_f^\epsilon)}+\epsilon\|\nabla\psi_\epsilon\|_{L^p(\mathcal{B}_m^\epsilon)}\le c(\|G_\epsilon\mathcal{X}_{\mathcal{B}_f^\epsilon}+g_\epsilon\mathcal{X}_{\mathcal{B}_m^\epsilon}\|_{L^p(\mathcal{B})}$$
$$+\|\Pi_{\mathcal{B}}^o(V_\epsilon\mathcal{X}_{\mathcal{B}_f^\epsilon}+v_\epsilon\mathcal{X}_{\mathcal{B}_m^\epsilon})\|_{W^{-1,p}([-\mathbf{M},2\mathbf{M}]^n)}+\epsilon^{-1}\|v_\epsilon\|_{W^{-1,p}(\mathcal{B}_m^\epsilon)}), \tag{2.3}$$

where $p\in(1,\infty)$, c is independent of ϵ, and $\vec{\mathbf{n}}^\epsilon$ is the unit normal vector on $\partial\mathcal{B}_f^\epsilon$.

(2) In addition to A3 and

A4. $\int V_\epsilon\mathcal{X}_{\mathcal{B}_f^\epsilon}+v_\epsilon\mathcal{X}_{\mathcal{B}_m^\epsilon}\,dx=0$,

the solution of

$$\begin{cases}-\nabla\cdot(\mathbf{K}_\epsilon\nabla\Psi_\epsilon+G_\epsilon)=V_\epsilon & \text{in } \mathcal{B}_f^\epsilon,\\ -\epsilon\nabla\cdot(\epsilon\mathbf{K}_\epsilon\nabla\psi_\epsilon+g_\epsilon)=v_\epsilon & \text{in } \mathcal{B}_m^\epsilon,\\ (\mathbf{K}_\epsilon\nabla\Psi_\epsilon+G_\epsilon)\cdot\vec{\mathbf{n}}^\epsilon=\epsilon(\epsilon\mathbf{K}_\epsilon\nabla\psi_\epsilon+g_\epsilon)\cdot\vec{\mathbf{n}}^\epsilon & \text{on } \partial\mathcal{B}_m^\epsilon,\\ \Psi_\epsilon=\psi_\epsilon & \text{on } \partial\mathcal{B}_m^\epsilon,\\ (\mathbf{K}_\epsilon\nabla\Psi_\epsilon+G_\epsilon)\cdot\vec{\mathbf{n}}^\epsilon=0 & \text{on } \partial\mathcal{B},\\ \int\Psi_\epsilon\mathcal{X}_{\mathcal{B}_f^\epsilon}+\psi_\epsilon\mathcal{X}_{\mathcal{B}_m^\epsilon}\,dx=0,\end{cases} \tag{2.4}$$

satisfies

$$\|\nabla\Psi_\epsilon\|_{L^p(\mathcal{B}_f^\epsilon)}+\epsilon\|\nabla\psi_\epsilon\|_{L^p(\mathcal{B}_m^\epsilon)}\le c(\|G_\epsilon\mathcal{X}_{\mathcal{B}_f^\epsilon}+g_\epsilon\mathcal{X}_{\mathcal{B}_m^\epsilon}\|_{L^p(\mathcal{B})}$$
$$+\|\Pi_{\mathcal{B}}^e(V_\epsilon\mathcal{X}_{\mathcal{B}_f^\epsilon}+v_\epsilon\mathcal{X}_{\mathcal{B}_m^\epsilon})\|_{W^{-1,p}([-\mathbf{M},2\mathbf{M}]^n)}+\epsilon^{-1}\|v_\epsilon\|_{W^{-1,p}(\mathcal{B}_m^\epsilon)}), \tag{2.5}$$

where $p\in(1,\infty)$, c is independent of ϵ, and $\vec{\mathbf{n}}^\epsilon$ is the unit normal vector on $\partial\mathcal{B}_f^\epsilon$.

Clearly if the right hand side of (2.1) (resp. (2.3) and (2.5)) is bounded, L^p norm for the gradient of the elliptic solutions of (1.1) (resp. (2.2) and (2.4)) in the connected sub-region is bounded uniformly in ϵ. But this is not the case for the elliptic solutions in the disconnected sub-region.

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A numerical method for elliptic equations in highly heterogeneous media

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The approximation of the solutions of elliptic equations in highly heterogeneous media is concerned. The media consist of a connected fractured subregion with high permeability and a disconnected matrix block subset with low permeability. Let ω denote the size ratio of the matrix blocks to the whole domain and let the permeability ratio of the matrix block subset to the fractured subregion be of the order $\omega^{2\lambda}$ for some $\lambda > 0$. The solutions of elliptic equations in highly heterogeneous media change smoothly in the fractured subregion but change rapidly in the disconnected matrix block subset. Indeed, it is shown that in the fractured subregion, the elliptic solutions can be bounded uniformly in ω in Lipschitz norm; but in the matrix block subset, the elliptic solutions may be unbounded in ω in general. Besides, a numerical method is proposed to find the approximation of the solutions of elliptic equations in highly heterogeneous media. The convergence rate of the numerical method in L^∞ norm is also derived.

Keywords: heterogeneous media, fractured region.

AMS Subject Classification: 35J25, 35J67, 35M20

1. Introduction

A numerical method is proposed to find the approximation of the solutions of elliptic equations in highly heterogeneous media. The media $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ contain two subsets, a connected subregion with high permeability and a disconnected matrix block subset with low permeability. Let $Y \equiv (0, 1)^n$ be a cell consisting of a sub-domain Y_m completely surrounded by another connected sub-domain Y_f $(\equiv Y \setminus \overline{Y}_m)$, $\mathcal{X}_{Y_m}$ be the characteristic function of Y_m and be extended Y-periodically to $\mathbb{R}^n$, and $\omega \in (0, 1)$ be the size ratio of the matrix blocks to the whole domain. If $\Omega(2\omega) \equiv \{x \in \Omega | dist(x, \partial\Omega) > 2\omega\}$, the disconnected matrix block subset is $\Omega^\omega_m \equiv \{x | x \in \omega(Y_m + j) \subset \Omega(2\omega) \text{ for some } j \in \mathbb{Z}^n\}$, the connected subregion is $\Omega^\omega_f \equiv \Omega \setminus \overline{\Omega}^\omega_m$, and the boundary of Ω (resp. Ω^ω_m) is $\partial\Omega$ (resp. $\partial\Omega^\omega_m$). The problem that we considered is

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\omega\nabla\Psi_{\epsilon,\omega})+\gamma\Psi_{\epsilon,\omega}=G & \text{in } \Omega_f^\omega, \\ -\nabla\cdot(\epsilon^{2\lambda}\mathbf{k}_\omega\nabla\psi_{\epsilon,\omega})+\epsilon^\tau\gamma\psi_{\epsilon,\omega}=\epsilon^\tau g_{\epsilon,\omega} & \text{in } \Omega_m^\omega, \\ \mathbf{K}_\omega\nabla\Psi_{\epsilon,\omega}\cdot\vec{\mathbf{n}}_\omega=\epsilon^{2\lambda}\mathbf{k}_\omega\nabla\psi_{\epsilon,\omega}\cdot\vec{\mathbf{n}}_\omega & \text{on } \partial\Omega_m^\omega, \\ \Psi_{\epsilon,\omega}=\psi_{\epsilon,\omega} & \text{on } \partial\Omega_m^\omega, \\ \Psi_{\epsilon,\omega}=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\epsilon,\omega\in(0,1)$, $\lambda,\tau>0$, and $\gamma\in[0,\mathbf{M}]$ are constants, $\mathbf{K}_\omega(x)\equiv\mathbf{K}(\frac{x}{\omega})$, $\mathbf{k}_\omega(x)\equiv\mathbf{k}(\frac{x}{\omega})$, $\mathbf{K}(1-\mathcal{X}_{Y_m})+\mathbf{k}\mathcal{X}_{Y_m}$ is a periodic positive function in $\mathbb{R}^n$ with period Y, and $\vec{\mathbf{n}}_\omega$ is a unit normal vector on $\partial\Omega_m^\omega$. It is known that if $\mathbf{K},\mathbf{k},G,g_{\epsilon,\omega}$ are smooth, a piecewise smooth solution of (1.1) exists uniquely and, by energy method, the H^1 norm of the solution in the high permeability subregion Ω_f^ω is bounded uniformly in ϵ,ω, but not in the low permeability subregion Ω_m^ω. However, the higher order norm of the solution of (1.1) may not be bounded uniformly in ϵ,ω. Indeed, the higher order norm may grow fast even in the high permeability subregion Ω_f^ω when ϵ,ω become small. Therefore, if standard finite element or finite difference method is used to compute the approximation of the solution of (1.1), then the mesh size should be very small in order to obtain good approximation of the solution of (1.1) when ϵ,ω are small. It is expensive to obtain the good approximation by classical methods. On the other hand, homogenization theory tells us that when ϵ,ω become small, the solution of (1.1) approaches to some function which satisfies a simple elliptic differential equation. So one may expect that an approximation of the simple differential equation is a good approximation of the solution of (1.1) when ϵ,ω become small.

There are some literatures related to this work. Lipschitz estimate and $W^{2,p}$ estimate for uniform elliptic equations with discontinuous coefficients had been proved in [22, 26]. Uniform Hölder, L^p, and Lipschitz estimates in ϵ for uniform elliptic equations with smooth oscillatory coefficients were proved in [5, 4]. Uniform Lipschitz estimate in ϵ for the Laplace equation in perforated domains was considered in [30]. Uniform Hölder and Lipschitz estimates in ϵ for non-uniform elliptic equations with discontinuous oscillatory coefficients were shown in [34]. By homogenization theory, the solutions of elliptic equations in perforated domains in general converge to a solution of some homogenized elliptic equation with convergence rate ϵ in L^2 norm and with convergence rate $\epsilon^{1/2}$ in H^1 norm as ϵ closes to 0 (see [6, 20, 28] and references therein). Higher order asymptotic expansion for the solutions of elliptic equations in perforated domains could be found in [7, 23]. Rigorous proof of higher order convergence rate for the solution of (2.3) in Hilbert spaces was considered in [6, 10, 28]. In this work, we shall derive uniform Hölder and Lipschitz estimates in ω for the solution of (1.1) and derive some convergence results of the approximation for the solution of (1.1) in L^∞ norm and Lipschitz norm.

2. Notation and main result

For any set D, $\overline{D}$ denotes the closure of D, $|D|$ is the volume of D, and $\mathcal{X}_D$ is the characteristic function on D. Let $C^{k,\alpha}$ denote the Hölder space with norm $\|\cdot\|_{C^{k,\alpha}}$ and L^p (resp. $H^s, W^{s,p}, H_0^s$) denote the Sobolev space with norm $\|\cdot\|_{L^p}$ (resp. $\|\cdot\|_{H^s}, \|\cdot\|_{W^{s,p}}, \|\cdot\|_{H^s}$) for $k \geq 0, \alpha \in (0,1], s \geq -1, p \in (1,\infty)$ [18]. $B_r(x)$ is a ball centered at x with radius $r > 0$. If D is a bounded set in $\mathbb{R}^n$, we define $D(x,r) \equiv D \cap B_r(x)$. For any $\varphi \in L^1(D)$ and $r > 0$,

$$(\varphi)_{x,r} \equiv \fint_{D(x,r)} \varphi(y)dy \equiv \frac{1}{|D(x,r)|} \int_{D(x,r)} \varphi(y)dy.$$

Remark 2.1. Next we recall an extension result [1].

For $1 \leq p < \infty$, there is a constant $\gamma_1(Y_f, p)$ and a linear continuous extension operator $\Pi_\omega : W^{1,p}(\Omega_f^\omega) \to W^{1,p}(\Omega)$ such that
1) If $\varphi \in W^{1,p}(\Omega_f^\omega)$, then

$$\begin{cases} \Pi_\omega \varphi = \varphi \quad \text{in } \Omega_f^\omega \text{ almost everywhere,} \\ \|\Pi_\omega \varphi\|_{W^{1,p}(\Omega)} \leq \gamma_1 \|\varphi\|_{W^{1,p}(\Omega_f^\omega)}, \\ \gamma_2 \leq \Pi_\omega \varphi \leq \gamma_3 \qquad \text{if } \varphi \in L^\infty(\Omega_f^\omega) \text{ and } \gamma_2 \leq \varphi \leq \gamma_3, \\ \Pi_\omega \varphi = \zeta \quad \text{in } \Omega \text{ if } \varphi = \zeta|_{\Omega_f^\omega} \text{ for some linear function } \zeta \text{ in } \Omega. \end{cases}$$

2) For any constant $r > 0$, $\Pi_{\omega/r}\zeta(x) = (\Pi_\omega \varphi)(rx)$ where $\zeta(x) \equiv \varphi(rx)$.

It is well-known that if $\tau \geq \lambda$, $\mathbf{K}, \mathbf{k} \in C^{0,1}(\mathbb{R}^n)$ are positive functions, and $\|G\|_{L^2(\Omega_f^\omega)} + \|g_{\epsilon,\omega}\|_{L^2(\Omega_m^\omega)}$ are bounded independent of ϵ, ω, the solution of (1.1) exists uniquely and satisfy $\|\Psi_{\epsilon,\omega}\|_{H^1(\Omega_f^\omega)} + \|\epsilon^{\tau/2}\psi_{\epsilon,\omega}, \epsilon^\lambda \nabla \psi_{\epsilon,\omega}\|_{L^2(\Omega_m^\omega)} \leq c$ (independent of ϵ, ω). By compactness principle [2, 18, 20],

$$\begin{cases} \Pi_\omega \Psi_{\epsilon,\omega} \to \Psi & \text{in } L^2(\Omega) \text{ strongly} \\ \mathbf{K}_\omega \nabla \Psi_{\epsilon,\omega} \mathcal{X}_{\Omega_f^\omega} \to \mathcal{K} \nabla \Psi & \text{in } L^2(\Omega) \text{ weakly} \qquad \text{as } \epsilon, \omega \to 0, \\ G\mathcal{X}_{\Omega_f^\omega} + \epsilon^\tau g_{\epsilon,\omega} \mathcal{X}_{\Omega_m^\omega} \to |Y_f| G & \text{in } L^2(\Omega) \text{ weakly} \end{cases} \tag{2.1}$$

where $\mathcal{K}$ is a constant symmetric positive definite matrix. Moreover, the function Ψ in (2.1) satisfies

$$\begin{cases} -\nabla \cdot (\mathcal{K} \nabla \Psi) + |Y_f|\gamma \Psi = |Y_f| G & \text{in } \Omega, \\ \Psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $|Y_f|$ is the volume of Y_f.

For any $\delta > 0$, define $\mu \equiv \frac{\delta}{n+\delta}$ and take $\alpha \in (\mu, 1)$. We assume

A1. $\Omega \subset \mathbb{R}^n$ for $n = 2, 3$ is $C^{3,\alpha}$, Y_m is a smooth simply connected domain,
A2. $\mathbf{K}, \mathbf{k} \in C^{0,1}(\mathbb{R}^n)$ are positive periodic functions in $\mathbb{R}^n$ with period Y and $\|\nabla \mathbf{K}\|_{L^\infty(Y)} + \|\nabla \mathbf{k}\|_{L^\infty(Y)}$ is small compared with $\min_{x \in Y}\{\mathbf{K}, \mathbf{k}\}$,
A3. $\epsilon, \omega \in (0,1)$, $2\lambda \geq \tau \geq \lambda > 0$, $\gamma \in [0, \mathbf{M}]$,

A4. $\|G\|_{L^{n+\delta}(\Omega_f^\omega)} + \|g_{\epsilon,\omega}\|_{L^{n+\delta}(\Omega_m^\omega)}$ for some $\delta \in (0,3)$ is bounded independent of ϵ, ω.

We have the following convergence result.

Theorem 2.1. *Under A1-4, the solutions of (1.1) and (2.2) satisfy*

$$\|\Psi_{\epsilon,\omega} - \Psi\|_{L^\infty(\Omega_f^\omega)} + \epsilon^{2\lambda}\|\psi_{\epsilon,\omega} - \Psi\|_{L^\infty(\Omega_m^\omega)} \le c\max\{\omega, \epsilon^\tau\},$$

where constant c is independent of ϵ, ω.

Define $\mathcal{B} \equiv (0,1)^n$, $\mathcal{B}_m^\omega \equiv \{x | x \in \omega(Y_m + j) \subset \mathcal{B} \text{ for some } j \in \mathbb{Z}^n\}$, and $\mathcal{B}_f^\omega \equiv \mathcal{B} \setminus \overline{\mathcal{B}}_m^\omega$. Suppose $\gamma > 0$ and

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\omega \nabla\Psi_{\epsilon,\omega}) + \gamma\Psi_{\epsilon,\omega} = G & \text{in } \mathcal{B}_f^\omega, \\ -\nabla\cdot(\epsilon^{2\lambda}\mathbf{k}_\omega\nabla\psi_{\epsilon,\omega}) + \epsilon^\tau\gamma\psi_{\epsilon,\omega} = \epsilon^\tau g_{\epsilon,\omega} & \text{in } \mathcal{B}_m^\omega, \\ \mathbf{K}_\omega\nabla\Psi_{\epsilon,\omega}\cdot\vec{\mathbf{n}}_\omega = \epsilon^{2\lambda}\mathbf{k}_\omega\nabla\psi_{\epsilon,\omega}\cdot\vec{\mathbf{n}}_\omega & \text{on } \partial\mathcal{B}_m^\omega, \\ \Psi_{\epsilon,\omega} = \psi_{\epsilon,\omega} & \text{on } \partial\mathcal{B}_m^\omega, \\ \Psi_{\epsilon,\omega}\mathcal{X}_{\Omega_f^\omega} + \psi_{\epsilon,\omega}\mathcal{X}_{\Omega_m^\omega} \text{ satisfies periodic boundary condition.} \end{cases} \tag{2.3}$$

Under A1–3 and $\|G\|_{L^2(\mathcal{B}_f^\omega)} + \|g_{\epsilon,\omega}\|_{L^2(\mathcal{B}_m^\omega)}$ is bounded independent of ϵ, ω, the solution of (2.3) exists uniquely and satisfies

$$\|\Psi_{\epsilon,\omega}\|_{H^1(\mathcal{B}_f^\omega)} + \|\epsilon^{\tau/2}\psi_{\epsilon,\omega}, \epsilon^\lambda\nabla\psi_{\epsilon,\omega}\|_{L^2(\mathcal{B}_m^\omega)} \le c,$$

where c is independent of ϵ, ω. By compactness principle [2, 18, 20],

$$\begin{cases} \Pi_\omega\Psi_{\epsilon,\omega} \to \Psi & \text{in } L^2(\mathcal{B}) \text{ strongly} \\ \mathbf{K}_\omega\nabla\Psi_{\epsilon,\omega}\mathcal{X}_{\mathcal{B}_f^\omega} \to \mathcal{K}\nabla\Psi & \text{in } L^2(\mathcal{B}) \text{ weakly} \\ G\mathcal{X}_{\mathcal{B}_f^\omega} + \epsilon^\tau g_{\epsilon,\omega}\mathcal{X}_{\mathcal{B}_m^\omega} \to |Y_f|G & \text{in } L^2(\mathcal{B}) \text{ weakly} \end{cases} \quad \text{as } \epsilon, \omega \to 0,$$

where $\mathcal{K}$ is a constant symmetric positive definite matrix. Moreover, the function Ψ satisfies

$$\begin{cases} -\nabla\cdot(\mathcal{K}\nabla\Psi) + |Y_f|\gamma\Psi = |Y_f|G & \text{in } \mathcal{B}, \\ \Psi \text{ satisfies periodic boundary condition.} \end{cases} \tag{2.4}$$

Next we consider Lipschitz estimate.

Theorem 2.2. *Under A1-3, $\gamma > 0$, and $\|G\|_{L^{n+\delta}(\mathcal{B}_f^\omega)} + \|g_{\epsilon,\omega}\|_{L^{n+\delta}(\mathcal{B}_m^\omega)}$ is bounded independent of ϵ, ω, the solutions of (2.3) and (2.4) satisfy*

$$\begin{aligned} &\sup_{x\in\mathcal{B}_f^\omega} \left|\nabla\Psi_{\epsilon,\omega}(x) - (I - \nabla\mathbb{X}(\frac{x}{\omega}))\nabla\Psi(x)\right| \\ &\qquad + \epsilon^{2\lambda}\sup_{x\in\mathcal{B}_m^\omega}\left|\nabla\psi_{\epsilon,\omega}(x) - (I - \nabla\mathbb{X}(\frac{x}{\omega}))\nabla\Psi(x)\right| \\ &\quad \le c\max\{\omega^{\mu/2}, \epsilon^\tau\}, \end{aligned}$$

where constant c is independent of ϵ, ω.

Next we consider the following

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\omega\nabla\Psi_{\epsilon,\omega}) = G & \text{in } \mathcal{B}_f^\omega, \\ -\nabla\cdot(\epsilon^{2\lambda}\mathbf{k}_\omega\nabla\psi_{\epsilon,\omega}) = \epsilon^\tau g_{\epsilon,\omega} & \text{in } \mathcal{B}_m^\omega, \\ \mathbf{K}_\omega\nabla\Psi_{\epsilon,\omega}\cdot\vec{\mathbf{n}}_\omega = \epsilon^{2\lambda}\mathbf{k}_\omega\nabla\psi_{\epsilon,\omega}\cdot\vec{\mathbf{n}}_\omega & \text{on } \partial\mathcal{B}_m^\omega, \\ \Psi_{\epsilon,\omega} = \psi_{\epsilon,\omega} & \text{on } \partial\mathcal{B}_m^\omega, \\ \Psi_{\epsilon,\omega} \text{ satisfies periodic boundary condition,} \\ \int G\mathcal{X}_{\mathcal{B}_f^\omega} + \epsilon^\tau g_{\epsilon,\omega}\mathcal{X}_{\mathcal{B}_m^\omega}\,dx = \int \Psi_{\epsilon,\omega}\mathcal{X}_{\mathcal{B}_f^\omega} + \psi_{\epsilon,\omega}\mathcal{X}_{\mathcal{B}_m^\omega}\,dx = 0. \end{cases} \tag{2.5}$$

Under A1–3 and $\|G\|_{L^2(\mathcal{B}_f^\omega)} + \|g_{\epsilon,\omega}\|_{L^2(\mathcal{B}_m^\omega)}$ is bounded independent of ϵ,ω, the solution of (2.5) exists uniquely and satisfies $\|\Psi_{\epsilon,\omega}\|_{H^1(\mathcal{B}_f^\omega)} + \|\epsilon^\lambda\nabla\psi_{\epsilon,\omega}\|_{L^2(\mathcal{B}_m^\omega)} \le c$ (independent of ϵ,ω). By compactness principle [2, 18, 20],

$$\begin{cases} \Pi_\omega\Psi_{\epsilon,\omega} \to \Psi & \text{in } L^2(\mathcal{B}) \text{ strongly} \\ \mathbf{K}_\omega\nabla\Psi_{\epsilon,\omega}\mathcal{X}_{\mathcal{B}_f^\omega} \to \mathcal{K}\nabla\Psi & \text{in } L^2(\mathcal{B}) \text{ weakly} \\ G\mathcal{X}_{\mathcal{B}_f^\omega} + \epsilon^\tau g_{\epsilon,\omega}\mathcal{X}_{\mathcal{B}_m^\omega} \to |Y_f|G & \text{in } L^2(\mathcal{B}) \text{ weakly} \end{cases} \quad \text{as } \epsilon,\omega\to 0,$$

where $\mathcal{K}$ is a constant symmetric positive definite matrix. Moreover, the function Ψ satisfies

$$\begin{cases} -\nabla\cdot(\mathcal{K}\nabla\Psi) = |Y_f|G & \text{in } \mathcal{B}, \\ \int_{\mathcal{B}}\Psi dx = \int_{\mathcal{B}} G dx = 0, \\ \Psi \text{ satisfies periodic boundary condition.} \end{cases} \tag{2.6}$$

Theorem 2.3. *Under A1-3 and* $\|G\|_{L^{n+\delta}(\mathcal{B}_f^\omega)} + \|g_{\epsilon,\omega}\|_{L^{n+\delta}(\mathcal{B}_m^\omega)}$ *is bounded independent of* ϵ,ω*, the solutions of (2.5) and (2.6) satisfy*

$$\begin{aligned} &\sup_{x\in\mathcal{B}_f^\omega}\left|\nabla\Psi_{\epsilon,\omega}(x) - (I-\nabla\mathbb{X}(\frac{x}{\omega}))\nabla\Psi(x)\right| \\ &\qquad + \epsilon^{2\lambda}\sup_{x\in\mathcal{B}_m^\omega}\left|\nabla\psi_{\epsilon,\omega}(x) - (I-\nabla\mathbb{X}(\frac{x}{\omega}))\nabla\Psi(x)\right| \\ &\le c\max\{\omega^{\mu/2},\epsilon^\tau\}, \end{aligned}$$

where constant c is independent of ϵ,ω.

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