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Abstract

This report provides an overview of major results we have obtained in the research project *"Constructions of Diversity-Multiplexing Tradeoff Optimal Codes for Multiuser MIMO Systems with Applications to MIMO Mobile Communications"* supported by *National Science Council* under contract number *NSC98 - 2221 - E - 009 - 045 - MY3* during August 2009 - July 2012.

In this report, we are concentrating explicit code constructions for multiple-input multiple-output (MIMO) multiple-access channels (MAC) with K users. The first construction is dedicated to the case of symmetric MIMO-MAC where all the users have the same number of transmit antennas n_t and transmit at the same level of per-user multiplexing gain r. Furthermore, we assume that the users transmit in an independent fashion and do not cooperate. The construction is systematic for any values of K, n_t and r. It is proved that this newly proposed construction achieves the optimal MIMO-MAC diversity-multiplexing gain tradeoff (DMT) provided by Tse *et al.* at high-SNR regime. We next take a further step to investigate the MAC-DMT of a general MIMO-MAC where the users are allowed to have different numbers of transmit antennas and can transmit at different levels of multiplexing gain. The exact optimal MAC-DMT of such channel is explicitly characterized in this report. Interestingly, in the general MAC-DMT, some users might not be able to achieve their single-user DMT performance as in the symmetric case, even when the multiplexing gains of the other users are close to 0. Detailed explanations of such unexpected result are provided in this report. Finally, by generalizing the code construction for the symmetric MIMO-MAC, explicit code constructions are provided for the general MIMO-MAC and are proved to be optimal in terms of the general MAC-DMT.

We also answer several open questions related to diversity-multiplexing tradeoffs (DMTs) for point-to-point and multiple-access (MAC) MIMO channels. By analyzing the DMT performance of a simple code, we show that the optimal MAC-DMT holds even when the channel remains fixed for less than $Kn_t + n_r - 1$ channel uses, where K is the number of users, n_t is the number of transmit antennas of each user, and n_r is the number of receive antennas at receiver. We also prove that the simple code is MAC-DMT optimal. A general code design criterion for constructing MAC-DMT optimal codes that is much more relaxed than the previously known design criterion is provided. Finally, by changing some design parameters, the simple code is modified for use in point-to-point MIMO channels. We show the modified code achieves the same DMT performance as the Gaussian random code.

Keywords: Diversity-multiplexing gain tradeoff (DMT), multiple access channel (MAC), cyclic division algebras (CDAs), multiple-input multiple-output (MIMO) channel, space-time block codes (STBCs).

Referred Papers Supported by Granted Project

Under the support of this three-years project, we have so-far successfully produced the following **17** papers (**6 Journal Papers**, and 11 conference papers published in the highest quality conferences):

1. C. Hollanti and H. F. Lu, “Construction methods for asymmetric and multi-block space-time codes,” *IEEE Trans. Inform. Theory*, vol. 55, no. 2, pp. 1086-1103, Mar. 2009.

2. H. F. Lu, R. Vehkalahti, C. Hollanti, J. Lahtonen, Y. Hong, and E. Viterbo, “New space-time code constructions for two-user multiple access channels,” *IEEE Journal of Selected Topics in Signal Processing*, vol. 3, nol. 6, pp. 939-957, Dec. 2009.

3. H. F. Lu and C. Hollanti, “Optimal diversity multiplexing tradeoff and code constructions of constrained asymmetric MIMO systems,” *IEEE Trans. Inform. Theory*, vol. 56, no. 5, pp.2121-2129, May 2010.

4. H. F. Lu, “Constructions of diversity-multiplexing tradeoff optimal vector codes for asynchronous cooperative networks using decode-and-forward protocols,” *IEEE Trans. Wireless Communications*, May 2010

5. H. F. Lu, C. Hollanti, R. Vehkalahti, and J. Lahtonen, “DMT optimal codes constructions for multiple-access MIMO channel,” *IEEE Trans. Inform. Theory*, vol. 57, no. 6, Jun. 2011.

6. H. F. Lu, ”Remarks on diversity-multiplexing tradeoffs for multiple-access and point-to-point MIMO channels,” *IEEE Trans. Inform. Theory*, vol. 58, no. 2, Feb. 2012.

7. H. F. Lu and C. Hollanti “Diversity-multiplexing tradeoff-optimal code constructions for symmetric MIMO multiple access channels,” *Proc. 2009 IEEE Int. Symp. on Inform. Theory* (ISIT), Seoul, Korea.

8. C. Hollanti, H.F. Lu, and R. Vehkalahti, “An algebraic tool for obtaining conditional non-vanishing determinants,” *Proc. 2009 IEEE Int. Symp. on Inform. Theory* (ISIT), Seoul, Korea.

9. J. Lahtonen, R. Vehkalahti, H. F. Lu, C. Hollanti, and E. Vitero, “On the decay of the determinants of multiuser MIMO lattice codes,” *Proc. ITW 2010*, Cairo, Egypt, Jan. 2010.

10. H. F. Lu, J. Lahtonen, R. Vehkalahti, and Camilla Hollanti, “Remarks on the criteria of constructing MIMO-MAC DMT optimal codes,” *Proc. ITW 2010*, Cairo, Egypt, Jan. 2010.

11. H. F. Lu, “Diversity-multiplexing tradeoff in MIMO Gaussian interference channels,” *Proc. 2010 IEEE Int. Symp. Inf. Theory* (ISIT 2010), Austin, TX, Jun. 2010.

12. H. F. Lu, “Approximately Universal MIMO Diversity Embedded Codes”, *Proc. 2010 ISITA*, Taichung, Taiwan, Oct. 2010.

13. R. Vehkalahti, C. Hollanti, J. Lahtonen, and H. F. Lu, ”Some Simple Observations on MISO Codes”, *Proc. 2010 ISITA*, Taichung, Taiwan, Oct. 2010.

14. R. Vehkalahti and H. F. Lu, “An algebraic look into MAC-DMT of lattice space-time codes,” *Proc. 2011 IEEE Int. Symp. on Inform. Theory* (ISIT), St. Petersburg, Russia.

15. T.W. Tang, M. K. Chen, and H. F. Lu, “Improving the DMT Performances of MIMO Linear Receivers,” *Proc. 2011 IEEE Int. Symp. on Inform. Theory* (ISIT), St. Petersburg, Russia.

16. R. Vehkalahti and H. F. Lu, “Diversity-multiplexing gain tradeoff: a tool in algebra?” in *Proc. ITW 2011*, pp. 135 - 139 , Paraty, Brazil, Oct. 2011.

17. S. M. Huang, H. F. Lu, and S. M. Moser, “Minimal-Rate Description for Multiple-Access Channels,” in *Proc. ISITA 2012*, Hawaii, Oct. 2012.

Chapter 1

Introduction

During the last decade extensive research has been carried out in the design of point-to-point space-time (ST) codes [1, 2] for multiple-input multiple-output (MIMO) communication systems. ST codes based on cyclic division algebras (CDAs) [3–7] that can also be regarded as a kind of algebraic lattice codes and/or as a kind of linear dispersion ST codes [8] have been shown to perform extremely well. The error performance of these codes have been shown to be very close to the outage bound not only for practical numbers of antennas but also at moderate SNR values.

For high-SNR regime, the same point-to-point CDA-based ST codes have been shown [4] to be optimal in terms of the diversity-multiplexing tradeoff (DMT) proposed by Zheng and Tse [9]. Specifically, let n_t and n_r be respectively the numbers of transmit and receive antennas at transmitter and receiver ends. Let r, $0 \leq r \leq \min\{n_t, n_r\}$, denote the multiplexing gain such that the actual transmission rate equals

$$R \;=\; r\log_2 \mathrm{SNR} \quad \text{(bits per channel use)}. \tag{1.1}$$

Assuming a MIMO Rayleigh block fading channel, it was shown [4] that at multiplexing gain r, the CDA-based ST codes achieve the optimal codeword error probability

$$P_{\text{cwe}}(r) \;\doteq\; \mathrm{SNR}^{-d^*_{n_t,n_r}(r)} \tag{1.2}$$

at high-SNR regime, where by $\doteq$ we mean the exponential equality defined in [9]. That is, we write $f(\mathrm{SNR}) \doteq \mathrm{SNR}^b$ if

$$\lim_{\mathrm{SNR}\to\infty} \frac{\log f(\mathrm{SNR})}{\log \mathrm{SNR}} \;=\; b.$$

The notations of $\dot{\geq}$ and $\dot{\leq}$ are similarly defined. The exponent $d^*_{n_t,n_r}(r)$ is commonly known as the DMT [9] and is given by a piecewise linear function connecting the points $(r, (n_t - r)(n_r - r))$ for $r = 0, 1, \cdots, \min\{n_t, n_r\}$. Furthermore, $d^*_{n_t,n_r}(r)$ represents the largest diversity gain that can be achieved by any point-to-point ST codes under Rayleigh block fading channel whenever the channel remains static for at least a block of n_t channel uses [4] and varies independently from one block to another.

For other types of fading statistics, the CDA-based ST codes are also known [4] to be capable of achieving the optimal error performance in such channels that include Rician, Weibull and Nakagami as special cases. ST codes that are optimal in all fading statistics are coined *approximately universal codes* [4, 10].

If coding across independent fading blocks is allowed, the multi-block CDA code [6] has been shown to be approximately universal as well. In particular, it achieves codeword error probability

$$P_{\text{cwe}}(r) \;\doteq\; \mathrm{SNR}^{-m\cdot d^*_{n_t,n_r}(r)}, \tag{1.3}$$

at multiplexing gain r, where m is the number of independent fading blocks occupied by the code. The exponent $m \cdot d^*_{n_t,n_r}(r)$ is known as the *multi-block DMT* [6, 9] when coding is applied over

m independent fading blocks. Therefore, the multi-block CDA-based ST code is optimal in terms of the multi-block DMT at high-SNR regime. More important, (1.3) indicates that the code has error probability decreasing to zero as m approaches infinity whenever $d^*_{n_t,n_r}(r) > 0$. Hence, the multi-block ST code could potentially achieve the MIMO ergodic channel capacity at high-SNR regime and simultaneously be optimal in terms of the multi-block DMT at every discrete value m.

Motivated by the promising outcome in the point-to-point scenario, the aim of this report is to investigate the code construction for the multiple-access channel (MAC) scenario. We will concentrate on the uplink transmission from multiple mobile users to a common base station (or access point). Both the mobile users and the base station may be equipped with multiple antennas.

Consider a MIMO-MAC with K mobile users. For simplicity, we first focus on the case of *symmetric MIMO-MAC* [11], where each user is equipped with n_t transmit antennas and communicates independently to the base station that has n_r receive antennas. Furthermore, we assume that all the users transmit at the same level of multiplexing gain. With a slight abuse of notation, hereafter we will denote by r the *per-user multiplexing gain* in the symmetric MIMO-MAC. Let $\mathcal{S}_0, \cdots, \mathcal{S}_{K-1}$, be respectively the ST codes used by the kth user, $k = 0, 1, \cdots, K-1$. Each code $\mathcal{S}_k$, $k = 0, 1, \cdots, K-1$, consists of $(n_t \times T)$ matrices and satisfies the following power constraint:

$$\mathbb{E}_{\,S\in\mathcal{S}_k} \|S\|_F^2 \ \le\ T\cdot\text{SNR}, \tag{1.4}$$

where by $\|S\|_F$ we mean the Frobenius norm of matrix S. Furthermore, we require $|\mathcal{S}_k| = \text{SNR}^{rT}$ for all k such that every user transmits at the same multiplexing gain r. Let H_k be the $(n_r \times n_t)$ channel matrix of the kth user. We assume H_k is fixed for a block of T channel uses. H_k is known completely to the receiver at base station but unknown to all the users. Entries of H_k are modeled as i.i.d. $\mathbb{C}\mathcal{N}(0,1)$ complex Gaussian random variables to model the MIMO Rayleigh block fading channel. Let $S_k \in \mathcal{S}_k$ be the signal matrix transmitted by the kth user; then the signal matrix received at base station is given by

$$Y \ =\ \sum_{k=0}^{K-1} H_k S_k + W, \tag{1.5}$$

where W is the $(n_r \times T)$ noise matrix with i.i.d. $\mathbb{C}\mathcal{N}(0,1)$ entries. When each user's information is encoded independently, Tse *et al.* [11] proved that the tradeoff between the diversity gain d and multiplexing gain r in a symmetric MIMO-MAC is governed by the following theorem.

Theorem 1 (Symmetric MAC-DMT [11])**.** *In a symmetric MIMO-MAC with K users, each having n_t transmit antennas and transmitting independently at multiplexing gain r, the maximal possible diversity gain is given by*

$$\begin{aligned} d^*_{n_t,n_r,K}(r) &:= \min_{1\le k\le K} d^*_{kn_t,n_r}(kr) \\ &= \begin{cases} d^*_{n_t,n_r}(r), & \textit{if } r \in \left[0, \min\left\{n_t, \frac{n_r}{K+1}\right\}\right], \\ d^*_{Kn_t,n_r}(Kr), & \\ \qquad \textit{if } r \in \left[\min\left\{n_t, \frac{n_r}{K+1}\right\}, \min\left\{n_t, \frac{n_r}{K}\right\}\right], & \end{cases} \end{aligned} \tag{1.6}$$

where $d^*_{kn_t,n_r}(kr)$ is the point-to-point DMT for kn_t transmit antennas, n_r receive antennas and multiplexing gain kr defined as before (or see [9,11]). Equation (1.6) *is termed* optimal symmetric MAC-DMT. *The multiplexing gain r for nonnegative diversity gain is bounded between*

$$0 \le r \le \min\left\{n_t, \frac{n_r}{K}\right\} = r_{\max}. \tag{1.7}$$

$\square$

Compared with the point-to-point scenario, the decrease of maximal multiplexing gain by a factor of K (see $\frac{n_r}{K}$ in $r_{\max}$ of (1.7)) is due to the sharing of n_r receive antennas among K users and the fact that $d^*_{Kn_t,n_r}(Kr_{\max}) = 0$. Equation (1.6) also shows that when the level of multiplexing gain is low such that $r \in \left[0, \min\left\{n_t, \frac{n_r}{K+1}\right\}\right]$, each user is able to retain his *single-user performance*, i.e., $d^*_{n_t,n_r,K}(r) = d^*_{n_t,n_r}(r)$, as if there were no other users in the channel. On the other hand, when the level of multiplexing gain is high and $r \in \left[\min\left\{n_t, \frac{n_r}{K+1}\right\}, \min\left\{n_t, \frac{n_r}{K}\right\}\right]$, the MIMO-MAC system would operate in the *antenna pooling region* [11], and single-user performance can no longer be maintained. As a consequence, a much lower diversity gain $d^*_{Kn_t,n_r}(Kr)$ dominates the error performance in this region.

In Fig. 1.1 we demonstrate the above facts of the symmetric MAC-DMT for the case of $K = 3$ users, $n_t = 2$ and $n_r = 2$. It can be clearly seen that the turning point between the single-user and antenna pooling regions is at $r = \min\left\{n_t, \frac{n_r}{K+1}\right\} = \frac{1}{2}$ and the cut-off point of r is at $\min\{n_t, \frac{n_r}{K}\} = \frac{2}{3}$.

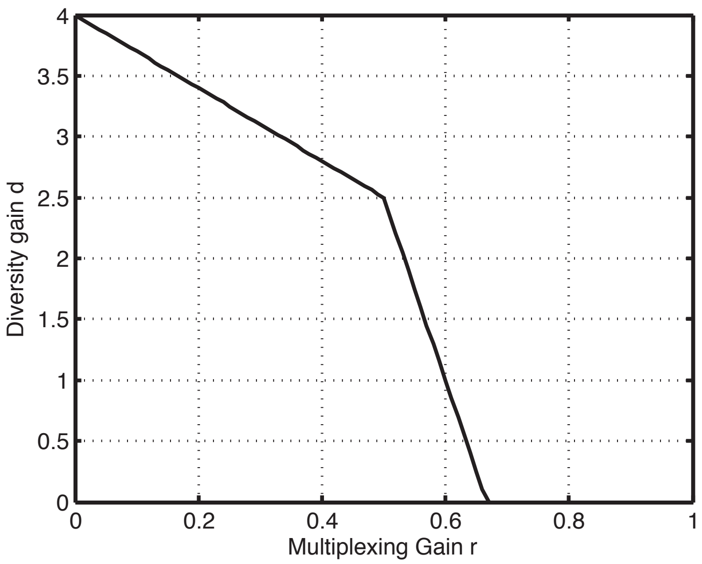

Figure 1.1: The MAC DMT for $K = 3$ users with $n_t = 2$ and $n_r = 2$.

The construction of MAC-DMT optimal codes calls for a coding scheme that independently encodes, but simultaneously transmits, each mobile user's information over the MIMO channel such that at receiver end, the decoding of all users' signals achieves the best possible error performance dictated by the MAC-DMT. Thus, a coding scheme is called *MAC-DMT optimal* if it achieves the following error performance under joint decoding

$$P_{\text{cwe}}(r) \doteq \text{SNR}^{-d^*_{n_t,n_r,K}(r)}.$$

1.1 Prior Work

Several works have been reported in this area. Nam *et al.* [12] presented the first MAC-DMT optimal scheme using a class of structured multiple-access random lattice ST codes. For the constructions of deterministic codes, below we briefly review some relevant earlier papers. Almost all deal exclusively with the two-user symmetric MIMO-MAC case, i.e., $K = 2$.

1. [13] extended the pairwise-error-probability-based design criteria of point-to-point ST codes to the MAC case for $K = 2$ users and $n_t = 2$, $n_r = 2$. An explicit (4×4) two-user MIMO

code[1], i.e., a (2×4) code for each user, based on independent Alamouti blocks [2] is also introduced in [13]. Yet, we remark that such code does not achieve the optimal symmetric MAC-DMT (1.6).

2. In [14] Badr and Belfiore proposed an explicit algebraic code for $K = 2$ and $n_t = 1$. The idea can be extended to bigger values of K. The determinant of the code matrix is nonzero thanks to a "twisting element." However, the determinant is vanishing. The decay of determinants of this two-user MIMO-MAC code was carefully studied in [15]. It was shown that the code is MAC-DMT optimal, when $r \leq \frac{1}{5}$. Whether this code achieves the optimal MAC-DMT also when $r > 1/5$ remains an open question. In [15] it was shown, however, that the code fails to satisfy the criteria for achieving optimal MAC-DMT set forth in [16], when $r > 1/5$. This alone does not mean that their code could not be optimal, as the criteria in [16] is sufficient, but not necessary (see [17] for justification of this claim).

3. Some explicit, algebraic code constructions for $n_t > 1$ and $K = 2$ were introduced by Hong and Viterbo in [18]. A design criterion based on an approximation of truncated union bound was proposed. With such criterion they constructed a code that outperforms in error performance the aforementioned (4×4) two-user code [13].

4. Badr and Belfiore [19] proposed another (4×4) two-user MIMO-MAC code which is obtained by adding a twist matrix Γ to the (2×2) Golden ST code [20, 21] such that the overall code matrix is nonsingular whenever all the submatrices associated with each user are nonzero. However, because of this additional Γ matrix, the overall code matrix, though nonsingular, could be ill-conditioned at high-SNR regime, thereby resulting in a vanishing determinant, similarly as did their earlier one-antenna code [14] already discussed above.

5. [22] addressed the problem of whether there exists a two-user MIMO-MAC code satisfying the non-vanishing determinant (NVD) property. This problem concerns whether the twisted Golden MIMO-MAC code [19] can be further improved to avoid the disadvantage of having a vanishing determinant. The answer is negative. [22] shows that if all the overall code matrices are nonsingular whenever the submatrices from each user are nonzero, then some of them must have determinant arbitrarily close to zero, i.e., have vanishing determinants.

6. By removing the Γ matrix, [22] reported another code construction and proved its MAC-DMT optimality for $K = 2$ and for any values of n_t and n_r. Computer simulations showed that this code outperforms the (4×4) code of [19] at all SNR values. Another important contribution reported in [22] was that, for the two-user MAC case, one does not need the whole code matrix to be nonsingular, and hence introducing the additional Γ rotation matrix is not necessary from the MAC-DMT point of view.

7. In [16], Coronel *et al.* studied the optimal DMT performance of a selective fading MIMO-MAC and provided a sufficient criterion for designing MAC-DMT optimal codes for any K and n_t. Noting that the Rayleigh block fading channel is a flat fading channel, a simplification of their criterion requires the product concatenation of codes from any subsets of K users to satisfy the NVD property such that the error probabilities associated with these subcodes do not exceed the corresponding outage probability. However, as already pointed out in [22], such codes do not exist for the case of $K = 2$. A further investigation of their criterion can be found in [17].

[1] In this report, by an $(m \times n)$ code we mean a code consisting of $(m \times n)$ code matrices, where m is the number of transmit antennas required for transmission, and n is the number of channel uses. The number m can be either n_t or Kn_t, depending on the discussion. When $m = n_t$, the code is for each user's use. When $m = Kn_t$, we mean the vertical concatenation of all users' codes as an overall code. Notation $n_t \times n_r$ without parenthesis is used for the channel dimensions.

1.2 Complete Construction of DMT-Optimal Multiuser MIMO Codes

A complete solution to the problem of constructing MIMO-MAC codes for K users that are MAC-DMT optimal in Rayleigh MIMO-MAC is presented in this report.

We first provide the constructions of MAC-DMT optimal codes for the symmetric MIMO-MAC. Later, we will give the code construction for the general MIMO-MAC where the users are allowed to have different numbers of transmit antennas and can transmit at different levels of multiplexing gain.

A general result on the nonexistence of NVD MIMO-MAC codes is presented in Chapter 2. This result suggests that the design criterion proposed by Coronel *et al.* [16] might be too strict to yield any MAC-DMT optimal codes. A relaxed design criterion is then provided in this section.

In Chapter 3, we present a new code construction for the symmetric MIMO-MAC for any K, n_t and n_r. Several nice properties of the proposed code are presented in this section. We prove that this newly proposed construction is MAC-DMT optimal and meets the relaxed design criterion given in Chapter 2. For ease of reading, the proof of MAC-DMT optimality is relegated to Chapter 6.

In Chapter 4 we investigate the MAC-DMT in a general MIMO-MAC where the users are allowed to have different numbers of transmit antennas and transmit at different levels of multiplexing gain. The exact general MAC-DMT in such channel will be given in Section 4.2, and it will be seen that unlike the symmetric case, some users in the general MIMO-MAC are no longer able to achieve their single-user performance even if the multiplexing gains of other users are extremely close to zero. The reasons for such unexpected result will be carefully explained therein. Finally, in Section 4.4 the newly proposed code construction for symmetric channels will be extended to cater to the general MIMO-MAC. The MAC-DMT optimality of the generalized construction will be presented in Chapter 7.

1.3 Several Open Problems in Multiuser MIMO Communication

It is known that using multiple antennas at both transmitting and receiving ends in a point-to-point multiple-input-multiple-output (MIMO) channel can increase the transmission rate and simultaneously provide higher diversity gain. Assuming there are n_t transmit antennas and n_r receive antennas, it has been shown that the ergodic channel capacity of such MIMO Rayleigh block fading channel is approximately $\min\{n_t, n_r\} \log_2 \mathrm{SNR}$ in bits per channel use [23], and the maximal achievable diversity gain is $n_t n_r$ [1, 24], provided that the channel remains fixed for at least n_t channel uses. Let $R = r \log_2 \mathrm{SNR}$ be the actual transmission rate, where r is termed the multiplexing gain. Zheng and Tse [9] showed there is a fundamental tradeoff between multiplexing gain r and diversity value d. Such tradeoff is commonly known as the diversity-multiplexing gain tradeoff (DMT) and is reproduced below.

Theorem 2 ([9]). *In a MIMO Rayleigh block fading channel with n_t transmit and n_r receive antennas, assuming the transmitter transmits at multiplexing gain r, the maximal diversity gain $d^*(r)$ can be achieved by any coding schemes is a piecewise linear function connecting the points $(r, (n_t - r)(n_r - r))$ for $r = 0, 1, \cdots, \min\{n_t, n_r\}$, when the channel is fixed for at least $T \geq n_t + n_r - 1$ channel uses.* □

If the MIMO channel cannot hold static for at least $n_t + n_r - 1$ channel uses, some lower bounds on DMT based on Gaussian random coding schemes are provided in [9]. By using space-time codes constructed from cyclic division algebra (CDA) [25], Elia *et al.* [4] proved that the same DMT $d^*(r)$ holds whenever the channel is static for at least $T \geq n_t$ channel uses. However, such result cannot be further improved, and the exact DMT for $T < n_t$ is still uncertain.

In both DMT results, Theorems 1 and 2, the proofs proceed by first establishing an upper bound on DMT based on an outage formulation, and then by using a Gaussian random coding scheme to show the converse based on a union bound argument. It should be noted that in both point-to-point and MAC cases the requirement on the channel coherence time T for the optimal DMT to hold actually comes from the union bound, not the outage. When $T \geq Kn_t + n_t - 1$, Coronel *et al.* [16] presented a criterion for constructing MAC-DMT optimal codes. For any coding schemes, let $\mathcal{E}_k$ denote the error event that only the messages from k users are erroneously decoded. Coronel *et al.* showed that for any k-subsets of users, $1 \leq k \leq K$, if $\Pr\{\mathcal{E}_k\}$ is upper bounded by the probability of the corresponding outage event formulated by these k users, i.e. if one can show

$$\Pr\{\mathcal{E}_k\} \ \dot{\leq} \Pr\left\{\log\det\left(I_{n_r} + \mathrm{SNR} H_k H_k^\dagger\right) \leq kr \log \mathrm{SNR}\right\}, \tag{1.8}$$

where $H = [H_{i_1} \ \cdots H_{i_k}]$ is the overall channel matrix and H_{i_j} is the $(n_r \times n_t)$ channel matrix of the jth user, then the code is MAC-DMT optimal. Notions of exponential inequalities $\dot{\geq}$, $\dot{\leq}$, $\dot{>}$, $\dot{<}$, and equality $\doteq$ are defined in [9]. Specifically, in terms of code design, the above criterion (1.8) means that the $(kn_t \times T)$ matrix obtained by vertically concatenating the signal matrices from k users must be of full row rank and should perhaps satisfy the nonvanishing determinant (NVD) criterion [4, 26]. This full NVD design criterion was explicitly given in [16].

The aim of this report is to answer the following questions.

1. Is it possible to achieve the optimal MAC-DMT $d^*_{n_t,n_r,K}(r)$ when $T < Kn_t + n_r - 1$?

2. Is design criterion (1.8) necessary? or is it only sufficient?

3. In order to be MAC-DMT optimal, is it necessary for a code to satisfy the NVD criterion for any $(kn_t \times T)$ submatrix formed by any k-subsets of users?

4. In point-to-point MIMO channel, can one design a non-random DMT optimal code for $T < n_t$? Also, will the resulting DMT be the same as $d^*_{n_t,n_r}(r)$? In other words, when $T < n_t$, it relates to the question of whether the outage event will dominate the error performance.

The major contribution of this report is not to provide constructions of codes having performance better than the previously known DMT optimal codes, for example, the CDA based codes [4], the Golden perfect codes [20], the max-order codes [7], or the multi-block codes [6]. Instead, we aim to address the above four questions that none of these codes can answer.

By analyzing the DMT performance of a very simple code, we will provide answers to all the above questions. We will consider a MIMO-MAC channel with $K = 2$ users, each having only $n_t = 1$ transmit antenna, and we will assume there are $n_r = 2$ receive antennas at receiving end. While Theorem 1 holds for codes with $T \geq Kn_t + n_r - 1 = 3$ channel uses, we will prove this simple code achieves the same optimal MAC-DMT $d^*_{1,2,2}(r)$ with only $T = 1$ or 2 channel uses. Furthermore, from the DMT analysis of this code we will see that criterion (1.8) is only sufficient, not necessary, and one does not need full NVD in order to achieve the optimal MAC-DMT. By slightly modifying the parameters of this code, we will show in the point-to-point MIMO scenario this simple code achieves the same DMT performance as the Gaussian random code over the fast Rayleigh fading channel, i.e. the case when $T = 1$, which relates to the fourth question in the above list.

In Chapter 8 we will present the simple code as well as the corresponding DMT performance analysis. Inferences from the DMT analysis will be given in Chapter 9 and will answer all the above questions of interest.

Chapter 2

Relaxed Design Criterion of MAC-DMT Optimal Codes

In this section, we first present a rigorous, yet negative, result on the nonexistence of a MIMO-MAC lattice code that has the NVD property. This result suggests that the design criteria proposed by Coronel *et al.* [16] might be too strict to yield any MAC-DMT optimal codes. Following this, a relaxed design criterion will be presented and will be met by all subsequent constructions of MIMO-MAC codes in this report.

Consider a symmetric MIMO-MAC with K users, each having n_t transmit antennas and communicating independently to the base station at the same level of multiplexing gain r. Let $\mathcal{S}_0, \cdots,$ $\mathcal{S}_{K-1}$, be respectively the $(n_t \times T)$ space-time codes used by the kth user, $k = 0, 1, \cdots, K-1$, all satisfying the power constraint (1.4). If independent Gaussian random codebooks were used, i.e., the entries of code matrices $S_k \in \mathcal{S}_k$ are i.i.d. $\mathbb{C}\mathcal{N}\left(0, \frac{\text{SNR}}{n_t}\right)$ random variables for all k, Tse *et al.* [11] showed that the event $\mathcal{E}_m$ of m users in error has probability upper bounded by

$$\Pr\{\mathcal{E}_m\} \;\leq\; \Pr\{\mathcal{O}_m\} \;\doteq\; \text{SNR}^{-d^*_{mn_t,n_r}(mr)}, \tag{2.1}$$

where $\mathcal{O}_m$ is the event of m users in outage. Note that the overall error event $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \cdots \cup \mathcal{E}_K$. The union bound on $\mathcal{E}$ gives

$$\Pr\{\mathcal{E}\} \;\leq\; \sum_{m=1}^{K} \Pr\{\mathcal{E}_m\} \;\dot{\leq}\; \max_m \Pr\{\mathcal{O}_m\}. \tag{2.2}$$

Since the right-hand-side of (2.2) has a negative SNR-exponent equal to $d^*_{n_t,n_r,K}(r)$ defined in (1.6), (2.2) proved the achievability of MAC-DMT claimed by Theorem 1 based on the argument of Gaussian random codebooks.

We next turn our attention to the deterministic ST codes. From the point-to-point perspective, it is known [4] that ST codes satisfying the NVD property have the same error probability as the outage events. Thus, for any MIMO-MAC code $\{\mathcal{S}_0, \cdots, \mathcal{S}_{K-1}\}$, set

$$\mathcal{C}_k \;=\; \left\{\frac{1}{\kappa} S_k : S_k \in \mathcal{S}_k\right\}$$

where

$$\kappa^2 \;\doteq\; \text{SNR}^{1-\frac{r}{n_t}}.$$

To see how κ is chosen, we offer the following insight. For each k, the code $\mathcal{C}_k$ has size $|\mathcal{C}_k| = |\mathcal{S}_k| = \text{SNR}^{rT}$ so that it is of multiplexing gain r. An explicit construction of $\mathcal{C}_k$ was given in [4] where the code is seen as a real algebraic ST lattice code with dimension $2n_tT$. Hence there are $|\mathcal{C}_k|^{\frac{1}{2n_tT}} = \text{SNR}^{\frac{r}{2n_t}}$ PAM signals selected from each dimension and $\|C_k\|_F^2 \;\dot{\leq}\; \text{SNR}^{\frac{r}{n_t}}$ for all

$C_k \in \mathcal{C}_k$. Thus, the constant κ is chosen such that the code $\mathcal{S}_k = \kappa\, \mathcal{C}_k$ satisfies the power constraint (1.4).

From [4], it is easy to prove the following theorem which in turn gives a sufficient criterion for designing MAC-DMT optimal codes. We remark that this theorem is an alternative statement of the result given in [16] under certain restrictions, and we refer the interested readers to [17] for the connections.

Theorem 3 ([16]). *Let $\mathcal{C}_0, \cdots, \mathcal{C}_{K-1}$ be given as above. For any $\mathcal{I}_m = \{i_0, i_1, \cdots, i_{m-1}\} \subseteq \{0, 1, \cdots, K-1\}$, let $\mathcal{C}_{\mathcal{I}_m}$ be the product concatenation of $\mathcal{C}_{i_0}, \cdots, \mathcal{C}_{i_{m-1}}$, defined by*

$$\mathcal{C}_{\mathcal{I}_m} = \left\{ C_{\mathcal{I}_m} = \begin{bmatrix} C_{i_0} \\ \vdots \\ C_{i_{m-1}} \end{bmatrix} : C_{i_j} \in \mathcal{C}_{i_j}, i_j \in \mathcal{I}_m \right\}.$$

If for all pairs of distinct code matrices $C_{i_j} \neq C'_{i_j} \in \mathcal{C}_{i_j}$, $j = 0, 1, \cdots, m-1$, the difference matrix

$$\Delta C_{\mathcal{I}_m} = \begin{bmatrix} C_{i_0} - C'_{i_0} \\ \vdots \\ C_{i_{m-1}} - C'_{i_{m-1}} \end{bmatrix}, \tag{2.3}$$

satisfies $\det(\Delta C_{\mathcal{I}_m} \Delta C_{\mathcal{I}_m}^\dagger) \,\dot{\geq}\, 1$, where by $C^\dagger$ we mean the Hermitian transpose of matrix C, then the codes $\mathcal{C}_0, \cdots, \mathcal{C}_{K-1}$ are jointly MAC-DMT optimal.

Proof. Note that the imposed condition implies that the code $\mathcal{C}_{\mathcal{I}_m}$ satisfies the NVD property for any $\mathcal{I}_m$. Along similar lines as in [4], it can be shown that the error event $\mathcal{E}(\mathcal{I}_m)$ associated with code $\mathcal{C}_{\mathcal{I}_m}$, i.e., the error event of users in $\mathcal{I}_m$ in error, has probability upper bounded by

$$\Pr\{\mathcal{E}(\mathcal{I}_m)\} \;\dot{\leq}\; \Pr\{\mathcal{O}(\mathcal{I}_m)\} \doteq \text{SNR}^{-d^*_{mn_t,n_r}(mr)},$$

where $\mathcal{O}(\mathcal{I}_m)$ is the event of users in $\mathcal{I}_m$ in outage. Now taking union bound over all possible $\mathcal{I}_m$ as in (2.2) completes the proof. □

Remark 1. *The condition of* $\det(\Delta C_{\mathcal{I}_m} \Delta C_{\mathcal{I}_m}^\dagger) \,\dot{\geq}\, 1$ *for all $\mathcal{I}_m$ is called the* full NVD criterion *and is actually equivalent to the criterion given by Coronel et al. in [16] with certain restrictions, see [17] for details. It should be noted that this full NVD condition is only sufficient, not necessary. However, the following result suggests that this condition might be too strong and precludes the existence of codes meeting the criterion. We call the stronger condition* $\det(\Delta C_{\mathcal{I}_m} \Delta C_{\mathcal{I}_m}^\dagger) \geq 1$ *the* exactly full NVD criterion.

Theorem 4. *For any $K > 1$ and for any $n_t \geq 1$, there do not exist any linear MIMO-MAC codes*[1] *that satisfy the exactly full NVD criterion.*

Proof. For ease of reading, the proof is relegated to Chapter 5. □

[1] Here by linear codes we mean codes having linear dispersion forms [8] or having a lattice structure. Almost all existing ST codes are linear, for example, the Alamouti codes [2], the CDA-based ST codes [3–7, 13, 18–21, 27], etc.

Roughly speaking, the proof of Theorem 4 shows that while it is possible to construct DMT-optimal codes $\mathcal{C}_0, \cdots, \mathcal{C}_{K-1}$ for each user, as the existing CDA-based ST codes [4] would do, it is impossible for the product code $\mathcal{C}_0 \times \cdots \times \mathcal{C}_{K-1}$ to have an exactly full NVD. Any such product code would have difference matrices $\Delta C_{\mathcal{I}_m}$ such that $\det(\Delta C_{\mathcal{I}_m} \Delta C_{\mathcal{I}_m}^\dagger)$ is extremely close to zero at high-SNR regime. In terms of conventional rank and coding gain design criteria of ST codes, this means that even if the code achieves full diversity gain, it necessarily loses significantly in coding gain. Therefore, it becomes meaningless to say that the code achieves full rank and full diversity. We may conclude that the exactly full NVD condition is in practice too strict to yield MAC-DMT optimal codes.

Another implication from the proof of Theorem 4 is that the exactly full NVD condition can be met only if the users cooperate in their transmission. Without cooperation, the exactly full NVD condition can never be met and the determinant must be vanishing.

On the other hand, we may relax the exactly full NVD condition without adversely affecting the DMT performance. To do so, we will partition the error events in a different manner. Given the set of users $\mathcal{I}_m$, let $\mathcal{E}_n(\mathcal{I}_m)$, $1 \leq n \leq m$, denote the error event when the users in $\mathcal{I}_m$ are in error and the corresponding error matrix $\Delta C_{\mathcal{I}_m}$ (cf. (2.3)) has rank exactly nn_t. Clearly event $\mathcal{E}(\mathcal{I}_m)$ defined in the proof of Theorem 3 is a disjoint union of $\mathcal{E}_1(\mathcal{I}_m), \cdots, \mathcal{E}_m(\mathcal{I}_m)$. Now the codes $\mathcal{C}_0, \cdots, \mathcal{C}_{K-1}$ are jointly MAC-DMT optimal if the following holds.

Theorem 5 (Relaxed design criterion)**.** *Let $\mathcal{C}_0, \cdots, \mathcal{C}_{K-1}$ be defined as above. Then they are jointly MAC-DMT optimal if the error events $\mathcal{E}_n(\mathcal{I}_m)$ have probabilities upper bounded by*

$$\Pr\{\mathcal{E}_n(\mathcal{I}_m)\} \ \dot{\leq} \ \text{SNR}^{-d^*_{nn_t,n_r}(nr)} \tag{2.4}$$

for all $1 \leq n \leq m \leq K$ and for all $\mathcal{I}_m \subseteq \{0, 1, \cdots, K-1\}$. Furthermore, as for design of MAC-DMT optimal codes we require at least that

$$\Pr\{\mathcal{E}_n(\mathcal{I}_m)\} \ \dot{\leq} \ \text{SNR}^{-\min\{d^*_{n_t,n_r}(r), d^*_{Kn_t,n_r}(Kr)\}} \tag{2.5}$$

for all $1 \leq n \leq m \leq K$ and for all $\mathcal{I}_m$. □

While (2.5) might be the most relaxed condition for designing MAC-DMT optimal codes, in this report we will focus on condition (2.4). The rationale behind the above theorem is the observation that the error probabilities $\text{SNR}^{-d^*_{mn_t,n_r}(mr)}$ with $1 < m < K$ are not dominant in the overall DMT performance. Hence we could relax the conditions such that the event

$$\mathcal{E}(\mathcal{I}_m) \ = \ \bigcup_{n=1}^{m} \mathcal{E}_n(\mathcal{I}_m)$$

has probability larger than the corresponding outage probability, but no larger than the dominant error probability. That is, we could allow

$$\Pr\{\mathcal{E}(\mathcal{I}_m)\} \ \gg \ \Pr\{\mathcal{O}(\mathcal{I}_m)\} \ \doteq \text{SNR}^{-d^*_{mn_t,n_r}(mr)}, \tag{2.6}$$

but would still require

$$\Pr\{\mathcal{E}(\mathcal{I}_m)\} \ \dot{\leq} \ \text{SNR}^{-\min\{d^*_{n_t,n_r}(r), d^*_{Kn_t,n_r}(Kr)\}}.$$

Relaxation (2.6) would not affect the overall DMT performance. Compared with the exactly full NVD condition required by Theorems 3, Theorem 5 relaxes greatly the code design criterion in the following ways.

1. We do not require the difference matrix $\Delta C_{\mathcal{I}_m}$ to be nonsingular and to satisfy the NVD property when all the component matrices $C_{i_j} - C'_{i_j}$ are nonzero, which has been shown to be impossible by Theorem 4.

2. Should the difference matrix $\Delta C_{\mathcal{I}_m}$ happen to be singular, (2.4) requires the resulting error performance must be no worse than $\mathrm{SNR}^{-d^*_{nn_t,n_r}(nr)}$ for some n, $1 \leq n \leq m$, in order to maintain the MAC-DMT optimality.

3. In Theorem 3, events $\mathcal{E}_n(\mathcal{I}_m)$ with $n < m$ were required to have probability absolutely zero. This is too strict and would preclude the existence of MAC-DMT optimal codes.

Chapter 3

MAC-DMT Optimal Code Construction for Symmetric MIMO-MAC Channels

For the symmetric MIMO-MAC coded system with K users, each having n_t transmit antennas and transmitting at multiplexing gain r, in this section we will propose a systematic code construction that is MAC-DMT optimal for any combinations of K, n_t, n_r, and r. The construction does not assume any cooperation among the users. Furthermore, compared with the MAC-DMT optimal two-user code proposed in [22] where a sign change is required in the code matrices, here in the proposed method each user encodes his own information using an identical encoder. This greatly simplifies the hardware implementation of these encoders.

3.1 Proposed Construction

Given the number of users K, let K_o be the smallest odd integer such that $K_o \geq K$, i.e.,

$$K_o = \begin{cases} K+1, & \text{if } K \text{ even,} \\ K, & \text{if } K \text{ odd.} \end{cases} \tag{3.1}$$

The construction calls for the following number fields. Let $\mathbb{K}_o = \mathbb{F}(\eta_o)$ be a number field that is a cyclic Galois extension of $\mathbb{F} = \mathbb{Q}(\imath)$ with degree K_o, where $\imath = \sqrt{-1}$. Let $\mathbb{L} = \mathbb{F}(\theta)$ be another cyclic Galois extension of $\mathbb{F}$ with degree n_t. Let σ and τ_o be the generators of Galois groups Gal($\mathbb{L}/\mathbb{F}$) and Gal($\mathbb{K}_o/\mathbb{F}$) with degrees n_t and K_o, respectively. The fields $\mathbb{K}_o$ and $\mathbb{L}$ are chosen[1] such that $\mathbb{K}_o \cap \mathbb{L} = \mathbb{F}$. Let $\mathbb{E}_o = \mathbb{K}_o\mathbb{L} = \mathbb{F}(\eta_o, \theta)$ be the compositum of $\mathbb{K}_o$ and $\mathbb{L}$. See Fig.3.1 for the relation among the required number fields. The readers are referred to [4,22,28] for the constructions of such number fields.

Let $\mathfrak{D}_o := (\mathbb{E}_o/\mathbb{K}_o, \sigma, \zeta)$ be a cyclic division algebra with

$$\mathfrak{D}_o = \mathbb{E}_o \oplus z\mathbb{E}_o \oplus \cdots \oplus z^{n_t-1}\mathbb{E}_o, \tag{3.2}$$

where

$$\zeta = \frac{\gamma}{\gamma^*}, \tag{3.3}$$

$$xz = z\sigma(x) \tag{3.4}$$

for $x \in \mathbb{E}_o$. The element z is an indeterminate satisfying $z^{n_t} = \zeta \in \mathbb{F}^*$, and $0 \neq \gamma \in \mathcal{O}_\mathbb{F}$ is some suitable nonnorm element [2]. By γ^* we mean the complex conjugate of γ and $\mathcal{O}_\mathbb{F}$ is the algebraic closure of $\mathbb{Z}$ in $\mathbb{F}$ [25, 29, 30]. Notice that $\|\zeta\| = 1$ and ζ is unimodular. It has been shown [5] that with such unimodular ζ, $\mathfrak{D}_o$ is always a cyclic division algebra.

[1] A more general condition on $\mathbb{K}_o$ and $\mathbb{L}$ is that the automorphisms σ and τ_0 commute.

[2] A sufficient criterion for finding a suitable nonnorm element γ is given in [26, Theorem 1]. Also, we refer the interested readers to [4, Theorems 10 and 11] for two explicit constructions of γ.

$$
\begin{tikzcd}
 & \mathbb{E}_o = \mathbb{F}(\theta, \eta_o) \arrow[dl, dash, "K_o"'] \arrow[dd, dash, "n_t K_o"'] \arrow[dr, dash, "n_t"] & \\
\mathbb{L} = \mathbb{F}(\theta) \arrow[dr, dash, "n_t"'] & & \mathbb{K}_o = \mathbb{F}(\eta_o) \arrow[dl, dash, "K_o"] \\
 & \mathbb{F} = \mathbb{Q}(\imath) &
\end{tikzcd}
$$

Figure 3.1: Field extensions required by the proposed code constructions.

Remark 2. *While in the above we have set ζ to be of form $\zeta = \frac{\gamma}{\gamma^*}$ such that ζ is unimodular, it might be possible that in some CDAs, the nonnorm element γ is actually an nth root of unity for some integer n and is already unimodular. See [31] for such example construction. Should it be the case, we could set $\zeta = \gamma$, and the discussion below can be easily modified to show that the MAC-DMT optimality of the proposed constructions remains to hold. Therefore, for simplicity, here we will focus only on the case of $\zeta = \frac{\gamma}{\gamma^*}$.* $\square$

Remark 3. *We note that by construction the Galois groups of the numbers fields are*

$$
\begin{aligned}
\text{Gal}(\mathbb{E}_o/\mathbb{K}_o) &= \langle \sigma \rangle , \\
\text{Gal}(\mathbb{E}_o/\mathbb{L}) &= \langle \tau_o \rangle , \\
\text{Gal}(\mathbb{E}_o/\mathbb{F}) &= \langle \tau_o, \sigma \rangle = \langle \tau_o \rangle \times \langle \sigma \rangle ,
\end{aligned}
$$

where in the last line $\langle \tau_o \rangle \times \langle \sigma \rangle$ denotes the direct product of the groups generated by τ_o and σ, respectively. It should also be noted that the automorphisms τ_o and σ commute, i.e.,

$$
\tau_o \sigma \;=\; \sigma \tau_o
$$

due to the direct product of two groups. $\square$

Given multiplexing gain r, let $\mathcal{A}(\text{SNR})$ be the base alphabet defined as

$$
\mathcal{A}(\text{SNR}) = \left\{ a + b\imath \; : \; \begin{array}{r} -\text{SNR}^{\frac{r}{2n_t}} \le a, b \le \text{SNR}^{\frac{r}{2n_t}}, \\ a, b \in \mathbb{Z}, \;\; a, b \text{ odd} \end{array} \right\};
$$

then the corresponding information set is

$$
\mathfrak{A}_o(\text{SNR}) = \left\{ \sum_{i=0}^{n_t-1} z^i \sum_{k=0}^{K_o n_t - 1} x_{i,k} e_k : x_{i,k} \in \mathcal{A}(\text{SNR}) \right\}, \tag{3.5}
$$

where $\{e_0, \cdots, e_{K_o n_t - 1}\}$ is an integral basis of $\mathbb{E}_o/\mathbb{F}$. It should be noted that $\sum_{k=0}^{K_o n_t - 1} x_{i,k} e_k \in \mathbb{E}_o$ for $x_{i,k} \in \mathcal{A}(\text{SNR}) \subset \mathcal{O}_{\mathbb{F}}$ and that $\mathfrak{A}_o(\text{SNR}) \subset \mathfrak{D}_o$. Let

$$
\psi_o : \mathfrak{D}_o \to M_{n_t}(\mathbb{E}_o)
$$

be the left-regular map that maps elements in $\mathfrak{D}_o$ into $(n_t \times n_t)$ square matrices with entries in $\mathbb{E}_o$. Specifically, given $u \in \mathfrak{D}_o$ with

$$
u \;=\; \sum_{i=0}^{n_t-1} z^i u_i, \quad u_i \in \mathbb{E}_o,
$$

$\psi_o(u)$ is given by

$$\psi_o(u) := \begin{bmatrix} u_0 & \zeta\sigma(u_{n_t-1}) & \cdots & \zeta\sigma^{n_t-1}(u_1) \\ u_1 & \sigma(u_0) & \cdots & \zeta\sigma^{n_t-1}(u_2) \\ \vdots & \vdots & \ddots & \vdots \\ u_{n_t-1} & \sigma(u_{n_t-2}) & \cdots & \sigma^{n_t-1}(u_0) \end{bmatrix}. \tag{3.6}$$

Note that the field $\mathbb{K}_o$ is the center of the division algebra $\mathfrak{D}_o$, meaning that $uk = ku$ for any $u \in \mathfrak{D}_o$ and $k \in \mathbb{K}_o$. Equivalently we have

$$\psi_o(u)\psi_o(k) = \psi_o(k)\psi_o(u),$$

showing that the matrix-product commutes.

Proposition 6 ([4, 25])**.** *Let $\mathfrak{D}_o$ and ψ_o be defined as above. Then*

$$\det\left(\psi_o(u)\right) \in \mathbb{K}_o^*$$

for all $0 \neq u \in \mathfrak{D}_o$, where $\mathbb{K}_o^* = \mathbb{K}_o \setminus \{0\}$. □

Having defined the above, the encoding of each user's data stream proceeds as follows. Given the multiplexing gain r, the ith user first partitions his binary data steam into blocks of $rK_on_t \log_2$ SNR bits. Then using the integral basis $\{e_0, \cdots, e_{K_on_t-1}\}$ and set $\mathfrak{A}_o(\text{SNR})$ defined above, each block of binary bits is mapped in a one-one fashion to a symbol $x_i \in \mathfrak{A}_o(\text{SNR}) \subset \mathfrak{D}_o$. The encoding is performed independently at each user's end.

Given $x_i \in \mathfrak{A}_o(\text{SNR})$, the ith user actually sends out the following $(n_t \times K_on_t)$ signal matrix S_i through his n_t transmit antenna array in K_on_t channel uses

$$S_i \;=\; \kappa\left[\begin{array}{cccc} X_i & \tau_o(X_i) & \cdots & \tau_o^{K_o-1}(X_i) \end{array}\right], \tag{3.7}$$

where $X_i = \psi_o(x_i)$ and where κ is a normalizing constant such that

$$\mathbb{E}\,\|S_i\|_F^2 \;=\; n_tK_o\text{SNR} \;\doteq\; \text{SNR}.$$

Hence we have

$$\kappa^2 \doteq \text{SNR}^{1-\frac{r}{n_t}} \tag{3.8}$$

Remark 4. *The above construction of the MIMO-MAC codes is reminiscent of the multi-block ST code presented in [6]. Some key differences are highlighted below.*

1. *In the proposed construction we require the length of the code to be $n_t \cdot K_o$ where K_o must be an odd integer.*

2. *The number fields $\mathbb{K}_o$ and $\mathbb{L}$ are required such that the automorphisms σ and τ_o commute. This was not needed in [6].*

3. *The element ζ of the CDA $\mathfrak{D}_o$ must be unimodular, and we have set $\zeta = \frac{\gamma}{\gamma^*}$.*

□

We use the following example to illustrate the proposed construction.

Example 1. *We consider the case of $K = 2$ and $n_t = 2$. By construction $K_o = 3$ is the smallest odd integer such that $K_o \geq K$. Then it can be shown that with $\theta = e^{\imath \frac{\pi}{8}}$ and $\eta_o = 2\cos\left(\frac{2\pi}{7}\right)$ the number fields $\mathbb{L} = \mathbb{F}(\theta)$ and $\mathbb{K}_o = \mathbb{F}(\eta_o)$ meet the required conditions of $[\mathbb{L} : \mathbb{F}] = 2$, $[\mathbb{K}_o : \mathbb{F}] = 3$ and $\mathbb{L} \cap \mathbb{K}_o = \mathbb{F}$. Furthermore, we have $\eta_o^3 + \eta_o^2 - 2\eta_o - 1 = 0$. The generators σ and τ_o for the Galois groups $\mathrm{Gal}(\mathbb{L}/\mathbb{F})$ and $\mathrm{Gal}(\mathbb{K}_o/\mathbb{F})$ are given respectively by*

$$\sigma : \theta \mapsto -\theta \quad \text{and } \tau_o : \eta_o \mapsto \left(\eta_o^2 - 2\right) = 2\cos\left(\frac{4\pi}{7}\right)$$

The set $\{1, \theta, \eta_o, \theta\eta_o, \eta_o^2, \theta\eta_o^2\}$ is an integral basis for $\mathbb{E}_o/\mathbb{F}$.

As the prime ideal $(2 + \imath)$ of $\mathbb{Z}[\imath]$ remains inert in $\mathcal{O}_{\mathbb{K}_o}$ and $\mathcal{O}_{\mathbb{L}}$, following from [4] this gives an appropriate nonnorm element $\gamma = 2 + \imath$. Hence we have $\zeta = \frac{2+\imath}{2-\imath}$. With $\mathbb{E}_o = \mathbb{F}(\theta, \eta_o)$, $\mathfrak{D}_o = (\mathbb{E}_o/\mathbb{K}_o, \sigma, \zeta)$ is a CDA of index 2 which is also a central simple $\mathbb{K}_o$-algebra [25]. Next let

$$u_i \;=\; x_{i,0} + \theta x_{i,1} + \eta_o x_{i,2} + \theta\eta_o x_{i,3} + \eta_o^2 x_{i,4} + \theta\eta_o^2 x_{i,5}$$

for $i = 0, 1$ with $x_{i,j} \in \mathcal{A}(\mathrm{SNR})$. The Galois conjugates of u_i are for example given by

$$\begin{aligned}\sigma(u_i) &= x_{i,0} - \theta x_{i,1} + \eta_o x_{i,2} - \theta\eta_o x_{i,3} + \eta_o^2 x_{i,4} - \theta\eta_o^2 x_{i,5},\\ \tau_o(u_i) &= x_{i,0} + \theta x_{i,1} + \eta_o' x_{i,2} + \theta\eta_o' x_{i,3} + \eta_o'^2 x_{i,4} + \theta\eta_o'^2 x_{i,5}\end{aligned}$$

where $\eta_o' = \eta_o^2 - 2 = 2\cos\left(\frac{4\pi}{7}\right)$ and $\tau_o(\eta_o') = 1 - \eta_o - \eta_o^2 = 2\cos\left(\frac{8\pi}{7}\right)$. With the above, the signal matrix of the first user is given by $S_0 = \kappa \begin{bmatrix} X_0 & \tau_o(X_0) & \tau_o^2(X_0) \end{bmatrix}$, where $\kappa^2 = \mathrm{SNR}^{1-\frac{r}{2}}$ and

$$X_0 \;=\; \begin{bmatrix} u_0 & \zeta\sigma(u_1) \\ u_1 & \sigma(u_0) \end{bmatrix}.$$

□

By vertically concatenating the signal matrices from all users, the overall MIMO-MAC code of the K users is

$$\mathcal{S} = \left\{ \begin{array}{l} S = \kappa \begin{bmatrix} X_0 & \cdots & \tau_o^{K_o-1}(X_0) \\ \vdots & \ddots & \vdots \\ X_{K-1} & \cdots & \tau_o^{K_o-1}(X_{K-1}) \end{bmatrix} : \\ \\ \hfill X_i = \psi_o(x_i), x_i \in \mathfrak{A}_o(\mathrm{SNR}) \end{array} \right\}. \tag{3.9}$$

For ease of code performance analysis that comes later we set $\mathcal{C} = \frac{1}{\kappa}\mathcal{S}$, i.e.,

$$\mathcal{C} = \left\{ \begin{array}{l} C = \begin{bmatrix} X_0 & \cdots & \tau_o^{K_o-1}(X_0) \\ \vdots & \ddots & \vdots \\ X_{K-1} & \cdots & \tau_o^{K_o-1}(X_{K-1}) \end{bmatrix} : \\ \\ \hfill X_i = \psi_o(x_i), x_i \in \mathfrak{A}_o(\mathrm{SNR}) \end{array} \right\}. \tag{3.10}$$

Remark 5. *Below we briefly compare the proposed construction of $\mathcal{S}$ with another MAC-DMT optimal code constructed for $K = 2$ users in [22]. The latter MIMO-MAC code takes the following form*

$$\mathcal{S}_2 \;=\; \left\{ \begin{array}{l} S_2 = \kappa \begin{bmatrix} X_0 & \tau(X_0) \\ X_1 & -\tau(X_1) \end{bmatrix} : \\ \\ X_i = \psi(x_i), x_i \in \mathfrak{A}(\mathrm{SNR}) \end{array} \right\}. \tag{3.11}$$

The construction of $\mathcal{S}_2$ requires a number field $\mathbb{K} = \mathbb{F}(\eta)$ with $[\mathbb{K} : \mathbb{F}] = 2$ and $\mathrm{Gal}(\mathbb{K}/\mathbb{F}) = \{1, \tau\}$ such that $\mathbb{E} = \mathbb{K}\mathbb{L} = \mathbb{F}(\theta, \eta)$, $[\mathbb{E} : \mathbb{F}] = 2n_t$ and $\mathrm{Gal}(\mathbb{E}/\mathbb{F}) = \mathrm{Gal}(\mathbb{L}/\mathbb{F}) \times \mathrm{Gal}(\mathbb{K}/\mathbb{F})$. Here by "1" of $\mathrm{Gal}(\mathbb{K}/\mathbb{F})$ we mean the trivial automorphism. The field $\mathbb{L}$ and the element θ are defined as before. The element x_i is taken from the cyclic division algebra $\mathfrak{D} = \mathbb{E} \oplus z'\mathbb{E}$ for some indeterminate z'. $\mathfrak{A}(\mathrm{SNR})$ is the base-information set defined similarly as $\mathfrak{A}_o(\mathrm{SNR})$ in (3.5). Thus, compared with the present proposed construction, we see that $\mathcal{S}_2$ requires an additional sign change at the second block matrix of the second user's code. This sign change is essential to ensure an NVD-like property. It also endows $\mathcal{S}_2$ with another nice property that the transmission of code matrices in $\mathcal{S}_2$ takes only $2n_t$ channel uses, less than that required by $\mathcal{S}$. However, this additional sign change might complicate system design as the system must constantly check which user requires a sign change and which user does not. Such disadvantage does not exist in the proposed construction of $\mathcal{S}$. Everything works perfectly after patching an extra block of transmission when K is even. Another drawback of $\mathcal{S}_2$ is the difficulty of generalization to the cases of $K > 2$. □

Let H_i be the $(n_r \times n_t)$ channel matrix of the ith user. We assume H_i is fixed for a block of $n_t K_o$ channel uses. Following (1.5), given the overall transmitted code matrix $S \in \mathcal{S}$, the received signal matrix at receiver end is

$$\begin{bmatrix} Y_0 & \cdots & Y_{K_o-1} \end{bmatrix} = \begin{bmatrix} H_0 & \cdots & H_{K-1} \end{bmatrix} S + W. \tag{3.12}$$

W is the noise matrix whose entries are i.i.d. $\mathbb{C}\mathcal{N}(0,1)$ random variables, and Y_j is the jth block received signal matrix given by

$$Y_j := \kappa \sum_{i=0}^{K-1} H_i \tau_o^j(X_i) + W_j, \quad j = 0, 1, \cdots, K_o - 1,$$

and

$$W = \begin{bmatrix} W_0 & W_1 & \cdots & W_{K_o-1} \end{bmatrix}.$$

3.2 Properties of the Proposed Construction

To simplify the analysis of code performance, below we define the extended versions of $\mathcal{S}$ and $\mathcal{C}$.

$$\mathcal{C}_o := \left\{ \begin{array}{c} C_o = \begin{bmatrix} X_0 & \cdots & \tau_o^{K_o-1}(X_0) \\ \vdots & \ddots & \vdots \\ X_{K_o-1} & \cdots & \tau_o^{K_o-1}(X_{K_o-1}) \end{bmatrix} : \\ \\ X_i = \psi_o(x_i), x_i \in \mathfrak{A}_o(\mathrm{SNR}) \end{array} \right\}, \tag{3.13}$$

$$\mathcal{S}_o := \{ S_o = \kappa C_o \ : \ C_o \in \mathcal{C}_o \}. \tag{3.14}$$

Given the overall signal matrix $S \in \mathcal{S}$, let $S_o \in \mathcal{S}_o$ be any signal matrix such that the upper $(Kn_t \times K_o n_t)$ submatrix of S_o equals S. Then we can rewrite (3.12) as

$$\begin{bmatrix} Y_0 & \cdots & Y_{K_o-1} \end{bmatrix} = \begin{bmatrix} H_0 & \cdots & H_{K_o-1} \end{bmatrix} S_o + W, \tag{3.15}$$

where

$$H_{K_o-1} = \begin{cases} H_{K-1}, & \text{if } K \text{ odd}, \\ \mathbf{0}, & \text{if } K \text{ even}. \end{cases}$$

By $\mathbf{0}$ we mean the all-zero matrix of proper size. Noting (3.12) and (3.15) are equivalent, henceforth we will work only with the extended codes $\mathcal{S}_o$ and $\mathcal{C}_o$, rather than $\mathcal{S}$ and $\mathcal{C}$. We next show several nice properties possessed by $\mathcal{S}_o$ and $\mathcal{C}_o$.

Property 1. *For any $C_o \in \mathcal{C}_o$, we have*

$$\left[(\gamma^*)^{K_o(n_t-1)} \det(C_o)\right] \in \mathbb{Z}[\imath]. \tag{3.16}$$

Proof. We first claim

$$\tau_o(\det(C_o)) = \det(C_o). \tag{3.17}$$

To see this, notice that

$$\begin{aligned}
\tau_o(\det(C_o)) &= \det\begin{pmatrix} \tau_o(X_0) & \cdots & \tau_o^{K_o}(X_0) \\ \vdots & \ddots & \vdots \\ \tau_o(X_{K_o-1}) & \cdots & \tau_o^{K_o}(X_{K_o-1}) \end{pmatrix} \\
&= \det\begin{pmatrix} \tau_o(X_0) & \cdots & X_0 \\ \vdots & \ddots & \vdots \\ \tau_o(X_{K_o-1}) & \cdots & X_{K_o-1} \end{pmatrix} \\
&= (-1)^{n_t(K_o-1)} \det(C_o) = \det(C_o),
\end{aligned}$$

where the last equality follows from the fact that $K_o - 1$ is even, hence the claim (3.17) is proved. Next, we show

$$\sigma(\det(C_o)) = \det(C_o). \tag{3.18}$$

To this end, define

$$Z = \psi_o(z), \tag{3.19}$$

where z is the indeterminate defined as in (3.2). Since from (3.4) $xz = z\sigma(x)$ for all $x \in \mathbb{E}_o$, it is clear that $\sigma(X) = Z^{-1}XZ$, where $X = \psi_o(x)$. Now we have

$$\begin{aligned}
&\sigma(\det(C_o)) \\
&= \begin{vmatrix} Z^{-1}X_0Z & \cdots & \tau_o^{K_o-1}(Z^{-1}X_0Z) \\ \vdots & \ddots & \vdots \\ Z^{-1}X_{K_o-1}Z & \cdots & \tau_o^{K_o-1}(Z^{-1}X_{K_o-1}Z) \end{vmatrix} \\
&= \begin{vmatrix} Z^{-1}X_0Z & \cdots & Z^{-1}\tau_o^{K_o-1}(X_0)Z \\ \vdots & \ddots & \vdots \\ Z^{-1}X_{K_o-1}Z & \cdots & Z^{-1}\tau_o^{K_o-1}(X_{K_o-1})Z \end{vmatrix} \\
&= \begin{vmatrix} Z^{-1} & & \\ & \ddots & \\ & & Z^{-1} \end{vmatrix} \times \\
&\quad \begin{vmatrix} X_0 & \cdots & \tau_o^{K_o-1}(X_0) \\ \vdots & \ddots & \vdots \\ X_{K_o-1} & \cdots & \tau_o^{K_o-1}(X_{K_o-1}) \end{vmatrix} \begin{vmatrix} Z & & \\ & \ddots & \\ & & Z \end{vmatrix} \\
&= \det(C_o),
\end{aligned}$$

where we have used the fact that $\tau_o(Z) = Z$ since $0 \neq \zeta \in \mathbb{F}$ by construction. Thus, as $\det(C_o)$ is fixed by both τ_o and σ, we see that $\det(C_o) \in \mathbb{F} = \mathbb{Q}(\imath)$.

Finally, from the definition of ψ_o (3.6), the matrix

$$\tau_o^j(X_i) \begin{bmatrix} 1 & & & \\ & \gamma^* & & \\ & & \ddots & \\ & & & \gamma^* \end{bmatrix}$$

has entries in $\mathcal{O}_{\mathbb{E}_o}$ for all $i = 0, 1, \cdots, K_o - 1$ and $j = 0, 1, \cdots, n_t - 1$ since

$$\mathfrak{A}_o(\mathrm{SNR}) \subset \mathcal{O}_{\mathbb{E}_o} \oplus z\mathcal{O}_{\mathbb{E}_o} \oplus \cdots \oplus z^{n_t-1}\mathcal{O}_{\mathbb{E}_o}.$$

$\mathcal{O}_{\mathbb{E}_o}$ is the ring of algebraic integers in number field $\mathbb{E}_o$. It then follows that

$$\left[(\gamma^*)^{K_o(n_t-1)} \det(C_o)\right] \in \mathcal{O}_{\mathbb{E}_o}.$$

Summarizing the above results, we conclude that

$$\left[(\gamma^*)^{K_o(n_t-1)} \det(C)\right] \in \mathcal{O}_{\mathbb{E}_o} \cap \mathbb{Q}(\imath) = \mathbb{Z}[\imath],$$

and this completes the proof. $\square$

Property 2. *Let*

$$\mathfrak{C} = \begin{bmatrix} \underline{x}_0^\top \\ \vdots \\ \underline{x}_{K_o-1}^\top \end{bmatrix} = \begin{bmatrix} x_0 & \cdots & \tau_o^{K_o-1}(x_0) \\ \vdots & \ddots & \vdots \\ x_{K_o-1} & \cdots & \tau_o^{K_o-1}(x_{K_o-1}) \end{bmatrix} \tag{3.20}$$

and

$$C_o = \begin{bmatrix} X_0 & \cdots & \tau_o^{K_o-1}(X_0) \\ \vdots & \ddots & \vdots \\ X_{K_o-1} & \cdots & \tau_o^{K_o-1}(X_{K_o-1}) \end{bmatrix} \in \mathcal{C}_o$$

with $X_i = \psi_o(x_i)$, $x_i \in \mathfrak{A}_o(\mathrm{SNR})$, where by $\underline{x}^\top$ we mean the transpose of vector $\underline{x}$. Let m be the maximal number of rows in $\mathfrak{C}$ that are linearly independent as a left $\mathfrak{D}_o$-module; then

$$\mathrm{rank}(C_o) \;=\; mn_t \tag{3.21}$$

where the rank is measured in the complex number field $\mathbb{C}$.

Proof. To find out the rank of matrix C_o, we use the elementary row operations from Gaussian elimination method. Note that the same row operations can be performed on $\mathfrak{C}$ whose entries are in $\mathfrak{D}_o$. Extra care must be taken because multiplication in $\mathfrak{D}_o$ is non-commutative. Further, we note that elementary row operations on $\mathfrak{C}$ are equivalent to the block elementary row operations on C_o. By this we mean that, say P is a $(K_o \times K_o)$ elementary matrix with entries in $\mathfrak{D}_o$; then it is clear

$$\Psi_o(P\mathfrak{C}) \;=\; \Psi_o(P)C_o,$$

where Ψ_o is the natural extension of ψ_o to the $(K_o \times K_o)$ central simple matrix algebra $M_{K_o}(\mathfrak{D}_o)$ over $\mathfrak{D}_o$ [25], i.e.,

$$\Psi_o(P) = \left[\psi_o(P_{i,j})\right]. \tag{3.22}$$

From hypothesis, assume $\{\underline{x}_{i_0}^\top, \cdots, \underline{x}_{i_{m-1}}^\top\}$ is the maximal subset of the rows of $\mathfrak{C}$ that are linearly independent over $\mathfrak{D}_o$. Then it follows that there are m leading ones in the row-reduced matrix of $\mathfrak{C}$. Equivalently, the same block elementary operations $\Psi_o(P)$ would reduce matrix C_o into a matrix whose main diagonal consists of m identity matrices, each of size $(n_t \times n_t)$, after permuting the columns if necessary. This completes the proof. $\square$

Property 2 shows that the overall code matrix $C_o \in \mathcal{C}_o$ might not always have full rank $K_o n_t$, and the rank of C_o is always a multiple of n_t. This is not too much of a surprise as it is straightforward to see that in (3.13) if some X_i's are identical, then the overall code matrix C_o cannot be nonsingular.

Compared with the constructions proposed in [18,19], the matrix C_o of the present construction could be singular even when the component matrices X_i are all distinct and nonzero as shown by Property 2. Nevertheless, we will prove in Chapter 6 that in order to achieve the optimal MAC-DMT performance at high-SNR regime, it is unnecessary to construct codes such that C_o is nonsingular whenever all the component matrices X_i are distinct and nonzero.

Before rigorously proving the above statement, a heuristic way to see this is the following. Since the users communicate independently to the base station, for any overall MIMO-MAC code $\mathcal{C}$ it is impossible for all the code matrices $C \in \mathcal{C}$ to be nonsingular as some component matrices C_k of the kth user could be zero. Also, from the pairwise error probability point of view, for any $C \neq C' \in \mathcal{C}$, $C - C'$ can be singular at least when the information symbols transmitted by some users are the same. The rank of overall code matrices C_o is at best a multiple of n_t. Therefore, intuitively speaking, perhaps it would not hurt to make things a bit worse in the sense that the difference matrix $C - C'$ can be singular in other cases. By this we mean that if there are m distinct information symbols in the difference matrix $C - C'$, the maximal possible rank of $C - C'$ is mn_t. We claim that it would not hurt in the DMT sense if the construction can provide only rank nn_t for some n with $1 < n < m$. The reason for this actually follows from Theorem 5 that the error events $\mathcal{E}_n(\mathcal{I}_m)$ of m users in error but getting only rank distance nn_t do not dominate the error performance in the final DMT performance. Therefore, we strongly speculate that such difference matrices $C - C'$ do not have to achieve the same rank mn_t as the Gaussian random code does. The rank can be less, as long as the resulting error performance is not worse than those of $m = 1$ and $m = K_o$.

Although we do not need the whole code $\mathcal{C}_o$ to satisfy the full NVD property as in the point-to-point scenario, an alternative NVD-like property is preferred and is given as below.

Property 3. *Let $\mathfrak{C}$ be defined as in* (3.20) *and assume that* $\{\underline{x}_{i_0}^\top, \cdots, \underline{x}_{i_{m-1}}^\top\}$ *is a subset of rows of $\mathfrak{C}$ that are linearly independent as a left $\mathfrak{D}_o$-module. Define*

$$\mathfrak{C}_s := \begin{bmatrix} \underline{x}_{i_0}^\top \\ \vdots \\ \underline{x}_{i_{m-1}}^\top \end{bmatrix} \qquad \text{and } C_s := \Psi_o\left(\mathfrak{C}_s\right), \tag{3.23}$$

i.e., C_s is the submatrix of C_o consisting of the corresponding linearly independent mn_t rows, where Ψ_o is the natural extension of ψ_o. Then

$$1 \leq \left[|\gamma|^{2mn_t} \cdot \det\left(C_s C_s^\dagger\right)\right] \in \mathbb{Z}, \tag{3.24}$$

where by $A^\dagger$ we mean the hermitian transpose of matrix A.

Proof. First, it follows from Property 2 that

$$\left[|\gamma|^{2mn_t} \cdot \det\left(C_s C_s^\dagger\right)\right] \; > \; 0$$

since C_s has full row rank mn_t and $\gamma \neq 0$ by assumption. To show $|\gamma|^{2mn_t} \cdot \det\left(C_s C_s^\dagger\right) \in \mathbb{Z}$, we shall first verify that $\det\left(C_s C_s^\dagger\right)$ is fixed under automorphisms τ_o and σ. For τ_o, it can be seen from the proof of Property 1 that

$$\tau_o\left(\det\left(C_s C_s^\dagger\right)\right) = \det\left(\tau_o\left(C_s\right)\left[\tau_0\left(C_s\right)\right]^\dagger\right)$$

and

$$\tau_o(C_s) = \begin{bmatrix} \tau_o(X_{i_0}) & \cdots & \tau_o^{K_o}(X_{i_0}) \\ \vdots & \ddots & \vdots \\ \tau_o(X_{i_{m-1}}) & \cdots & \tau_o^{K_o}\left(X_{i_{m-1}}\right) \end{bmatrix}$$
$$= \begin{bmatrix} \tau_o(X_{i_0}) & \cdots & X_{i_0} \\ \vdots & \ddots & \vdots \\ \tau_o(X_{i_{m-1}}) & \cdots & X_{i_{m-1}} \end{bmatrix}$$
$$= C_s P$$

for some column permutation matrix P of size $(K_o n_t \times K_o n_t)$, where $X_{i_j} = \psi_o(x_{i_j,0})$ and $\underline{x}_{i_j}^\top = \left[x_{i_j,0}, \cdots, x_{i_j,K_o-1}\right]$, $j = 0, 1, \cdots, m-1$. Now it follows that

$$\det\left(\tau_o\left(C_s\right)\left[\tau_0\left(C_s\right)\right]^\dagger\right) = \det\left(C_s P P^\dagger C_s^\dagger\right) = \det\left(C_s C_s^\dagger\right)$$

as $PP^\dagger = I_{K_o n_t}$, and we have proved $\det\left(C_s C_s^\dagger\right)$ is fixed by τ_o.

For σ, again from the proof of Property 1 we see that

$$\sigma\left(\det\left(C_s C_s^\dagger\right)\right) = \det\left(\sigma\left(C_s\right)\left[\sigma\left(C_s\right)\right]^\dagger\right)$$

and

$$\sigma(C_s) = \begin{bmatrix} Z^{-1}X_{i_0}Z & \cdots & Z^{-1}\tau_o^{K_o-1}(X_{i_0 1})Z \\ \vdots & \ddots & \vdots \\ Z^{-1}X_{i_{m-1}}Z & \cdots & Z^{-1}\tau_o^{K_o-1}(X_{i_{m-1}})Z \end{bmatrix}$$
$$= \begin{bmatrix} Z^{-1} & & \\ & \ddots & \\ & & Z^{-1} \end{bmatrix} C_s \begin{bmatrix} Z & & \\ & \ddots & \\ & & Z \end{bmatrix},$$

where

$$Z = \psi_o(z) = \begin{bmatrix} 0 & 0 & 0 & \cdots & \zeta \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{3.25}$$

From (3.25) it is clear that $ZZ^\dagger = I_{n_t}$ since $\zeta\zeta^* = 1$ by construction. Therefore, we see that

$$\sigma(C_s)\left[\sigma(C_s)\right]^\dagger$$
$$= \text{diag}(Z^{-1}, \cdots, Z^{-1})\, C_s\, C_s^\dagger\, \text{diag}((Z^{-1})^\dagger, \cdots, (Z^{-1})^\dagger).$$

Taking into account that $\det\left(Z^{-1}\left(Z^{-1}\right)^\dagger\right) = 1$ it follows that $\sigma \det(C_s C_s^\dagger) = \det(C_s C_s^\dagger)$. So far, we have proved that $\det(C_s C_s^\dagger)$ is fixed by both τ_o and σ. This in turn implies that $\det(C_s C_s^\dagger) \in \mathbb{Q} \cap \mathbb{R} = \mathbb{Q}$. Finally, the proof is complete after noting that $\gamma^* C_s$ has entries in $\mathcal{O}_{\mathbb{E}_o}$. $\square$

In Property 2 we have shown that the overall code matrix C_o might not have full rank, and when that happens, its rank always equals mn_t for some m. The number m indicates the number of users whose transmitted signal vectors, when regarded as rows of matrix $\mathfrak{C}$ in (3.20), are linearly independent over $\mathfrak{D}_o$. Further, Property 3 shows that even when C_o is singular and fails to have NVD, i.e., fails to satisfy $\det(C_o C_o^\dagger) \geq 1$, the submatrix C_s formed by the transmitted signal matrices of those m users still satisfies the NVD property. Such result can be further extended to yield the following property on the nonzero eigenvalues of $C_o C_o^\dagger$.

Property 4. *Let C_o and C_s be defined as above with* $\text{rank}(C_o) = \text{rank}(C_s) = mn_t$. *Let $\beta_1, \cdots, \beta_{mn_t}$ be the nonzero eigenvalues of $C_oC_o^\dagger$. Then*

$$\prod_{i=1}^{mn_t} \beta_i \;\geq\; \det\left(C_sC_s\right) \,\dot{\geq}\, 1. \tag{3.26}$$

Proof. Here we take an information theoretic approach to prove the first inequality. To this end, let $\underline{N} = [N_1, \cdots, N_{K_on_t}]^\top$ be a complex Gaussian random vector of length K_on_t with zero mean and covariance matrix

$$\mathbb{E}\,\underline{N}\,\underline{N}^\dagger \;=\; C_oC_o^\dagger.$$

Without loss of generality we can assume that m linearly independent users are the first m users and i_j corresponds to the jth user, $j = 0, 1, \cdots, m-1$. Hence the covariance matrix of the sub-vector $\underline{N}_s = [N_1, \cdots, N_{mn_t}]^\top$ equals

$$\mathbb{E}\,\underline{N}_s\,\underline{N}_s^\dagger \;=\; C_sC_s^\dagger.$$

We have the following inequality for the differential entropies of $\underline{N}$ and $\underline{N}_s$

$$\begin{aligned} h\left(N_1, \cdots, N_{K_on_t}\right) &\geq h\left(N_1, \cdots, N_{mn_t}\right) \\ &= \log\det\left(C_sC_s^\dagger\right) + mn_t\log(2\pi e) \end{aligned} \tag{3.27}$$

Notice that the covariance matrix of $\underline{N}$ can be decomposed as

$$C_oC_o^\dagger \;=\; U\Sigma U^\dagger$$

for some $(K_on_t \times K_on_t)$ unitary matrix U. Σ is a diagonal matrix whose nonzero entries are the β_i's. Thus setting $\underline{N}' = U\underline{N}$ we have

$$h(\underline{N}) \;=\; h(U^\dagger\underline{N}') \;=\; \sum_{i=1}^{mn_t}\log\beta_i + mn_t\log(2\pi e).$$

Now combining the above results proves the first inequality in (3.26). The second inequality in (3.26) follows directly from Property 3 and from $|\gamma| \doteq 1$. $\square$

Remark 6. *The above property shows that despite C_o can be singular, the product of the nonzero eigenvalues of $C_oC_o^\dagger$ is always bounded from below by* 1. *This can be regarded as a relaxation of the conventional NVD property. In the design of ST codes, satisfying the NVD criterion is a sufficient condition to achieve the optimal point-to-point DMT performance. To guarantee NVD in the point-to-point MIMO, we require all the users to cooperate fully as already seen in Theorem 4. However, it is not allowed in MIMO-MAC where users transmit independently their own information to the common receiver. Thus, in MIMO-MAC we do not demand full NVD, and only partial NVD is required as shown in* (3.26). $\square$

3.3 MAC-DMT Optimality of the Proposed Construction

Armed with the properties discussed in the previous section, below we are able to show the proposed code is MAC-DMT optimal.

Theorem 7. *Given multiplexing gain r, the proposed code $\mathcal{S}$ defined as in* (3.9) *achieves the following diversity gain*

$$d(r) \ = \ \min_{1 \leq k \leq K} d^*_{kn_t,n_r}(kr) \tag{3.28}$$

over Rayleigh block fading channel with channel coherence time $T \geq K_o n_t$ channel uses. Thus, $\mathcal{S}$ is MAC-DMT optimal.

Proof. The proof is relegated to Chapter 6 for ease of reading. □

Chapter 4

MAC-DMT Optimal Codes for General MIMO-MAC Systems

In [11], Tse *et al.* focused on analyzing the DMT in a symmetric MIMO-MAC system. By symmetric we mean that every mobile user in the system has the same number of transmit antennas and transmits at the same level of multiplexing gain. However, the symmetric MIMO-MAC might not be practical enough. In the near future, the mobile communication is likely to be at a transition stage, migrating from conventional SISO (single-input single-output) to MIMO. In fact, such transition already takes place in wireless local area networks where some old laptops have single transmit antenna while the latest ones could have more than two transmit antennas. In the mixture of SISO and MIMO communication environment, one would expect the mobile users having different numbers of transmit antennas. Furthermore, in practice it is often possible that mobile users transmit at different rates because of the different plans they purchase from the service provider. The different rate implies a different level of multiplexing gain in the DMT sense. It is then of fundamental importance that we must have a general code construction that works for any MIMO-MAC systems where the mobile users are allowed to have different numbers of transmit antennas and can transmit at different levels of multiplexing gains. In the previous sections we have provided a systematic construction for the symmetric MIMO-MAC and have proved that it achieves the optimal MAC-DMT. Below we will extend these results to the general channel.

4.1 Decoding in General MIMO-MAC

There can be at least two decoding methods in the general MIMO-MAC, depending on how much computational complexity one can afford. The first decoder is the joint ML decoder, by which we mean the following. Assuming there are K users, each transmitting using a codebook $\mathcal{S}_i$ that consists of $(n_i \times T)$ ST code matrices, for $i = 0, 1, \cdots, K-1$. Let $S_i \in \mathcal{S}_i$ be the signal matrix transmitted by the ith user, and let

$$Y = \sum_{i=0}^{K-1} H_i S_i + W$$

be the received signal matrix; then the joint ML decoder seeks the optimal joint ML estimate $\left(\hat{S}_0, \cdots, \hat{S}_{K-1}\right)$ by

$$\begin{aligned}\left(\hat{S}_0, \cdots, \hat{S}_{K-1}\right) &= \arg\max_{S \in \mathcal{S}} \Pr\left\{S = (S_0, \cdots, S_{K-1}) \,|Y\right\} \\ &= \arg\min_{S \in \mathcal{S}} \left\| Y - \sum_{i=0}^{K-1} H_i S_i \right\|_F, \end{aligned} \tag{4.1}$$

where $\mathcal{S} = \mathcal{S}_0 \times \mathcal{S}_1 \times \cdots \times \mathcal{S}_{K-1}$. This joint ML decoder was used in [11] for analyzing the MAC-DMT performance in symmetric MIMO-MAC.

However, the above joint ML decoder might not be optimal in terms of the error performance of each user. For the ith user, the truly optimal decoder, though having extremely high computational complexity, is the individual ML decoder that seeks optimal ML estimate $\hat{S}_i$ by

$$\begin{aligned}\hat{S}_i &= \arg\max_{S_i \in \mathcal{S}_i} \Pr\{S_i | Y\} \\ &= \arg\max_{S_i \in \mathcal{S}_i} \sum_{S^{(i)} \in \mathcal{S}^{(i)}} \exp\left(- \left\| Y - \sum_{i=0}^{K-1} H_i S_i \right\|_F^2 \right), \end{aligned} \tag{4.2}$$

where $S^{(i)} = (S_0, S_1, \cdots, S_{i-1}, S_{i+1}, \cdots, S_{K-1})$ and $\mathcal{S}^{(i)} = \mathcal{S}_0 \times \mathcal{S}_1 \times \cdots \times \mathcal{S}_{i-1} \times \mathcal{S}_{i+1} \times \cdots \times \mathcal{S}_{K-1}$. The difference between the individual and joint ML decoders is analogous to that between the BCJR and Viterbi decoders [32] for the decoding of convolutional codes. It is easy to see that the individual ML decoder always outperforms the joint ML decoder.

In the next two sections we will examine the MAC-DMT performances of these two decoders. Obviously we expect there might exist certain performance loss in the joint ML decoder, compared to the individual ML decoder.

4.2 MAC-DMT for General MIMO-MAC with Joint Decoding

Consider a general MIMO-MAC system with K mobile users. Let n_i denote the number of transmit antennas of the ith user, $i = 0, 1, \cdots, K-1$, and let r_i be the corresponding multiplexing gain. Assuming n_r receive antennas at the base station, the first major result of this section is the following.

Theorem 8 (General joint MAC-DMT)**.** *Let K, n_i, r_i and n_r be defined as above. If joint decoding is performed at receiver end, the optimal MAC-DMT of such system is given by*

$$d^*_{\{n_0,\cdots,n_{K-1}\},n_r}(r_0, \cdots, r_{K-1}) \;=\; \min_{\mathcal{I}} d^*_{N_t(\mathcal{I}),n_r}\left(\sum_{i\in\mathcal{I}} r_i\right) \tag{4.3}$$

for i.i.d. Rayleigh block fading channel that is fixed for at least

$$T \;\geq\; \left\lceil \sum_{i=0}^{K-1} n_i \right\rceil + n_r - 1 \textit{ (channel uses).}$$

The minimization in (4.3) *is taken over all possible non-empty subsets $\mathcal{I} \subseteq \{0, 1, \cdots, K-1\}$, and*

$$N_t(\mathcal{I}) \;:=\; \sum_{i\in\mathcal{I}} n_i \tag{4.4}$$

is the total number of transmit antennas of users in $\mathcal{I}$. The notion of $d^*_{p,q}(r)$ is the conventional point-to-point DMT. □

Prior to proving Theorem 8, we shall give an example illustrating this theorem and in particular, show some unexpected effects resulting from joint decoding.

Example 2. *For simplicity, here we consider a general MIMO-MAC system with two users. The first user has* $n_0 = 1$ *transmit antenna and transmits at multiplexing gain* r_0*; the second user has* $n_1 = 2$ *transmit antennas and transmits at multiplexing gain* r_1*. Assume there are* $n_r = 2$ *receive antennas at receiver end. Using* (4.3) *the resulting MAC-DMT is shown in Fig. 4.1. First, it is interesting to note that unlike the symmetric MIMO-MAC where all users have same number of transmit antennas and transmit at same level of multiplexing gain, here the second user cannot achieve his single-user DMT performance even when* $r_0 = 0$*. This effect is shown in Fig. 4.2. While this is quite unexpected, such phenomenon can be easily explained. Recall that the DMT is an asymptotic result. Strictly speaking, the multiplexing gain* r_i *is defined as*

$$r_i \;=\; \lim_{\mathrm{SNR}\to\infty} \frac{R_i}{\log_2 \mathrm{SNR}},$$

and R_i *is the actual transmission rate. Therefore, when we say* $r_0 = 0$ *it does not necessarily mean* $R_0 = 0$*. It simply means that the rate of the first user grows much slower than* $\log_2 \mathrm{SNR}$*. For example, an ST code that is fixed and does not vary with* SNR *has multiplexing gain* 0 *since the rate* R_i *is a constant. But the rate* R_i *is bounded away from* 0*.*

Having learned the above, in our example given the multiplexing gain $r_0 = \epsilon$ *for some positive* ϵ *very close to* 0*, the DMT performance of joint decoder would be dominated by erroneous decoding of the first user's signals when* r_1 *is small. It is also easy to confirm this observation from pairwise error probability (PEP) analysis. Assume* $r_0 = r_1 = 0$*, but* $R_0, R_1 > 0$*, i.e., the codes are fixed and do not vary with* SNR*. Since the two users do not cooperate, for any distinct pairs of overall code matrices, the maximal possible rank is the minimum of* n_0 *and* n_1*. Hence the resulting maximal possible diversity gain equals*

$$d_{\max} \;=\; n_r \cdot \min\{n_0, n_1\},$$

which equals 2 *in this example. Therefore, the PEP analysis confirms that the single-user DMT performance* $d^*_{2,2}(r_1)$ *cannot be achieved for small values of* r_0 *as shown in Fig. 4.2.*

Before concluding this example we remark that the loss in DMT for the second user can in fact be recovered if an individual ML decoder is used. We will come back to this in Chapter 4.3. □

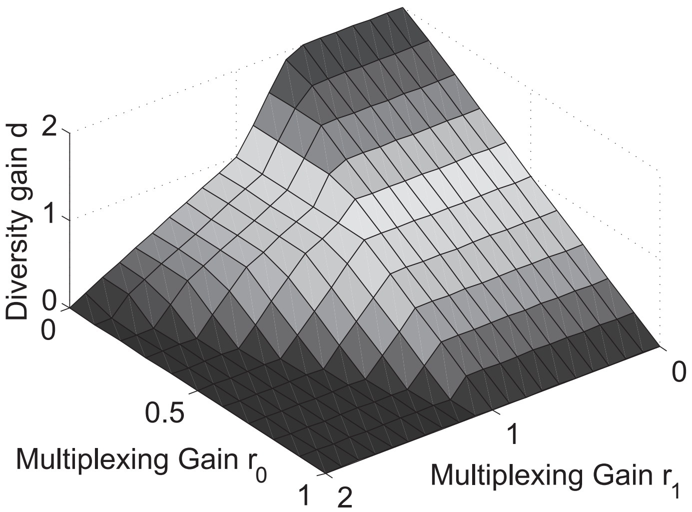

Figure 4.1: Joint MAC-DMT $d^*_{\{1,2\},2}(r_0, r_1)$ of general MIMO-MAC with two users.

The proof of Theorem 8 follows along similar lines of that of symmetric MAC-DMT provided by Tse *et al.* in [11]. Specifically, let

$$R_i \;:=\; r_i \log_2 \mathrm{SNR} \quad \text{(bits/channel use)}$$

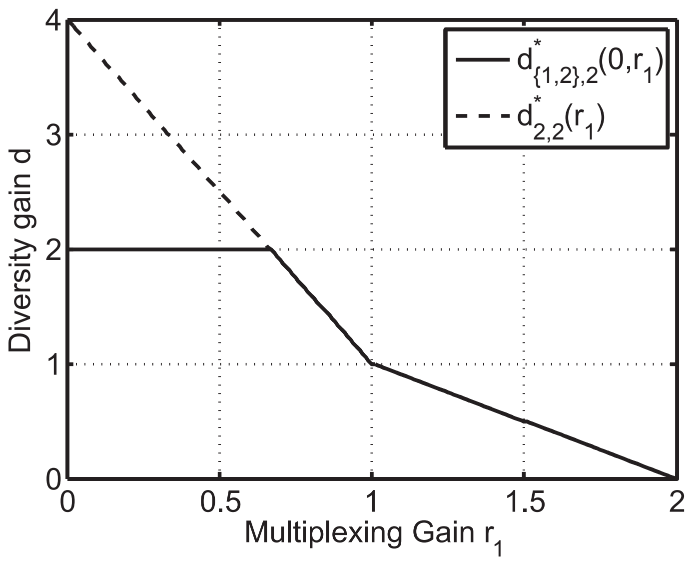

Figure 4.2: Joint MAC-DMT $d^*_{\{1,2\},2}(0, r_1)$ of general MIMO-MAC with two users.

denote the actual transmission rate of the ith user. Given the subset $\mathcal{I}$ of users, let $\mathcal{O}(\mathcal{I})$ denote the following outage event

$$\mathcal{O}(\mathcal{I}) \ := \ \left\{ H \in \mathbb{C}^{n_r \times N} : I\left(S_{\mathcal{I}}; \underline{y} | S_{\mathcal{I}^c}, H\right) \ \leq \ \sum_{i \in \mathcal{I}} R_i \right\}, \tag{4.5}$$

where

- $H = [H_0 \ \cdots \ H_{K-1}]$ is the overall channel matrix, H_i is the channel matrix of size $(n_r \times n_i)$ of the ith user,
- N is the total number of transmit antennas defined by

$$N \ := \ \sum_{i=0}^{K-1} n_i,$$

- $S_{\mathcal{I}}$ contains the transmitted signal vectors of users in $\mathcal{I}$ and is defined as

$$S_{\mathcal{I}} \ := \ \{\underline{s}_i : i \in \mathcal{I}\},$$

- $\underline{y}$ is the received signal vector given by

$$\underline{y} \ = \ \sum_{i=0}^{K-1} H_i \underline{s}_i + \underline{w},$$

  where $\underline{w}$ is the complex Gaussian random noise vector, and
- $S_{\mathcal{I}^c}$ consists of transmitted signals of users not in $\mathcal{I}$.

Let $\mathcal{O}$ denote the overall outage event. It is clear that

$$\mathcal{O} \ = \ \bigcup_{\mathcal{I}} \mathcal{O}(\mathcal{I}).$$

Following similar arguments as in [11] it is straightforward to see that the error probability of joint decoding $P_e(r_0, \cdots, r_{K-1})$ is lower bounded by

$$\begin{aligned} P_e(r_0, \cdots, r_{K-1}) \geq \Pr\{\mathcal{O}\} \ \geq \ \max_{\mathcal{I}} \Pr\{\mathcal{O}(\mathcal{I})\} \\ \doteq \text{SNR}^{-\min_{\mathcal{I}} d^*_{N_t(\mathcal{I}),n_r}\left(\sum_{i\in\mathcal{I}} r_i\right)}. \end{aligned} \tag{4.6}$$

To establish the converse, we take the random codebook approach similar to that used by Tse *et al.* in [11]. Let $\mathcal{S}_i$ be the codebook of the ith mobile user, consisting of $(n_i \times T)$ code matrices that are randomly generated by some complex Gaussian random generator. Further, $\mathcal{S}_i$ satisfies the desired multiplexing gain,

$$\frac{1}{T}\log_2|\mathcal{S}_i| \ = \ R_i \ = \ r_i \log_2 \text{SNR}.$$

Let $\mathcal{E}(\mathcal{I})$ denote the event that the signal matrices of users in $\mathcal{I}$ are erroneously decoded by the joint decoder. Then arguing similarly as in [11], it can be shown that

$$\Pr\{\mathcal{E}(\mathcal{I})\} \ \dot{\leq} \ \text{SNR}^{-d^*_{N_t(\mathcal{I}),n_r}\left(\sum_{i\in\mathcal{I}} r_i\right)}$$

whenever

$$T \ \geq \ N_t(\mathcal{I}) + n_r - 1.$$

Thus, using union bound we have

$$\begin{aligned} P_e(r_0, \cdots, r_{K-1}) \leq \sum_{\mathcal{I}} \Pr\{\mathcal{E}(\mathcal{I})\} \\ \doteq \text{SNR}^{-\min_{\mathcal{I}} d^*_{N_t(\mathcal{I}),n_r}\left(\sum_{i\in\mathcal{I}} r_i\right)}, \end{aligned}$$

provided that

$$T \ \geq \ \max_{\mathcal{I}} N_t(\mathcal{I}) + n_r - 1 \ = \ N + n_r - 1.$$

This proves Theorem 8.

4.3 MAC-DMT for General MIMO-MAC with Individual ML Decoding

In the previous section we investigated the MAC-DMT for a general MIMO-MAC with joint decoding at the receiver end. We also observed in Example 2 that certain DMT performance loss could result from the use of joint decoder. However, such loss can be safely avoided by the use of individual ML decoder.

Recall that for the ith user, the truly optimal decoder, though having extremely high computational complexity, is the individual ML decoder that seeks optimal ML estimate $\hat{S}_i$ by

$$\begin{aligned} \hat{S}_i &= \arg\max_{S_i\in\mathcal{S}_i} \Pr\{S_i|Y\} \\ &= \arg\max_{S_i\in\mathcal{S}_i} \sum_{S^{(i)}\in\mathcal{S}^{(i)}} \exp\left(-\left\|Y - \sum_{i=0}^{K-1} H_i S_i\right\|_F^2\right), \end{aligned} \tag{4.7}$$

where $S^{(i)} = (S_0, S_1, \cdots, S_{i-1}, S_{i+1}, \cdots, S_{K-1})$ and $\mathcal{S}^{(i)} = \mathcal{S}_0 \times \mathcal{S}_1 \times \cdots \times \mathcal{S}_{i-1} \times \mathcal{S}_{i+1} \times \cdots \times \mathcal{S}_{K-1}$. Clearly (4.7) outperforms (4.1) in error performance, but at a cost of much higher computational complexity.

Without loss of generality, below we focus on the error performance of the individual ML decoding for the ith user. To distinguish the DMT performances of decoders (4.1) and (4.7), we shall call the DMT of the latter the *individual MAC-DMT* and will denote it by $d^{(i)*}_{\{n_0,\cdots,n_{K-1}\},n_r}(r_0,\cdots,r_{K-1})$.

To characterize the DMT performance of the individual ML decoder, we only need to consider the outage events $\mathcal{O}(\mathcal{I})$ (cf. (4.5)) in which the ith user is a member of $\mathcal{I}$. Event $\mathcal{O}(\mathcal{I})$ with $i \notin \mathcal{I}$ is not counted as an outage for the ith user for obvious reasons. Thus, along similar lines as in the proof of Theorem 8 we can show the following.

Theorem 9 (General individual MAC-DMT)**.** *Let K, n_i, r_i and n_r be defined as before. If individual ML decoding is performed at receiver end for the ith user, the optimal individual MAC-DMT is given by*

$$d^{(i)*}_{\{n_0,\cdots,n_{K-1}\},n_r}(r_0,\cdots,r_{K-1}) \;=\; \min_{\mathcal{I}:i\in\mathcal{I}} d^*_{N_t(\mathcal{I}),n_r}\left(\sum_{i\in\mathcal{I}} r_i\right), \tag{4.8}$$

where the minimization is taken over all $\mathcal{I} \subseteq \{0,1,\cdots,K-1\}$ under the condition $i \in \mathcal{I}$ and $N_t(\mathcal{I})$ is defined in (4.4). □

Proof. For brevity we only outline the proof. Let $\mathcal{O}_i$ denote the outage event of the ith user; then following from the above discussion it can be seen that

$$\mathcal{O}_i \;=\; \bigcup_{\substack{\mathcal{I}\subseteq\{0,1,\cdots,K-1\}\\ i\in\mathcal{I}}} \mathcal{O}(\mathcal{I}),$$

since if $i \notin \mathcal{I}$, the ith user is not in outage. Now let $P_e^{(i)}(r_0,\cdots,r_{K-1})$ denote the error probability of the individual decoder for the ith user; then it can be shown that

$$\begin{aligned} P_e^{(i)}(r_0,\cdots,r_{K-1}) &\geq \Pr\{\mathcal{O}_i\} \\ &\dot{\geq} \max_{\substack{\mathcal{I}\subseteq\{0,1,\cdots,K-1\}\\ i\in\mathcal{I}}} \Pr\{\mathcal{O}(\mathcal{I})\} \\ &\doteq \mathrm{SNR}^{-d^{(i)*}_{\{n_0,\cdots,n_{K-1}\},n_r}(r_0,\cdots,r_{K-1})}, \end{aligned}$$

where the first inequality follows from [9, Lemma 5]. To show the converse, let $\mathcal{E}(\mathcal{I})$ denote the error event that the signal matrices of the users in $\mathcal{I}$ are erroneously decoded under joint decoding. Then simply note that the error probability of an individual ML decoder is upper bounded by that of a joint ML decoder, i.e.,

$$P_e^{(i)}(r_0,\cdots,r_{K-1}) \;\leq\; \Pr\left\{ \bigcup_{\substack{\mathcal{I}\subseteq\{0,1,\cdots,K-1\}\\ i\in\mathcal{I}}} \mathcal{E}(\mathcal{I}) \right\},$$

where the right-hand-side gives the probability of a joint ML decoder when the signal of the ith user is erroneously decoded. Now using the union bound argument and along similar lines as in the proof of Theorem 8 it can be shown that

$$\begin{aligned} P_e^{(i)}(r_0,\cdots,r_{K-1}) &\leq \sum_{\substack{\mathcal{I}\subseteq\{0,1,\cdots,K-1\}\\ i\in\mathcal{I}}} \Pr\{\mathcal{E}(\mathcal{I})\} \\ &\dot{\leq} \sum_{\substack{\mathcal{I}\subseteq\{0,1,\cdots,K-1\}\\ i\in\mathcal{I}}} \mathrm{SNR}^{-d^*_{N_t(\mathcal{I}),n_r}\left(\sum_{i\in\mathcal{I}} r_i\right)} \\ &\doteq \mathrm{SNR}^{-d^{(i)*}_{\{n_0,\cdots,n_{K-1}\},n_r}(r_0,\cdots,r_{K-1})}. \end{aligned}$$

This completes the proof. □

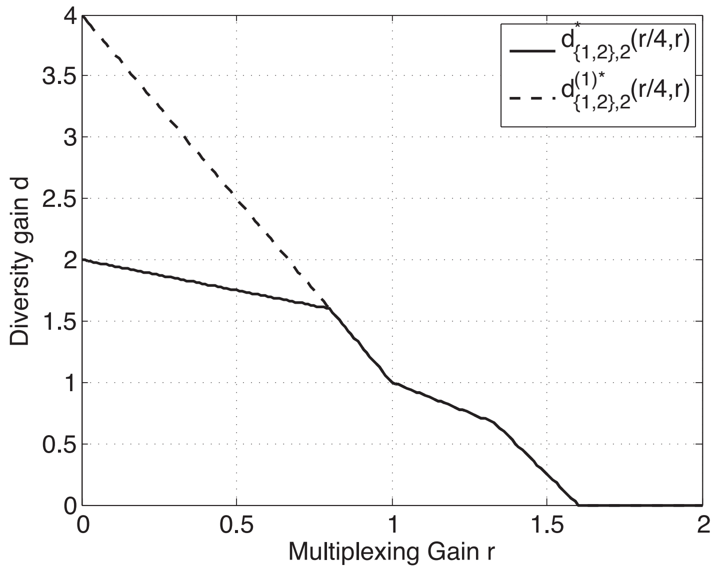

Figure 4.3: Comparison between the joint MAC-DMT and the individual MAC-DMT of the second user when $r_1 = 4r_0 = r$.

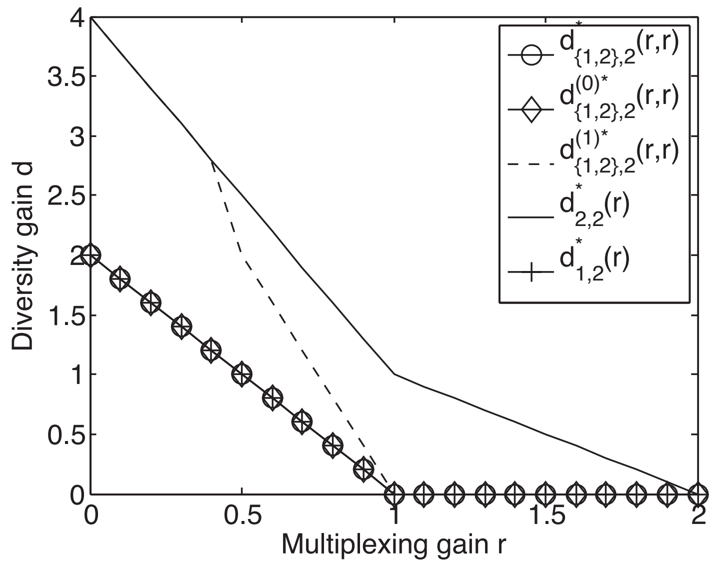

Figure 4.4: Comparison between the joint MAC-DMT and the individual MAC-DMT when $r_1 = r_0 = r$.

With the above result, we now come back to Example 2 to investigate the individual MAC-DMT of the second user.

Example 3 (Continued from Example 2)**.** *In Example 2 we have considered the specific case of $K = 2$, $n_0 = 1$, $n_1 = 2$, $n_r = 2$ and $r_0 = 0$. Assuming the second user transmits at multiplexing gain r_1, from Theorem 9 the individual MAC-DMT of the second user is*

$$d^{(1)*}_{\{1,2\},2}(0, r_1) \;=\; \min\{d^*_{2,2}(r_1), d^*_{3,2}(r_1)\} \;=\; d^*_{2,2}(r_1).$$

Hence we see that the single-user performance of the second user is recovered by the use of an individual ML decoder. To illustrate further the difference in MAC-DMT between (4.1) *and* (4.7), *in Fig. 4.3 we compare the MAC-DMT performances of joint and individual decoders at $r_1 = 4r_0 = r$. It can be clearly seen that the individual ML decoder outperforms significantly the joint ML decoder at low-multiplexing-gain regime.*

Another comparison between the DMT performances of both decoders at $r_1 = r_0 = r$ is given in Fig. 4.4. It shows that the joint ML decoder (given by $d^*_{\{1,2\},2}(r,r)$) is not optimal for the second user. The truly optimal individual ML decoder for the second user has DMT performance $d^{(1)*}_{\{1,2\},2}(r,r)$. Furthermore, the individual ML decoder for the second user achieves the single-user DMT performance $d^*_{2,2}(r)$ as long as $r \leq 0.4$. On the other hand, for the first user who has lesser number of transmit antennas, the DMT performances of the joint and individual decoders are the same and are actually equal to his single-user performance $d^*_{1,2}(r)$. □

Next we could apply Theorem 9 to the case of symmetric MIMO-MAC to see how the error probabilities of joint and individual ML decoders compare. The comparison is given in the following corollary. It shows that in the symmetric MIMO-MAC there is no difference in terms of MAC-DMT performance between the joint and individual ML decoders.

Corollary 10. *For symmetric MIMO-MAC with K users, each having n_t transmit antennas and transmitting at multiplexing gain $r_i = r$, let $P_e(r)$ denote the error probability of the joint ML decoder and $P_e^{(i)}(r)$ denote the error probability of the ith individual ML decoder. Then*

$$P_e(r) \doteq P_e^{(i)}(r)$$

and in terms of DMT we have

$$d^{(i)*}_{\{n_t,\cdots,n_t\},n_r}(r,\cdots,r) = d^*_{\{n_t,\cdots,n_t\},n_r}(r,\cdots,r)$$

for all $i = 0, 1, \cdots, K-1$.

Proof. It suffices to show only the equality in DMT. First, from Theorem 9 we have

$$d^{(i)*}_{\{n_t,\cdots,n_t\},n_r}(r,\cdots,r) = \min_{1\leq m\leq K} d^*_{mn_t,n_r}(mr)$$

and the proof is complete after noting that the right-hand-side of the above is the same as the MAC-DMT given in Theorem 1. □

Before concluding the section we have the following remarks. First, while the individual ML decoder could achieve a much higher DMT performance as seen in Examples 2 and 3, the computational complexity required by (4.7) is often extremely high. Thus, the individual ML decoder has widely been considered as being impractical in multiuser detections. The reason for including this receiver is only to clarify the unexpected DMT performance loss of the joint ML decoder in Example 2.

As the individual ML decoder is rarely used, below we will not consider this receiver anymore. We will regard the joint MAC-DMT given in Theorem 8 as the optimal MAC-DMT in practice, although it is now clear that it is not the best one can actually achieve.

4.4 General MAC-DMT Optimal Codes

So far we have provided the optimal MAC-DMT (4.3) for the general MIMO-MAC system with K users where the ith user has n_i transmit antennas and transmits at multiplexing gain r_i. To have a deterministic code for the general MIMO-MAC, we can extend the code construction given in Chapter 3 for symmetric MIMO-MAC to the present case.

Let K, n_i, r_i, and n_r be defined as before. For brevity we only present the construction when K is odd. Codes for K even can be constructed by simply patching an extra coded block to each user's code matrices, similar to that described in Chapter 3. Henceforth we will drop the subscript "o" in K_o, $\mathbb{K}_o$, τ_o, $\mathcal{C}_o$, $\mathcal{S}_o$, etc. for simplicity.

Given n_i and K, we first define

$$n_{\max} := \max_{0 \le i \le K-1} n_i \tag{4.9}$$

as the maximal number of transmit antennas among all users. In general, the number $n_{\max}$ can be either pre-known to all the users, or explicitly specified among any groups of users. Next, let $\mathbb{L} = \mathbb{F}(\vartheta)$ be the number field that is cyclic Galois over $\mathbb{F} = \mathbb{Q}(\imath)$ with degree $n_{\max}$, and let $\mathbb{K} = \mathbb{F}(\eta)$ be another cyclic Galois extension of $\mathbb{F}$ with degree K. Let σ be the generator of the Galois group Gal$(\mathbb{L}/\mathbb{F})$, and similarly let τ be the generator of Gal$(\mathbb{K}/\mathbb{F})$. The fields $\mathbb{L}$ and $\mathbb{K}$ are required to satisfy $\mathbb{L} \cap \mathbb{K} = \mathbb{F}$ or are required such that τ and σ commute. Finally, we set $\mathbb{E} = \mathbb{L}\mathbb{K}$ to be the compositum of fields $\mathbb{L}$ and $\mathbb{K}$. Similar to Chapter 3, with some suitable unimodular $\zeta \in \mathbb{F}^*$, we have

$$\mathfrak{D} = \mathbb{E} \oplus z\mathbb{E} \oplus \cdots \oplus z^{n_{\max}-1}\mathbb{E} \tag{4.10}$$

as an appropriate central simple division $\mathbb{K}$-algebra with $xz = z\sigma(x)$ for $x \in \mathbb{E}$, where z is an indeterminate satisfying $z^{n_{\max}} = \zeta$.

Given the multiplexing gain r_i of the ith user, we set the corresponding base alphabet and information set as follows:

$$\mathcal{A}_i(\text{SNR}) = \left\{ a + b\,\imath \,:\, \begin{array}{r} -\text{SNR}^{\frac{r_i}{2n_{\max}}} \le a, b \le \text{SNR}^{\frac{r_i}{2n_{\max}}}, \\ a, b \in \mathbb{Z}, \quad a, b \text{ odd} \end{array} \right\} \tag{4.11}$$

and

$$\mathfrak{A}_i(\text{SNR}) = \left\{ \sum_{j=0}^{n_{\max}-1} z^j \sum_{k=0}^{Kn_{\max}-1} x_{j,k} e_k : x_{j,k} \in \mathcal{A}_i(\text{SNR}) \right\}, \tag{4.12}$$

where $\{e_0, \cdots, e_{Kn_{\max}-1}\}$ is an integral basis of $\mathbb{E}/\mathbb{F}$. Unlike the construction for symmetrical MIMO-MAC, here the information set can be different among users as each user has different level of multiplexing gain.

Let ψ denote the left-regular map of elements in $\mathfrak{D}$ into matrices of size $(n_{\max} \times n_{\max})$ whose entries are in $\mathbb{E}$ (similar to ψ_o of (3.6)); then the ST code $\mathcal{S}_i$ of the ith user is given by

$$\mathcal{S}_i = \left\{ \begin{array}{r} S_i = \kappa_i \left[\begin{array}{cccc} X_i & \tau(X_i) & \cdots & \tau^{K-1}(X_i) \end{array} \right] : \\ X_i = \psi(x_i), x_i \in \mathfrak{A}_i(\text{SNR}) \end{array} \right\}, \tag{4.13}$$

where

$$\kappa_i^2 \doteq \text{SNR}^{1-\frac{r_i}{n_{\max}}} \tag{4.14}$$

such that the power constraint (1.4) is satisfied.

Given $\mathcal{S}_i$, $i = 0, 1, \cdots, K-1$, the overall code is obtained by vertically concatenating the code matrices from each user,

$$\begin{aligned} \mathcal{S} &:= \mathcal{S}_0 \times \mathcal{S}_1 \times \cdots \times \mathcal{S}_{K-1} \\ &= \left\{ S = \begin{bmatrix} S_0 \\ \vdots \\ S_{K-1} \end{bmatrix} : S_i \in \mathcal{S}_i \right\}. \end{aligned} \tag{4.15}$$

The overall code matrix S is a square matrix of size $(Kn_{\max} \times Kn_{\max})$. Below we will present some nice properties of $\mathcal{S}$ which are essential to proving its MAC-DMT optimality.

The first property extends Property 2 of the symmetric MAC code $\mathcal{S}_o$ in Chapter 3.2.

Property 5. *For any $x_i \in \mathfrak{A}_i(\mathrm{SNR})$, define*

$$\mathfrak{C} = \begin{bmatrix} \underline{x}_0^\top \\ \vdots \\ \underline{x}_{K-1}^\top \end{bmatrix} = \begin{bmatrix} x_0 & \cdots & \tau^{K-1}(x_0) \\ \vdots & \ddots & \vdots \\ x_{K-1} & \cdots & \tau^{K-1}(x_{K-1}) \end{bmatrix} \tag{4.16}$$

and let

$$S = \begin{bmatrix} \kappa_0 X_0 & \cdots & \kappa_0 \tau^{K-1}(X_0) \\ \vdots & \ddots & \vdots \\ \kappa_{K-1} X_{K-1} & \cdots & \kappa_{K-1}\tau^{K-1}(X_{K-1}) \end{bmatrix} \in \mathcal{S}$$

be the corresponding overall code matrix where $X_i = \psi(x_i)$. Then $\|S\|_F^2 \dot{\leq} \mathrm{SNR}$. Further, let m be the maximal number of rows of $\mathfrak{C}$ that are linearly independent as a left $\mathfrak{D}$-module. Then

$$\mathrm{rank}(S) \;=\; mn_{\max}, \tag{4.17}$$

where the rank is measured in the complex number field $\mathbb{C}$.

Proof. The first claim can be easily verified from the settings of κ_i and $\mathcal{A}_i(\mathrm{SNR})$, and is thus omitted for brevity. For the second, to determine the rank of S, it suffices to consider the rank of the unscaled code matrix

$$C = \begin{bmatrix} X_0 & \cdots & \tau^{K-1}(X_0) \\ \vdots & \ddots & \vdots \\ X_{K-1} & \cdots & \tau^{K-1}(X_{K-1}) \end{bmatrix}. \tag{4.18}$$

Notice that C is a code matrix of the code $\mathcal{C}_o$ defined in (3.14) for the symmetric MIMO-MAC when we set $n_t = n_{\max}$ and

$$r \;=\; \max_i r_i.$$

Now the result follows from Property 2. $\square$

The next property generalizes Property 3 in Chapter 3.2 where we were interested in the Gram determinant of the un-scaled code matrix. Here, for the purpose of analyzing the general MAC-DMT performance of the proposed code, we will seek directly the Gram determinant of the overall matrix S.

Property 6. *Let $\mathfrak{C}$ be defined as in* (4.16) *and assume that $\{\underline{x}_{i_0}^\top, \cdots, \underline{x}_{i_{m-1}}^\top\}$ is a subset of rows of $\mathfrak{C}$ that are linearly independent as a left $\mathfrak{D}$-module. Let*

$$S_s = \begin{bmatrix} \kappa_{i_0} X_{i_0} & \cdots & \kappa_{i_0}\tau^{K-1}(X_{i_0}) \\ \vdots & \ddots & \vdots \\ \kappa_{i_{m-1}} X_{i_{m-1}} & \cdots & \kappa_{i_{m-1}}\tau^{K-1}\left(X_{i_{m-1}}\right) \end{bmatrix} \tag{4.19}$$

be the submatrix of S consisting of the corresponding $mn_{\max}$ rows. Then

$$\left[\|\gamma\|^{2mn_t} \cdot \det\left(S_s S_s^\dagger\right)\right] \;\geq\; \mathrm{SNR}^{mn_{\max} - \sum_{j=0}^{m-1} r_{i_j}}. \tag{4.20}$$

Proof. Arguing similarly to the proof of Property 5, set

$$C_s = \begin{bmatrix} X_{i_0} & \cdots & \tau^{K-1}\left(X_{i_0}\right) \\ \vdots & \ddots & \vdots \\ X_{i_{m-1}} & \cdots & \tau^{K-1}\left(X_{i_{m-1}}\right) \end{bmatrix}. \tag{4.21}$$

Then we have

$$S_s = \begin{bmatrix} \kappa_{i_0} I_{n_{\max}} & & \\ & \ddots & \\ & & \kappa_{i_{m-1}} I_{n_{\max}} \end{bmatrix} C_s$$

and

$$\left\lceil \|\gamma\|^{2mn_{\max}} \cdot \det\left(C_s C_s^\dagger\right) \right\rceil \geq 1$$

by Property 3 since C_s is a submatrix of the code matrix C (cf. (4.18)) of the code $\mathcal{C}_o$ (cf. (3.14)) for the symmetric MIMO-MAC when setting $n_t = n_{\max}$ and $r = \max_i r_i$. The result now follows from

$$\begin{aligned} \det\left(S_s S_s^\dagger\right) &= \det\left(C_s C_s^\dagger\right) \prod_{j=0}^{m-1} \det\left(\kappa_{i_j} I_{n_{\max}}\right) \\ &\dot{\geq} \prod_{j=0}^{m-1} \det\left(\kappa_{i_j} I_{n_{\max}}\right) \end{aligned}$$

and from the definition of κ_{i_j} in (4.14). □

The two properties above are exactly what we need to prove the MAC-DMT optimality of the proposed general MIMO-MAC code $\mathcal{S}$ in (4.15). Hence, with these properties we can prove the following theorem.

Theorem 11. *Given n_i and r_i, $i = 0, 1, \cdots, K-1$ with K odd, the proposed code $\mathcal{S}$ defined in* (4.15) *achieves the general joint MAC-DMT*

$$d(r_0, \cdots, r_1) = \min_{\mathcal{I}_m} d^*_{N_t(\mathcal{I}), n_r}\left(\sum_{i \in \mathcal{I}} r_i\right) \tag{4.22}$$

over a Rayleigh block fading channel that remains static for at least $T \geq K n_{\max}$ channel uses. $\mathcal{S}$ is MAC-DMT optimal.

Proof. The proof is similar to that of Theorem 7 and is relegated to Chapter 7 for ease of reading. □

The proof to Theorem 11 can in fact be further extended to show that the proposed code $\mathcal{S}$ (4.15) achieves the optimal individual MAC-DMT (4.8), provided that an individual ML decoder for each user is used at the receiver end. This result along with the proof will be presented in Corollary 15 of Chapter 7.

Chapter 5

Proof of Theorem 4

Consider the multi-user set up with K users, each having n_t transmit antennas. The users are simultaneously and synchronously transmitting the matrix $C_k, k = 0, 1, \cdots, K-1$, and each is using a (user specific) code lattice $\mathbf{L}_k, k = 0, 1, \cdots, K-1$ of complex $n_t \times Kn_t$-matrices, so $C_k \in \mathbf{L}_k$ for all k. From the receiver's point of view, the overall $Kn_t \times Kn_t$ code matrix

$$C = M(C_0, C_1, \cdots, C_{K-1}) = \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_{K-1} \end{bmatrix}$$

then becomes interesting. Theorem 3 lists the following requirements that the group of lattices $\mathbf{L}_k$ should ideally have.

1. Each lattice $\mathbf{L}_k$ should have the full rank $2Kn_t^2$ in order to reach the maximum multiplexing gain. That is, assuming a PAM alphabet of size SNR$^{\frac{r}{2n_t}}$, the code $\mathcal{C}_k$ defined by $\mathbf{L}_k$ has size

   $$\left[\text{SNR}^{\frac{r}{2n_t}}\right]^{2Kn_t^2} = \text{SNR}^{rKn_t},$$

   which means a multiplexing gain of value r for Kn_t channel uses.

2. Given any $\mathcal{I}_m = \{i_0, i_1, \cdots, i_{m-1}\} \subseteq \{0, 1, \cdots, K-1\}$, whenever $C_{i_j} \neq C'_{i_j} \in \mathcal{C}_{i_j}$ for $j = 0, 1, \cdots, m-1$, $\Delta C_{\mathcal{I}_m} = M(C_{i_0} - C'_{i_0}, \cdots, C_{i_{m-1}} - C'_{i_{m-1}})$ should have full row rank mn_t, i.e., $\det(\Delta C_{\mathcal{I}_m} \Delta C_{\mathcal{I}_m}^\dagger) \neq 0$.

3. In the cases listed in item 2 the determinants $\det(\Delta C_{\mathcal{I}_m} \Delta C_{\mathcal{I}_m}^\dagger)$ should be bounded away from zero, i.e., NVD. In term of the notation of exponential equality, this means

   $$\det(\Delta C_{\mathcal{I}_m} \Delta C_{\mathcal{I}_m}^\dagger) \dot{\geq} 1.$$

The main idea behind our proof to Theorem 4 is that these ideal requirements are incompatible if in the third requirement we have $\det(\Delta C_{\mathcal{I}_m} \Delta C_{\mathcal{I}_m}^\dagger) \geq 1$. More precisely, we shall show that first two requirements imply that the determinants $\det(\Delta C_{\mathcal{I}_m} \Delta C_{\mathcal{I}_m}^\dagger)$ will necessarily become arbitrarily small, for all $m > 1$.

For simplicity, here it suffices to prove only the case of $m = K$ and $\mathcal{I}_K = \{0, 1, , \cdots, K-1\}$, and we will show that if the first two requirements are met, then for the overall $(Kn_t \times Kn_t)$ difference code matrix

$$\Delta C = M(C_0 - C'_0, C_1 - C'_1, \cdots, C_{K-1} - C'_{K-1}),$$

though invertible, the absolute of determinant $|\det(\Delta C)|$ can be arbitrarily small and be close to 0, hence the third requirement of NVD cannot be met.

To see this, set

$$\Delta C_k = C_k - C'_k,$$

with $\Delta C_k \neq \mathbf{0}$ from the second requirement, and let W be the complex vector space of $(n_t \times Kn_t)$ matrices. Let $V = L(\Delta C_1, \Delta C_2, \cdots, \Delta C_{K-1})$ be the complex vector space spanned by these $K-1$ matrices. From the second requirement we immediately see that $\dim_{\mathbb{C}} V' = K-1$. We shall work with the quotient space $Q = W/V$. It has a natural structure of a finite dimensional complex vector space. We also need its topology which is that of a Euclidean (or a Hermitian) space that is well known to also be equal to the quotient topology.

The following simple observation is the key to prove Theorem 4.

Lemma 12. *The mapping $f : Q \to \mathbb{C}$, given by*

$$X + L(\Delta C_1, \cdots, \Delta C_{K-1}) \mapsto \det(M(X, \Delta C_1, \cdots, \Delta C_{K-1}) \tag{5.1}$$

is well-defined and continuous.

Proof. Any two $(n_t \times Kn_t)$ matrices X and X' determine the same coset modulo V, if and only if the difference matrix $X - X'$ is a complex linear combination of the matrices $\Delta C_1, \Delta C_2, \cdots,$ ΔC_{K-1}. It is immediately clear that in that case

$$\begin{aligned} &\det(M(X, \Delta C_1, \cdots, \Delta C_{K-1}) \\ &= \det(M(X', \Delta C_1, \cdots, \Delta C_{K-1}). \end{aligned}$$

Therefore f is a well-defined function. Continuity of f follows from the continuity of the polynomial function $X \mapsto \det(M(X, \Delta C_1, \cdots, \Delta C_{K-1})$ and the basic properties of the quotient topology. □

Lemma 13. *A subgroup in $\mathbb{C}^n$ is a lattice if and only if it is discrete.* □

Having obtained Lemmas 12 and 13, we are now in position to prove Theorem 4.

Proof. As above, let us fix non-zero difference matrices $\Delta C_k \in \mathbf{L}_k, k = 1, 2, \cdots, K-1$ for all the other users. Let $\Delta C_0 \in \mathbf{L}_0$ be non-zero. Let $\pi : W \to Q$ denote the natural projection.

By the second requirement we have

$$\det(M(\Delta C_0, \cdots, \Delta C_{K-1})) \neq 0,$$

so ΔC_0 does not belong to the subspace V. Therefore $\Delta C_0 + V \neq 0_Q$, and we see that $\ker \pi$ intersects trivially with the lattice of the first user $\mathbf{L}_0$. So restricted to the free abelian group $\mathbf{L}_0$, π is an injection. Hence $G = \pi(\mathbf{L}_0)$ is a free abelian group of rank $2Kn_t^2 \subset Q$.

Because $\dim_{\mathbb{C}} Q < Kn_t^2$, the quotient space Q is not big enough to contain a free abelian group of rank $2Kn_t^2$ as a discrete subset. Therefore the set G must have an accumulation point in Q by Lemma 13. In other words, there are matrices in G that are arbitrarily close to each other. As G is closed under addition and negation, it follows that we can find a sequence of non-zero matrices $(S_i)_{i=1,2,\cdots}$ from the lattice $\mathbf{L}_0$ such that the sequence of their images in the space Q converges towards zero, or

$$\lim_{i\to\infty} \pi(S_i) = 0_Q.$$

The continuity of the function f of Lemma 12 then implies that

$$\lim_{i\to\infty} \det(M(S_i, \Delta C_1, \cdots, \Delta C_{K-1}) = f(0) = 0.$$

As all these matrices are of the form prescribed in condition 3), we see that this last condition cannot be met. □

Chapter 6

Proof of Theorem 7

Here we only prove the case of K odd. The case of K even and $K_o = K+1$ can be proved using similar arguments, and will therefore be briefly handled in a remark following the proof.

6.1 Proof Overview

In this section we provide an overview of the proof for the case of K odd, along with a few insights to the proof.

Given the overall channel matrix

$$H = \begin{bmatrix} H_0 & H_1 & \cdots & H_{K_o-1} \end{bmatrix} \tag{6.1}$$

we will provide an upper bound on the codeword error probability $P_{\text{cwe}}(r)$ of the joint decoder at receiver end. Let

$$\mathcal{I}_m := \{i_0, \cdots, i_{m-1}\} \subseteq \{0, 1, \cdots, K_o - 1\}$$

be a subset of K_o users, and let $\mathcal{E}_n(\mathcal{I}_m)$ denote the event that

1. the signal of the ith user is erroneously decoded if and only if $i \in \mathcal{I}_m$, and further that
2. the rank distance between overall transmitted code matrix and the erroneously decoded overall signal matrix is only nn_t for some $n \leq m$.

Specifically, let $x_i \in \mathfrak{A}_o(\text{SNR})$ denote the information symbol transmitted by the ith user, and let $\hat{x}_i$ be the corresponding decoding output at receiver; then the event $\mathcal{E}_n(\mathcal{I}_m)$ can be formulated as follows:

$$\mathcal{E}_n(\mathcal{I}_m) := \left\{ \begin{array}{l} x_i \neq \hat{x}'_i, \text{ for all } i \in \mathcal{I}_m, \\ x_i = \hat{x}'_i, \text{ for all } i \notin \mathcal{I}_m, \text{ and} \\ \text{rank}\,(C_o - C'_o) = nn_t \end{array} \right\}, \tag{6.2}$$

where from the proposed construction $\mathcal{S}_o$ (cf. (3.14)) we have

$$C_o = \begin{bmatrix} \psi_o(x_0) & \cdots & \tau_o^{K_o-1}(\psi_o(x_0)) \\ \vdots & \ddots & \vdots \\ \psi_o(x_{K_o-1}) & \cdots & \tau_o^{K_o-1}(\psi_o(x_{K_o-1})) \end{bmatrix}$$

and

$$C'_o = \begin{bmatrix} \psi_o(x'_0) & \cdots & \tau_o^{K_o-1}(\psi_o(x'_0)) \\ \vdots & \ddots & \vdots \\ \psi_o(x'_{K_o-1}) & \cdots & \tau_o^{K_o-1}\left(\psi_o(x'_{K_o-1})\right) \end{bmatrix}.$$

Note that the difference matrix $C_o - C_o'$ has exactly mn_t nonzero rows, and by Property 2 we see the rank distance $n_t \leq \text{rank}(C_o - C_o') \leq mn_t$. Hence it makes sense for the second requirement of error event $\mathcal{E}_n(\mathcal{I}_m)$ that $\text{rank}(C_o - C_o') = nn_t$ for some $1 \leq n \leq m$.

Thus, it can be seen from the union bound argument that the codeword error probability is upper bounded by

$$\begin{aligned} P_{\text{cwe}}(r) &= \Pr\left\{ \bigcup_{\mathcal{I}_m, n \leq m} \mathcal{E}_n(\mathcal{I}_m) \right\} \\ &\leq \sum_m \sum_{\mathcal{I}_m} \sum_{n \leq m} \Pr\left\{ \mathcal{E}_n(\mathcal{I}_m) \right\}. \end{aligned} \tag{6.3}$$

The event $\mathcal{E}_n(\mathcal{I}_m)$ is a further partition of the event considered by Tse *et al.* in [11]. We discuss this in more detail in the following remark.

Remark 7. *With regard to the Gaussian random codebook considered by Tse et al. [11], it is straightforward to see $\mathcal{E}_n(\mathcal{I}_m)$ is empty with probability one if $n < m$, since the component matrices associated with each user are complex Gaussian random matrices of size $(n_t \times T)$ for some $T \geq K_o n_t$. In other words, if $x_i \neq \hat{x}_i'$ for all $i \in \mathcal{I}_m$ and $x_i = \hat{x}_i'$ otherwise, then the error matrix $C_o - C_o'$ would have rank mn_t with probability one. Therefore, one can rewrite* (6.3) *as*

$$P_{cwe}(r) \leq \sum_m \sum_{\mathcal{I}_m} \Pr\left\{ \mathcal{E}_m(\mathcal{I}_m) \right\} \tag{6.4}$$

and recover the same union bound used in [11]. □

Unlike [11] where the authors analyzed each summand $\Pr\{\mathcal{E}_m(\mathcal{I}_m)\}$ of (6.4) by a union bound argument with a Gaussian random codebook, here we will focus on the error probability of a deterministic codebook $\mathcal{S}_o$ (cf. (3.14)), and attempt to upper bound the probability $\Pr\{\mathcal{E}_n(\mathcal{I}_m)\}$ by using a joint ML decoder. To this end, in Chapter 6.2 we will examine the minimum Euclidean distance among the noise-free received code matrices contained in $\mathcal{E}_n(\mathcal{I}_m)$. It should be noted that here by minimum Euclidean distance, we mean the minimum Euclidean distance among only the pairs of code matrices in $\mathcal{E}_n(\mathcal{I}_m)$, not the whole code $\mathcal{C}_o$. Thus, the minimum Euclidean distance will be a function of n, $\mathcal{I}_m$, and $\mathcal{C}_o$.

Once we obtain the minimum Euclidean distance, we will analyze the error performance of a bounded distance decoder, which will be used as an upper bound on that of the ML decoder. The bounded distance decoder results in an error only when the noise matrix has norm larger than half of the minimum Euclidean distance. More precisely, let H be the overall channel matrix defined in (6.1) and be known to the decoder; let $\mathcal{S}_o = \mathcal{S}_0 \times \cdots \times \mathcal{S}_{K_o - 1}$ be the overall MIMO-MAC code, where $\mathcal{S}_i$ is the codebook of the ith user. The minimum Euclidean distance $d_{\min}$ among all code matrices in $\mathcal{S}_o$ is defined as

$$d_{\min}(H) \;=\; \min_{S_o \neq S_o' \in \mathcal{S}_o} \left\| H\left(S_o - S_o'\right) \right\|,$$

which is dependent upon H. Given the received signal matrix $Y = HS_o + W$, the bounded distance decoder outputs $\hat{S}_o \in \mathcal{S}_o$ if $\left\| Y - H\hat{S}_o \right\| < \frac{d_{\min}(H)}{2}$, and declares a decoding failure otherwise. Thus, only the received signal matrices that are within distance $\frac{d_{\min}(H)}{2}$ from the original transmitted overall code matrix can be correctly decoded in the bounded distance decoder. Other received signal matrices would result in either a decoding error (i.e., decoding into an erroneous code matrix) or a decoding failure (i.e., cannot find a code matrix within distance $\frac{d_{\min}(H)}{2}$). Though this decoder is suboptimal compared to the ML decoder, its error performance can be mathematically analyzed.

The error performance analysis following this outline will be given in Chapter 6.3. Finally, in Chapter 6.6 we briefly discuss the proof for the case of even K.

6.2 Lower Bounds on the Minimum Distance Among Noise-Free Received Signal Matrices

For any $C_o \neq C'_o \in \mathcal{C}_o$ with

$$C_o = \begin{bmatrix} \psi_o(x_0) & \cdots & \tau_o^{K_o-1}(\psi_o(x_0)) \\ \vdots & \ddots & \vdots \\ \psi_o(x_{K_o-1}) & \cdots & \tau_o^{K_o-1}(\psi_o(x_{K_o-1})) \end{bmatrix}$$

and

$$C'_o = \begin{bmatrix} \psi_o(x'_0) & \cdots & \tau_o^{K_o-1}(\psi_o(x'_0)) \\ \vdots & \ddots & \vdots \\ \psi_o(x'_{K_o-1}) & \cdots & \tau_o^{K_o-1}\left(\psi_o(x'_{K_o-1})\right) \end{bmatrix}$$

let $S_o = \kappa C_o$, $S'_o = \kappa C'_o$, and let $H = [H_0 \cdots H_{K_o-1}]$, where H_i is the $(n_r \times n_t)$ channel matrix associated with the ith user.

Given the channel matrix H, below we provide a lower bound on the squared Euclidean distance between HS_o and HS'_o, i.e.,

$$d_E^2(S_o, S'_o) = \Bigg\| H \underbrace{(S_o - S'_o)}_{:=\Delta S_o} \Bigg\|_F^2 . \tag{6.5}$$

We distinguish the following two cases which correspond to the error events $\mathcal{E}_m(\mathcal{I}_m)$ and $\mathcal{E}_n(\mathcal{I}_m)$ with $n < m$, respectively.

1. For event $\mathcal{E}_m(\mathcal{I}_m)$ we have $x_\ell \neq x'_\ell$ for $\ell \in \{i_0, \cdots, i_{m-1}\}$, $x_\ell = x'_\ell$ otherwise, and $\mathrm{rank}(C_o - C'_o) = mn_t$. In this case, let C_s and C'_s be defined as in (3.23) and let

   $$H_s = \left[H_{i_0} \cdots H_{i_{m-1}}\right]$$

   be the equivalent $(n_r \times mn_t)$ channel matrix; then we have

   $$d_E^2(S_o, S'_o) = \|\kappa H_s (C_s - C'_s)\|_F^2 .$$

   Let $\lambda_{1,1}^{(m)} \leq \cdots \leq \lambda_{1,Q_m}^{(m)}$ be the set of ordered nonzero eigenvalues of $H_s H_s^\dagger$ where $Q_m = \min\{mn_t, n_r\}$, and let $\ell_{1,1} \geq \cdots \geq \ell_{1,mn_t} > 0$ be the ordered nonzero eigenvalues of $(C_s - C'_s)(C_s - C'_s)^\dagger$. Then we have

   $$d_E^2(S_o, S'_o) \geq \kappa^2 \sum_{i=1}^{Q_m} \lambda_{1,i}^{(m)} \ell_{1,mn_t - Q_m + i}. \tag{6.6}$$

   Note that

   $$\prod_{i=1}^{mn_t} \ell_{1,i} \geq \frac{1}{\|\gamma\|^{2mn_t}} \doteq 1, \tag{6.7}$$

   where the first inequality follows from Property 3, and the second exponential equality is because γ is fixed and is independent of SNR.

By repeatedly using the arithmetic mean-geometric mean inequality and (6.7) as in [4, 6] for $k = 1, 2, \cdots, Q_m$, we have

$$
\begin{aligned}
& d_E^2(S_o, S_o') && (6.8)\\
& \geq \kappa^2 \sum_{i=Q_m-k+1}^{Q_m} \lambda_{1,i}^{(m)} \ell_{1,mn_t-Q_m+i} \\
& \geq \kappa^2 \left[\prod_{i=Q_m-k+1}^{Q_m} \lambda_{1,i}^{(m)} \right]^{\frac{1}{k}} \times \\
& \quad \left[\prod_{i=Q_m-k+1}^{Q_m} \ell_{1,mn_t-Q_m+i} \right]^{\frac{1}{k}} \\
& \dot{\geq} \kappa^2 \left[\prod_{i=Q_m-k+1}^{Q_m} \lambda_{1,i}^{(m)} \right]^{\frac{1}{k}} \left[\prod_{i=1}^{mn_t-k} \ell_{1,i} \right]^{-\frac{1}{k}} && (6.9)\\
& \dot{\geq} \kappa^2 \left[\prod_{i=Q_m-k+1}^{Q_m} \lambda_{1,i}^{(m)} \right]^{\frac{1}{k}} \left[\sum_{i=1}^{mn_t-k} \ell_{1,i} \right]^{-\frac{mn_t-k}{k}} \\
& \dot{\geq} \kappa^2 \left[\prod_{i=Q_m-k+1}^{Q_m} \lambda_{1,i}^{(m)} \right]^{\frac{1}{k}} \| C_s - C_s' \|_F^{-\frac{mn_t-k}{k}} \\
& \dot{\geq} \mathrm{SNR}^{1-\frac{r}{n_t}} \left[\prod_{i=Q_m-k+1}^{Q_m} \lambda_{1,i}^{(m)} \right]^{\frac{1}{k}} \mathrm{SNR}^{-\frac{r}{n_t}\frac{mn_t-k}{k}} \\
& := d_{1,k}^{(m)}(\underline{\alpha}_1^{(m)}) \; = \; \mathrm{SNR}^{\delta_{1,k}^{(m)}(\underline{\alpha}_1^{(m)})}, && (6.10)
\end{aligned}
$$

where (6.9) follows from (6.7) and where in (6.10) we have set

$$
\begin{aligned}
\lambda_{1,i}^{(m)} &= \mathrm{SNR}^{-\alpha_{1,i}^{(m)}}, \\
\underline{\alpha}_1^{(m)} &= \left[\alpha_{1,1}^{(m)} \cdots \alpha_{1,Q_m}^{(m)} \right]^\top.
\end{aligned}
$$

Hence the SNR exponent of $d_E^2(S_o, S_o')$ is lower bounded by

$$
\delta_{1,k}^{(m)}(\underline{\alpha}_1^{(m)}) := \frac{1}{k} \left[\sum_{i=Q_m-k+1}^{Q_m} \left(1 - \alpha_{1,i}^{(m)} \right) \right] - \frac{rm}{k}. \tag{6.11}
$$

2. The second case corresponds to event $\mathcal{E}_n(\mathcal{I}_m)$ which means $x_\ell \neq x_\ell'$ for $\ell \in \mathcal{I}_m = \{i_0, \cdots, i_{m-1}\}$, $x_i = x_i'$ otherwise, and rank $(C_o - C_o') = nn_t < mn_t$. In other words, the m nonzero rows

$$
\left\{ \left[(x_\ell - x_\ell') \cdots \tau_o^{K_o-1}(x_\ell - x_\ell') \right] : \ell \in \mathcal{I}_m \right\}
$$

are not linearly independent over $\mathfrak{D}_o$. From Property 2 we can assume without loss of generality that

$$
\left\{ \left[(x_\ell - x_\ell') \cdots \tau_o^{K_o-1}(x_\ell - x_\ell') \right] : \ell = i_0, \cdots, i_{n-1} \right\}
$$

are linearly independent for some $n < m$.

Let $dx_\ell := x_\ell - x_\ell'$ and let C_s and C_s' be defined as in (3.23) with respect to the set $\{i_0, \cdots, i_{m-1}\}$. Set $\Delta C_s = C_s - C_s'$ and $\Delta X_\ell = \psi_o(dx_\ell)$. Property 2 in turn implies that

$$\Delta C_s = \begin{bmatrix} I_{n_t} & & \\ & \ddots & \\ & & I_{n_t} \\ P_{i_n,0} & \cdots & P_{i_n,n-1} \\ \vdots & \vdots & \vdots \\ P_{i_{m-1},0} & \cdots & P_{i_{m-1},n-1} \end{bmatrix} \mathbf{\Delta X}, \tag{6.12}$$

for some square matrices $P_{i,j}$, where

$$\mathbf{\Delta X} := \begin{bmatrix} \Delta X_{i_0} & \cdots & \tau_o^{K_o-1}(\Delta X_{i_0}) \\ \vdots & \ddots & \vdots \\ \Delta X_{i_{n-1}} & \cdots & \tau_o^{K_o-1}\left(\Delta X_{i_{n-1}}\right) \end{bmatrix}. \tag{6.13}$$

Similar to the previous case, let

$$H_s = \left[\begin{array}{ccc} H_{i_0} & \cdots & H_{i_{m-1}} \end{array} \right]$$

be the equivalent channel matrix; then the difference of the noise-free received signal matrices can be rewritten as

$$\kappa H_s \Delta C_s = \kappa \mathbf{H}_{eq} \mathbf{\Delta X}, \tag{6.14}$$

where

$$\mathbf{H}_{eq} = \left[\begin{array}{ccc} \tilde{H}_{i_0} & \cdots & \tilde{H}_{i_{n-1}} \end{array} \right] \tag{6.15}$$

is an alternative channel equivalent matrix and

$$\tilde{H}_{i_\ell} := H_{i_\ell} + \sum_{k=n}^{m-1} H_{i_k} P_{i_k,\ell} \tag{6.16}$$

for $\ell = 0, 1, \cdots, n-1$.

Let $\lambda_{2,1}^{(m,n)} \le \cdots \le \lambda_{2,Q_n}^{(m,n)}$ be the set of ordered nonzero eigenvalues of $\mathbf{H}_{eq}\mathbf{H}_{eq}^\dagger$, where $Q_n = \min\{nn_t, n_r\}$, and let $\ell_{2,1} \ge \cdots \ge \ell_{2,nn_t} > 0$ be the ordered nonzero eigenvalues of $\mathbf{\Delta X}\mathbf{\Delta X}^\dagger$. Notice that

$$\prod_{i=1}^{nn_t} \ell_{2,i} = \det\left(\mathbf{\Delta X}\mathbf{\Delta X}^\dagger\right) \ge \frac{1}{\|\gamma\|^{2nn_t}} \doteq 1$$

from Property 3. Arguing similarly as in the first case shows that

$$d_E^2(S_o, S_o') \ge d_{2,k}^{(m,n)}(\underline{\alpha}_2^{(m,n)}) := \text{SNR}^{\delta_{2,k}^{(m,n)}(\underline{\alpha}_2^{(m,n)})} \tag{6.17}$$

for $k = 1, 2, \cdots, Q_n$, where

$$\lambda_{2,i}^{(m,n)} := \text{SNR}^{-\alpha_{2,i}^{(m,n)}}, \tag{6.18}$$

$$\underline{\alpha}_2^{(m,n)} = \left[\alpha_{2,1}^{(m,n)} \cdots \alpha_{2,Q_n}^{(m,n)}\right]^\top, \tag{6.19}$$

and

$$\delta_{2,k}^{(m,n)}(\underline{\alpha}_2^{(m,n)}) := \frac{1}{k}\left[\sum_{i=Q_n-k+1}^{Q_n} \left(1 - \alpha_{2,i}^{(m,n)}\right)\right] - \frac{rn}{k}. \tag{6.20}$$

Remark 8. *We remark that* (6.20) *shows the last term is* $-\frac{rn}{k}$, *instead of being* $-\frac{rm}{k}$ *as in* (6.11). *For readers who may wonder why these two terms are different given both events concern the case of* m *users in error, the major reason is due to the distance bounding techniques, i.e., the repeated arithmetic mean-geometric mean inequalities, we have used in the above.*

In general, when the equivalent channel matrix $\mathbf{H}_{eq}$ *of* (6.15), *and similarly when the channel matrix* H_s *with* $n = m$, *has rank* Q_n, *the rank of the product matrix* $\mathbf{H}_{eq}\Delta\mathbf{X}$ *would be* Q_n *since* $\Delta\mathbf{X}$ *is of full rank* nn_t. *Thus our lower bound on the norm* $\|\mathbf{H}_{eq}\Delta\mathbf{X}\|$ *would only capture the* Q_n *smaller eigenvalues of* $\Delta\mathbf{X}\Delta\mathbf{X}^\dagger$, *which are all nonzero. Furthermore, one reason for introducing the equivalent channel matrix* $\mathbf{H}_{eq}$, *rather than working with* H_s *is that, algebraically speaking, the norm* $\|H_s\Delta C_s\|$ *could be zero as* ΔC_s *is singular and all the rows of* H_s *could lie in the left-null space of* C_s. *However, since* H_s *is random, this occurs with probability zero. In other words, if we apply the series of arithmetic mean-geometric mean inequalities to the matrix product* $H_s\Delta C_s$, *we could end up with the trivial algebraic inequality*

$$d_E(S_o, S_o') \;\geq\; \min_{H_s} \|H_s\Delta C_s\| = 0,$$

even the right-hand-side has probability 0. Whether the above could happen depends on the relations among n, m, n_t, *and* n_r. *While there is nothing wrong with the algebraic inequality itself, this bound can actually be further tightened by introducing the equivalent channel* $\mathbf{H}_{eq}$ *so that we can focus on error events that have probability larger than zero.* □

Remark 9. *Another heuristic way to see why the last term of* $\delta_{2,k}^{(m,n)}$ *equals* $-\frac{rn}{k}$ *follows from the base-alphabet* $\mathcal{A}(\mathrm{SNR})$ *defined in Chapter 3.1. Recall that in the construction of* $(nn_t \times T)$ *CDA-based ST code for point-to-point channel [4, 6], to achieve the DMT optimality therein we would set the base-alphabet as*

$$\mathcal{A}'(\mathrm{SNR}) = \left\{ a + b\imath \;:\; \begin{array}{r} -\mathrm{SNR}^{\frac{r}{2nn_t}} \leq a, b \leq \mathrm{SNR}^{\frac{r}{2nn_t}}, \\ a, b \in \mathbb{Z}, \quad a, b \textit{ odd} \end{array} \right\}$$

such that the resulting exponent equals

$$\delta_{2,k}^{\prime(m,n)}(\underline{\alpha}_2^{(m,n)}) := \frac{1}{k}\left[\sum_{i=Q_n-k+1}^{Q_n} \left(1 - \alpha_{2,i}^{(m,n)}\right)\right] - \frac{r}{k};$$

then along the same lines as in [4, 6] one can prove such code is approximately universal and achieves diversity gain $d^*_{nn_t,n_r}(r)$. *However, it is because we set the base-alphabet as* $\mathcal{A}(\mathrm{SNR})$, *which has size*

$$|\mathcal{A}(\mathrm{SNR})| \;=\; |\mathcal{A}'(\mathrm{SNR})|^n\,,$$

meaning an n*-fold increase in the multiplexing gain, we expect the error probability associated with event* $\mathcal{E}_n(\mathcal{I}_m)$ *has diversity gain* $d^*_{nn_t,n_r}(nr)$. □

6.3 Upper Bounds on Codeword Error Probability

Having obtained the squared minimum Euclidean distances $d_{1,k}^{(m)}(\underline{\alpha}_1^{(m)})$ among the signal matrices associated with error event $\mathcal{E}_m(\mathcal{I}_m)$, and $d_{2,k}^{(m,n)}(\underline{\alpha}_2^{(m,n)})$ among the signal matrices associated with error event $\mathcal{E}_n(\mathcal{I}_m)$, below we proceed to analyze the error performance of the proposed construction. The analysis resembles the sphere bounding technique used in [4, 6] which is essentially a bounded-distance decoding technique. That is, the bounded-distance decoder declares an error only when the noise has norm larger than half of the minimum Euclidean distance. Clearly, the error performance of a bounded-distance decoder serves as an upper bound on that of a joint ML decoder.

First, since the lower bounds on the Euclidean distance $d_E(S_o, S_o')$ hold for all k, we define

$$\begin{aligned} d_{1,\min}^{(m)}(\underline{\alpha}_1^{(m)}) &:= \max_{1\le k\le Q_m} d_{1,k}^{(m)}(\underline{\alpha}_1^{(m)}), \\ \delta_{1,\min}^{(m)}(\underline{\alpha}_1^{(m)}) &:= \max_{1\le k\le Q_m} \delta_{1,k}^{(m)}(\underline{\alpha}_1^{(m)}), \\ d_{2,\min}^{(m,n)}(\underline{\alpha}_2^{(m,n)}) &:= \max_{1\le k\le Q_n} d_{2,k}^{(m,n)}(\underline{\alpha}_2^{(m,n)}), \\ \delta_{2,\min}^{(m,n)}(\underline{\alpha}_2^{(m,n)}) &:= \max_{1\le k\le Q_n} \delta_{2,k}^{(m,n)}(\underline{\alpha}_2^{(m,n)}). \end{aligned}$$

Then, using the bounded distance decoder discussed in Chapter 6.1, the probability of error event $\mathcal{E}_m(\mathcal{I}_m)$ given channel matrix H can be upper bounded by

$$\begin{aligned} &\Pr\{\mathcal{E}_m(\mathcal{I}_m)|H\} \\ &\le \Pr\left\{ \|W\|_F^2 \ \ge \ \frac{\left[d_{1,\min}^{(m)}(\underline{\alpha}_1^{(m)})\right]^2}{4} \right\} \\ &= \exp\left(-\frac{\left[d_{1,\min}^{(m)}(\underline{\alpha}_1^{(m)})\right]^2}{4} \right) \sum_{j=0}^{K_o n_r n_t - 1} \frac{\left[d_{1,\min}^{(m)}(\underline{\alpha}_1^{(m)})\right]^{2j}}{j!}, \end{aligned} \tag{6.21}$$

where the inequality follows from the property of a bounded distance decoder. Hence we see that $\Pr\{\mathcal{E}_m(\mathcal{I}_m)|H\} \doteq 0$ if $\delta_{1,k}^{(m)}(\underline{\alpha}_1^{(m)}) > 0$. On the other hand, we may replace the above upper bound of $\Pr\{\mathcal{E}_m(\mathcal{I}_m)|H\}$ with the trivial upper bound $\Pr\{\mathcal{E}_m(\mathcal{I}_m)|H\} \le 1$ when $\delta_{1,\min}^{(m)}(\underline{\alpha}_1^{(m)}) \le 0$. Thus, it implies

$$\begin{aligned} \Pr\{\mathcal{E}_m(\mathcal{I}_m)\} &= \mathbb{E}_H \Pr\{\mathcal{E}_m(\mathcal{I}_m)|H\} \\ &\le \Pr\left\{ H : \delta_{1,\min}^{(m)}(\underline{\alpha}_1^{(m)}) \le 0 \right\} \\ &\le \Pr\left\{ H : \delta_{1,k}^{(m)}(\underline{\alpha}_1^{(m)}) \le 0, \quad 1 \le k \le Q_m \right\}. \end{aligned}$$

Similarly, for error event $\mathcal{E}_n(\mathcal{I}_m)$ with $n < m$ we have

$$\begin{aligned} \Pr\{\mathcal{E}_n(\mathcal{I}_m)\} &= \mathbb{E}_H \Pr\{\mathcal{E}_n(\mathcal{I}_m)|H\} \\ &\le \Pr\left\{ H : \delta_{2,k}^{(m,n)}(\underline{\alpha}_2^{(m,n)}), \quad 1 \le k \le Q_n \right\}. \end{aligned}$$

Since the above bounds do not depend on the specific choices of $\mathcal{I}_m$, from (6.3) the union bound on the codeword error probability $P_{\text{cwe}}(r)$ gives

$$\begin{aligned} P_{\text{cwe}}(r) \ &\le \ \sum_m \sum_{\mathcal{I}_m} \sum_{n\le m} \Pr\{\mathcal{E}_n(\mathcal{I}_m)\} \\ &\le \sum_{m=1}^{K_o} \binom{K_o}{m} \left[\Pr\left\{ H : \delta_{1,k}^{(m)}(\underline{\alpha}_1^{(m)}) \le 0, 1 \le k \le Q_m \right\} + \right. \\ &\left. \sum_{n=1}^{m-1} \binom{m}{n} \Pr\left\{ H : \delta_{2,k}^{(m,n)}(\underline{\alpha}_2^{(m,n)}), 1 \le k \le Q_n \right\} \right]. \end{aligned} \tag{6.22}$$

Remark 10. *One can regard the probability*

$$\binom{K_o}{m}\Pr\left\{H : \delta_{1,k}^{(m)}(\underline{\alpha}_1^{(m)}) \le 0, 1 \le k \le Q_m\right\}$$

as a further upper bound on the union bound $\sum_{\mathcal{I}_m} \Pr\{\mathcal{E}_m(\mathcal{I}_m)\}$ *in* (6.3)*, and the second type of probability*

$$\binom{K_o}{m}\sum_{n=1}^{m-1}\binom{m}{n}\Pr\left\{H : \delta_{2,k}^{(m,n)}(\underline{\alpha}_2^{(m,n)}), 1 \le k \le Q_n\right\} \tag{6.23}$$

as an upper bound on $\sum_{\mathcal{I}_m}\sum_{n<m}\Pr\{\mathcal{E}_n(\mathcal{I}_m)\}$*. Furthermore, the event* $\mathcal{E}_m$ *of* m *users in error has probability upper bounded by*

$$\begin{aligned}\Pr\{\mathcal{E}_m\} &\le \sum_{\mathcal{I}_m}\sum_{n\le m}\Pr\{\mathcal{E}_n(\mathcal{I}_m)\}\\ &\le \binom{K_o}{m}\Bigg[\Pr\left\{H : \delta_{1,k}^{(m)}(\underline{\alpha}_1^{(m)}) \le 0, 1 \le k \le Q_m\right\} +\\ &\sum_{n=1}^{m-1}\binom{m}{n}\Pr\left\{H : \delta_{2,k}^{(m,n)}(\underline{\alpha}_2^{(m,n)}), 1 \le k \le Q_n\right\}\Bigg].\end{aligned} \tag{6.24}$$

It should be noted that in (6.23) *we have over-estimated the number of choices of* n $\mathfrak{D}_o$*-linearly independent rows out of* m *nonzero rows in the difference matrix* $[\tau_o^j(x_i - x_i')]_{i=0}^{K_o-1}{}_{j=0}^{K_o-1}$ *that can happen in the event* $\mathcal{E}_n(\mathcal{I}_m)$*.* □

Even with this over-estimate, noting

$$\binom{K_o}{m}, \binom{m}{n} \doteq 1$$

for all m, n within the range of interest, we can rewrite (6.22) as

$$\begin{aligned}P_{\text{cwe}}(r) \,\dot{\le}\, \max_m \Pr\left\{H : \max_k \delta_{1,k}^{(m)}(\underline{\alpha}_1^{(m)}) \le 0\right\} +\\ \max_{n<m}\Pr\left\{H : \max_k \delta_{2,k}^{(m,n)}(\underline{\alpha}_2^{(m,n)}) \le 0\right\}.\end{aligned} \tag{6.25}$$

Below we investigate the diversity orders of each term in (6.25).

6.4 Diversity Gain of the First Case

For each m, $1 \le m \le K_0$, we have

$$\begin{aligned}&\Pr\left\{H : \max_k \delta_{1,k}^{(m)}(\underline{\alpha}_1^{(m)}) \le 0\right\}\\ &= \Pr\left\{H : \begin{array}{c}\frac{1}{k}\left[\sum_{i=Q_m-k+1}^{Q_m}\left(1-\alpha_{1,i}^{(m)}\right)\right] - \frac{rm}{k} \le 0,\\ \text{all } k, \text{ and } \alpha_{1,1}^{(m)} \ge \alpha_{1,2}^{(m)} \cdots \ge \alpha_{1,Q_m}^{(m)}\end{array}\right\}\\ &= \Pr\left\{H : \sum_{i=1}^{Q_m}\left(1-\alpha_{1,i}^{(m)}\right)^+ \le rm\right\} &(6.26)\\ &= \Pr\left\{H : \log\det\left(I_{n_r} + \text{SNR} H_s H_s^\dagger\right) \le rm \log \text{SNR}\right\}\\ &\doteq \text{SNR}^{-d^*_{mn_t,n_r}(rm)}, &(6.27)\end{aligned}$$

where $Q_m := \min\{mn_t, n_r\}$, $H_s = \left[H_{i_0} \cdots H_{i_{m-1}}\right]$, and where $(x)^+ = \max\{x, 0\}$ for $x \in \mathbb{R}$. Equation (6.26) follows from [33–35], and (6.27) is given in [9] since H_s is a matrix of size $(n_r \times mn_t)$ having entries that are i.i.d. $\mathbb{C}\mathcal{N}(0,1)$ complex Gaussian random variables. The quantity $d^*_{mn_t,n_r}(rm)$ represents the point-to-point DMT of an $mn_t \times n_r$ MIMO Rayleigh fading channel at multiplexing gain rm.

6.5 Diversity Gain of the Second Case

Similarly, for the second set of maximizations in (6.25) we have for each $1 \le n < m \le K_o$ that

$$\begin{aligned}
&\Pr\left\{H : \max_k \delta_{2,k}^{(m,n)}(\underline{\alpha}_2^{(m,n)}) \le 0\right\} \\
&= \Pr\left\{H : \begin{array}{l} \frac{1}{k}\left[\sum_{i=Q_n-k+1}^{Q_n}\left(1-\alpha_{2,i}^{(m,n)}\right)\right] - \frac{rn}{k} \le 0, \\ \text{all } k, \text{ and } \alpha_{2,1}^{(m,n)} \ \ge\ \alpha_{2,2}^{(m,n)} \ \cdots \ \ge \alpha_{2,Q_n}^{(m,n)} \end{array}\right\} \\
&= \Pr\left\{H : \sum_{i=1}^{Q_n}\left(1-\alpha_{2,i}^{(m,n)}\right)^+ \le rn\right\} \\
&= \Pr\left\{H : \log\det\left(I_{n_r} + \text{SNR}\mathbf{H}_{eq}\mathbf{H}_{eq}^\dagger\right) \le rn\log\text{SNR}\right\},
\end{aligned} \tag{6.28}$$

where $\mathbf{H}_{eq}$ is defined in (6.15) and is of size $(n_r \times nn_t)$. Noting that the entries of $\mathbf{H}_{eq}$ are correlated complex Gaussian random variables, we invoke the following result which was shown independently in [36, Corollary 1] and [37, Theorem 3] to simplify the analysis.

Theorem 14 ([36,37])**.** *The diversity order of outage probability for Rayleigh fading channels with arbitrary full rank correlations is unchanged from the case of i.i.d. Rayleigh fading. Moreover, if the channel matrix H can be decoupled as $\Sigma_L\tilde{H}\Sigma_R$ where $\tilde{H}$ has independent and regular entries, then the optimal DMT for channel H is the same as that for $\tilde{H}$.* □

Armed with Theorem 14, the analysis of the diversity gain of the second case is now easy. A direct application of the above theorem gives

$$\begin{aligned}
&\Pr\left\{H : \max_k \delta_{2,k}^{(m,n)}(\underline{\alpha}_2^{(m,n)}) \le 0\right\} \\
&= \Pr\left\{H : \log\det\left(I_{n_r} + \text{SNR}\mathbf{H}_{eq}\mathbf{H}_{eq}^\dagger\right) \le rn\log\text{SNR}\right\} \\
&\doteq \ \text{SNR}^{-d^*_{nn_t,n_r}(nr)}.
\end{aligned} \tag{6.29}$$

Summarizing results of (6.25), (6.27) and (6.29) gives

$$P_{\text{cwe}}(r) \dot{\le} \text{SNR}^{-d(r)}$$

and

$$\begin{aligned}
d(r) &:= \min_{n<m}\left\{d^*_{mn_t,n_r}(mr), d^*_{nn_t,n_r}(nr)\right\} \\
&= \min_m\left\{d^*_{mn_t,n_r}(mr)\right\} \ = \ d^*_{n_t,n_r,K}(r).
\end{aligned}$$

This completes the proof.

Remark 11. *The above proof shows that*

$$\begin{aligned}
\Pr\left\{\mathcal{E}_m(\mathcal{I}_m)\right\} &\le \Pr\left\{H : \max_k \delta_{1,k}^{(m)}(\underline{\alpha}_1^{(m)}) \le 0\right\} \\
&\doteq \text{SNR}^{-d^*_{mn_t,n_r}(mr)}
\end{aligned}$$

for error events $\mathcal{E}_m(\mathcal{I}_m)$, $m = 1, 2, \cdots, K$, and

$$\begin{aligned}\Pr\{\mathcal{E}_n(\mathcal{I}_m)\} &\le \Pr\left\{H : \max_k \delta_{2,k}^{(m,n)}(\underline{\alpha}_2^{(m,n)}) \le 0\right\} \\ &\doteq \text{SNR}^{-d^*_{nn_t,n_r}(nr)}\end{aligned}$$

for $1 \le n < m$. This is exactly what is shown in Theorem 3. Furthermore, in the event $\mathcal{E}_m$ of m users in error, the proposed code $\mathcal{S}_o$ has error probability

$$\begin{aligned}\Pr\{\mathcal{E}_m\} &\le \sum_{n=1}^{m} \Pr\{\mathcal{E}_n(\mathcal{I}_m)\} \\ &\dot{\le} \max\left\{\text{SNR}^{-d^*_{n_t,n_r}(r)}, \text{SNR}^{-d^*_{mn_t,n_r}(mr)}\right\}.\end{aligned}$$

□

6.6 Proof Outline for K Even

The proof of Theorem 7 can be adapted to cater to the case when the number of users K is even. Here we discuss only briefly what the changes are. Firstly, with

$$H = \left[\begin{array}{cccc} H_0 & \cdots & H_{K-1} & \mathbf{0} \end{array}\right]$$

in mind, i.e., $H_{K_o-1} = \mathbf{0}$, the result (6.5) of the squared Euclidean distance between S_o and S_o' remains to hold. Similarly, the further lower bounds on $d_E^2(S_o, S_o')$ in (6.11) and (6.20) stay without changes except that one should keep the following in mind.

1. The parameter m of the first case, where $\text{rank}(C_o - C_o') = mn_t$, and m out of K_o x_i's are distinct, has value from 1 up to $K_o - 1 = K$. This is because $H_{K_o-1} = \mathbf{0}$, and we can always assume $x_{K_o-1} = x'_{K_o-1}$ without affecting the value of $d_E^2(S_o, S_o')$. Thus the diversity gain resulting from the first case is
$$\min_{1 \le m \le K} d^*_{mn_t,n_r}(mr).$$
Compared with the case of odd K, (6.27) has m up to K_o.

2. The parameters m and n in the second case can be argued similarly as the above, and we have $1 \le n < m \le K_o - 1 = K$. Hence the diversity gain of this case is
$$\min_{1 \le n \le K-1} d^*_{nn_t,n_r}(nr).$$

Overall, it shows the MAC-DMT optimality of the proposed construction remains to hold.

Chapter 7

Proof of Theorem 11

The proof of Theorem 11 is similar to that of Theorem 7. Therefore, we will skip the most of the details and highlight only the key differences.

First, for any subset $\mathcal{I}_m = \{i_0, \cdots, i_{-1}\} \subseteq \{0, 1, \cdots, K-1\}$ of users, again let $\mathcal{E}_n(\mathcal{I}_m)$ denote the event that the decoder has made an error in decoding m users signals, but only the rows formed by the difference signal matrices of some n users, say user $i_0, \cdots, i_{n-1}$, are linearly independent over $\mathfrak{D}$. In other words, let (x_i, x'_i) be a pair of distinct information symbols of the ith user for $i \in \mathcal{I}_m$. Set the submatrices C_s and C'_s as in (4.21) with $X_i = \phi(x_i)$ and $X'_i = \phi(x'_i)$. Then the event $\mathcal{E}_n(\mathcal{I}_m)$ corresponds to the case when $\text{rank}\left(C_s - C'_s\right) = nn_{\max}$.

Let H_i denote the $(n_r \times n_i)$ channel matrix of the ith user that is known completely to receiver. Since the code matrices $S_i \in \mathcal{S}_i$ of the ith users are of size $(n_{\max} \times Kn_{\max})$, here we will assume without loss of generality that only the first n_i rows of S_i are used for transmission via the n_i transmit antennas of the ith user, and the remaining $(n_{\max} - n_i)$ rows are discarded during either encoding or transmission. On the other hand, we could extend the channel matrix H_i to an equivalent channel matrix $\tilde{H}_i$ of size $(n_r \times n_{\max})$ by adding on the right an all-zero matrix with appropriate size. That is, we set

$$\tilde{H}_i := \left[\begin{array}{cc} H_i & \mathbf{0}_{n_r \times (n_{\max} - n_i)} \end{array}\right],$$

and the received signal matrix Y can be written as

$$Y = \sum_{i=0}^{K-1} \tilde{H}_i S_i + W,$$

where W is the noise matrix of size $(n_r \times Kn_{\max})$.

For the event $\mathcal{E}_m(\mathcal{I}_m)$, let

$$\tilde{H}_s = \left[\begin{array}{ccc} \tilde{H}_{i_0} & \cdots & \tilde{H}_{i_{m-1}} \end{array}\right]$$

be the overall equivalent channel matrix, and let $\lambda_{1,1}^{(m)} \leq \cdots \leq \lambda_{1,Q_{\mathcal{I}_m}}^{(m)}$ be the ordered nonzero eigenvalues of $\tilde{H}_s \tilde{H}_s^\dagger$ with

$$Q_{\mathcal{I}_m}^{(m)} = \min\left\{ n_r, \sum_{j=0}^{m-1} n_{i_j} \right\}.$$

Similarly, let $\ell_{1,1} \geq \cdots \geq \ell_{1,mn_{\max}} > 0$ be the ordered nonzero eigenvalues of $\left(S_s - S'_s\right)\left(S_s - S'_s\right)^\dagger$.

Then the minimum squared Euclidean distance between S_s and S'_s is bounded by

$$
\begin{aligned}
d_E^2(S_s, S'_s) &\geq \sum_{i=Q_{\mathcal{I}_m}^{(m)}-k+1}^{Q_{\mathcal{I}_m}^{(m)}} \lambda_{1,i}^{(m)} \ell_{1,mn_{\max}-Q_{\mathcal{I}_m}^{(m)}+i} \\
&\dot{\geq} \left[\prod_{i=Q_{\mathcal{I}_m}^{(m)}-k+1}^{Q_{\mathcal{I}_m}^{(m)}} \lambda_{1,i}^{(m)} \right]^{\frac{1}{k}} \times \\
&\quad \mathrm{SNR}^{\frac{mn_{\max}-\sum_{j=1}^m r_{i_j}}{k}} \|S_s - S'_s\|_F^{-\frac{mn_{\max}-k}{k}} \qquad (7.1) \\
&\dot{\geq} \mathrm{SNR}^{\delta_{1,k}^{(m)}(\underline{\alpha}_1^{(m)})}, \qquad (7.2)
\end{aligned}
$$

where

$$\delta_{1,k}^{(m)}(\underline{\alpha}_1^{(m)}) \; = \; \frac{1}{k}\left[\sum_{j=Q_{\mathcal{I}_m}^{(m)}-k+1}^{Q_{\mathcal{I}_m}^{(m)}} \left(1-\alpha_{1,j}^{(m)}\right) - \sum_{j=1}^{m} r_{i_j} \right] \tag{7.3}$$

for $k = 1, \cdots, Q_{\mathcal{I}_m}^{(m)}$, and where

$$\alpha_{1,j}^{(m)} \; = \; -\frac{\log \lambda_{1,i}^{(m)}}{\log \mathrm{SNR}}. \tag{7.4}$$

(7.1) follows from Property 6 that

$$\prod_{j=1}^{mn_{\max}} \ell_{1,1} \;\dot{\geq}\; \mathrm{SNR}^{mn_{\max}-\sum_{j=1}^m r_{i_j}}$$

and (7.2) from Property 5 that

$$\sum_{j=1}^{mn_{\max}} \ell_{1,1} \;\dot{\leq}\; \mathrm{SNR}.$$

Now we see that

$$
\begin{aligned}
&\Pr\left\{\mathcal{E}_m(\mathcal{I}_m)\right\} \\
&\dot{\leq} \Pr\left\{\tilde{H}_s : \delta_{1,k}^{(m)}(\underline{\alpha}_1^{(m)}) < 0, \text{ all } k\right\} \\
&\doteq \Pr\left\{\log\det\left(I_{n_r} + \mathrm{SNR}\tilde{H}_s\tilde{H}_s^{\dagger}\right) \leq \sum_{j=0}^{m-1} r_{i_j} \log \mathrm{SNR}\right\} \\
&\doteq \mathrm{SNR}^{-d^*_{N_t(\mathcal{I}_m),n_r}\left(\sum_{i\in\mathcal{I}_m} r_i\right)}. \qquad (7.5)
\end{aligned}
$$

Similarly, techniques used in Chapters 6.2 and 6.5 can be modified accordingly to show

$$\Pr\left\{\mathcal{E}_n(\mathcal{I}_m)\right\} \;\dot{\leq}\; \mathrm{SNR}^{-d^*_{\sum_{i=0}^{n-1} n_i, n_r}\left(\sum_{i=0}^{n-1} r_i\right)}. \tag{7.6}$$

Combining the above results completes the proof of MAC-DMT optimality of the proposed construction.

Corollary 15. *Given n_i and r_i, $i = 0, 1, \cdots, K-1$ with K odd, the proposed code $\mathcal{S}$ defined in* (4.15) *achieves the general individual MAC-DMT of the ith user*

$$d^{(i)*}_{\{n_0,\cdots,n_{K-1}\},n_r}(r_0,\cdots,r_{K-1}) \; = \; \min_{\mathcal{I}:i\in\mathcal{I}} d^*_{N_t(\mathcal{I}),n_r}\left(\sum_{i\in\mathcal{I}} r_i\right),$$

over a Rayleigh block fading channel that remains static for at least $T \geq Kn_{\max}$ channel uses and for each i, where the minimization is taken over all $\mathcal{I} \subseteq \{0, 1, \cdots, K-1\}$ under the condition $i \in \mathcal{I}$. Thus, $\mathcal{S}$ is individual MAC-DMT optimal.

Proof. Following the proof outline of Theorem 9 it suffices to show only that the error probability of the ith individual decoder of the proposed code $\mathcal{S}$ meets the following bound

$$P_e^{(i)}(r_0, \cdots, r_{K-1}) \dot{\leq} \mathrm{SNR}^{-d^{(i)*}_{\{n_0, \cdots, n_{K-1}\}, n_r}(r_0, \cdots, r_{K-1})}.$$

Again, note that the error probability of ith individual decoder can be upper bounded by the joint decoder when the error events $\mathcal{E}_m(\mathcal{I}_m)$ and $\mathcal{E}_n(\mathcal{I}_m)$, $1 \leq n \leq m$, occur with $i \in \mathcal{I}_m$. It then follows from (7.5) and (7.6) that

$$\begin{aligned}
& P_e^{(i)}(r_0, \cdots, r_{K-1}) \\
\leq & \sum_{m=1}^{K} \sum_{\mathcal{I}_m : i \in \mathcal{I}_m} \left[\Pr\{\mathcal{E}_m(\mathcal{I}_m)\} + \sum_{n=1}^{m-1} \Pr\{\mathcal{E}_n(\mathcal{I}_m)\} \right] \\
\doteq & \max_{\mathcal{I}_m : i \in \mathcal{I}_m} \mathrm{SNR}^{-d^*_{N_t(\mathcal{I}_m), n_r}\left(\sum_{i \in \mathcal{I}_m} r_i\right)}.
\end{aligned}$$

This completes the proof. $\square$

Chapter 8

DMT Performance of A Simple Code

For simplicity, we will first present the code for use in a MIMO-MAC channel. For point-to-point MIMO channels, the same code can be easily modified and will be discussed later in the next section.

Consider a MIMO-MAC channel with $K = 2$ users, each having $n_t = 1$ transmit antenna and transmitting at multiplexing gain r. Assume there are $n_r = 2$ receive antennas at receiving end. The code to be analyzed is the following:

$$\mathcal{S} \;=\; \left\{ S = \kappa \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} : s_{ij} \in \mathcal{A}(\text{SNR}) \right\} \tag{8.1}$$

where

$$\mathcal{A}(\text{SNR}) = \left\{ a + b\,\imath \;:\; |a|, |b| \leq \text{SNR}^{\frac{r}{2}},\; a, b \text{ odd} \right\}, \tag{8.2}$$

$\imath = \sqrt{-1}$, and where

$$\kappa^2 \;\doteq\; \text{SNR}^{1-r}. \tag{8.3}$$

Entries s_{ij} are independently drawn from the QAM set $\mathcal{A}(\text{SNR})$. During transmission, the first user transmits the first row of S while the second user sends the second row. Clearly, the two users do not cooperate. Given $S \in \mathcal{S}$, the received signal matrix is

$$Y \;=\; HS + W, \tag{8.4}$$

where $H = [\underline{h}_1 \underline{h}_2]$ is the overall (2×2) channel matrix whose entries are modeled as i.i.d. complex Gaussian random variables $\mathbb{C}\mathcal{N}(0, 1)$ and where W is the (2×2) noise matrix. $\underline{h}_i$ is the channel vector associated with the $\imath$th user. We assume H is known to the receiver but is unknown to either of the users.

Obviously, the code $\mathcal{S}$ of (8.1) is uncoded since the entries are just plain QAM symbols with some scaling factor κ that is chosen to satisfy the power constraint $\mathbb{E}\;|\kappa s_{ij}|^2 \;\dot{\leq}\; \text{SNR}$. Nevertheless, below we will show that this uncoded scheme $\mathcal{S}$ achieves the optimal MAC-DMT (cf. (1.6))

$$d^*_{1,2,2}(r) = \min \left\{ d^*_{1,2}(r), d^*_{2,2}(2r) \right\}$$

over the two-user MIMO-MAC channel.

To prove the claim, we will partition the error event $\mathcal{E}$ into several subevents $\mathcal{E}_1, \cdots, \mathcal{E}_n$ for some n, and analyze the probability of each. Then we will apply the union bound

$$\Pr \left\{ \mathcal{E} = \bigcup_{i=1}^{n} \mathcal{E}_i \right\} \;\leq\; \sum_{i=1}^{n} \Pr \left\{ \mathcal{E}_i \right\}$$

to establish the claim. Although during the analysis some subevents can be combined, in order to be extra cautious we will analyze separately the error probabilities of these events. Below we distinguish five different kinds of error events.

8.1 Type-I Error Event

The first-type error event $\mathcal{E}_1$ corresponds to the case when only one entry in S is erroneously decoded. Without loss of generality, below we focus on a specific subevent of $\mathcal{E}_1$: for any $S \neq S' \in \mathcal{S}$,

$$\mathcal{E}_{1,1} := \left\{ S - S' = \kappa \begin{bmatrix} d_{11} & 0 \\ 0 & 0 \end{bmatrix} : 0 \neq d_{11} = s_{11} - s'_{11} \right\}. \tag{8.5}$$

That is, events with only one $d_{ij} \neq 0$ can be considered the same as $\mathcal{E}_{1,1}$ and we have $\Pr\{\mathcal{E}_1\} \leq 4 \Pr\{\mathcal{E}_{1,1}\}$.

To find out the error probability of $\mathcal{E}_{1,1}$, it suffices to note that the subcode $\mathcal{S}_{11} = \{\kappa s_{11} : s_{11} \in \mathcal{A}(\text{SNR})\}$ is exactly the CDA-based code proposed by Elia *et al.* [4] with $n_t = 1$ and $T = 1$. Hence we have

$$\Pr\{\mathcal{E}_{1,1}\} \;\dot{\leq}\; \text{SNR}^{-d^*_{1,2}(r)}. \tag{8.6}$$

To make the present report self-contained, below we briefly highlight some key steps in proving (8.6). The proof actually follows from the fact that the bounded-distance decoder would make an error if the noise vector $\underline{w}_1$ has norm larger than half the minimum Euclidean distance, i.e. if

$$\|\underline{w}_1\|^2 \;\geq\; \min_{S-S' \in \mathcal{E}_1} \|H(S-S')\|^2 \;=\; \min_{d_{11} \neq 0} \|\kappa \underline{h}_1 d_{11}\|^2,$$

where we have set the noise matrix $W = [\underline{w}_1 \; \underline{w}_2]$. By $\|A\|$ we mean the Frobenius norm of matrix A.

Given the channel vector $\underline{h}_1$, if

$$\min_{d_{11} \neq 0} \|\kappa \underline{h}_1 d_{11}\|^2 \;\dot{>}\; \text{SNR}^0,$$

then it can be shown that $\Pr\left\{\|\underline{w}_1\|^2 \;\dot{>}\; \text{SNR}^0\right\} \doteq 0$. Hence

$$\begin{aligned}
\Pr\{\mathcal{E}_{1,1}\} \dot{\leq} \Pr\left\{ \|\underline{w}_1\|^2 \dot{\leq} \text{SNR}^0 \,\middle|\, \min_{d_{11}\neq 0} \|\kappa \underline{h}_1 d_{11}\|^2 \dot{\leq} \text{SNR}^0 \right\} \times& \\
\Pr\left\{ \min_{d_{11}\neq 0} \|\kappa \underline{h}_1 d_{11}\|^2 \dot{\leq} \text{SNR}^0 \right\}& \\
\leq \Pr\left\{ \min_{d_{11}\neq 0} \|\kappa \underline{h}_1 d_{11}\|^2 \dot{\leq} \text{SNR}^0 \right\}& \\
\overset{(a)}{\leq} \Pr\left\{ \|\kappa \underline{h}_1\|^2 \dot{\leq} \text{SNR}^0 \right\}& \\
\doteq \text{SNR}^{-(2-2r)^+} \;=\; \text{SNR}^{-d^*_{1,2}(r)},&
\end{aligned}$$

where (a) follows from $0 \neq d_{11} \in \mathbb{Z}[\imath]$ and $\|d_{11}\|^2 \geq 1$. The notation $(x)^+$ is defined as $(x)^+ = \max\{x, 0\}$. Thus, we conclude $\Pr\{\mathcal{E}_1\} \leq 4\Pr\{\mathcal{E}_{1,1}\} \dot{\leq} \text{SNR}^{-d^*_{1,2}(r)}$.

8.2 Type-II Error Event

The second-type is the event when only the messages from exactly one of the two users are erroneously decoded in both channel uses, i.e. the case when s_{i1} and s_{i2} are both erroneously decoded for $i = 1$ or 2. Clearly for this specific code $\mathcal{S}$ we have

$$\Pr\{\mathcal{E}_2\} \;\leq\; 4\Pr\{\mathcal{E}_1\} \;\dot{\leq}\; \text{SNR}^{-d^*_{1,2}(r)}. \tag{8.7}$$

The factor of 4 comes from that the probability of both s_{11} and s_{12} are erroneously decoded is at most twice of $\Pr\{\mathcal{E}_1\}$ and the same holds for s_{21} and s_{22} in error.

The previous two types of error events concerns the case when only one user is in error. The remaining ones will deal with situations when both users are in error.

8.3 Type-III Error Event

The third-type error event is the case when both users are in error but only the one of the two transmissions is erroneously decoded. Again, without loss of generality, we focus on the case when the first transmission is erroneously decoded, i.e. it is of the following form:

$$\mathcal{E}_{3,1} := \left\{ S - S' = \kappa \begin{bmatrix} d_{11} & 0 \\ d_{21} & 0 \end{bmatrix} : 0 \neq d_{i1} = s_{i1} - s'_{i1} \right\}. \tag{8.8}$$

and we have $\Pr\{\mathcal{E}_3\} \ \leq \ 2\Pr\{\mathcal{E}_{3,1}\}$.

We remark that in this case the difference matrix $\Delta S = S - S'$ is of rank 1 and does not satisfy the full NVD criterion given in [16] (cf. (1.8)). Furthermore, if one applies the conventional mismatched bound on product of eigenvalues [38], which is subsequently used as a key ingredient for proving the DMT optimality of CDA-based codes [4], the resulting bound on the DMT of present event $\mathcal{E}_{3,1}$ would be too loose to become any useful. Thus, below we will use a novel technique to analyze the DMT performance of this case.

Let $\underline{s}_1 = [s_{11}\ s_{21}]^\top$, $\underline{s}'_1 = [s'_{11}\ s'_{21}]^\top$, $s_{i1} \neq s'_{i1} \in \mathcal{A}(\text{SNR})$, where by $\underline{a}^\top$ we mean the usual transpose of vector $\underline{a}$. Set $\underline{d}_1 = \underline{s}_1 - \underline{s}'_1$. Then from the pairwise error probability analysis [1,24], the probability of erroneously decoding $\underline{s}_1$ as $\underline{s}'_1$ is given by

$$\Pr\{\underline{s}_1 \to \underline{s}'_1\} \ \doteq \ \left[1 + \kappa^2 \left\|\underline{d}_1\right\|^2\right]^{-2}. \tag{8.9}$$

Fixing $\underline{s}_1$, we see that the number of $\underline{s}'_1$ such that $\left\|\underline{d}_1\right\|^2 \dot{\leq} \text{SNR}^z$ for some $0 \leq z \leq r$ can be upper bounded by

$$\left|\left\{ \underline{s}'_1 \ : \ \left\|\underline{s}_1 - \underline{s}'_1\right\|^2 \ \dot{\leq} \ \text{SNR}^z \right\}\right| \ \leq \ \text{SNR}^{2z} \tag{8.10}$$

due to the choice of $\mathcal{A}(\text{SNR})$. The exponent $2z$ comes from the independent choices of s'_{11} and s'_{21}. Now from the union bound we see

$$\begin{aligned} \Pr\{\mathcal{E}_{3,1}\} &\leq \sum_{\underline{s}'_1 : \mathcal{E}_3} \Pr\{\underline{s}_1 \to \underline{s}'_1\} \\ &\doteq \sup_{0 \leq z \leq r} \left[1 + \kappa^2 \text{SNR}^z\right]^{-2} \text{SNR}^{2z} \\ &\doteq \kappa^{-4} \ = \ \text{SNR}^{-(2-2r)^+} \ = \ \text{SNR}^{-d^*_{1,2}(r)}. \end{aligned}$$

Thus we conclude that the diversity gain achieved by $\mathcal{S}$ in $\mathcal{E}_3$ equals $d^*_{1,2}(r)$.

8.4 Type-IV Error Event

The fourth error event concerns the case when the messages from both users are erroneously decoded in both transmissions, but the difference matrix has only rank 1. That is, $\mathcal{E}_4$ can be formulated as

$$\mathcal{E}_4 := \left\{ S - S' = \kappa \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} : \begin{array}{l} 0 \neq d_{ij} = s_{ij} - s'_{ij}, \\ \text{rank}(S - S') = 1 \end{array} \right\}. \tag{8.11}$$

The conditions of $d_{ij} \neq 0$ for all $i, j = 1, 2$ and $\text{rank}(S - S') = 1$ distinguish this case from the remaining one, the type-V error event, the case when $\text{rank}(S - S') = 2$. Specifically, if $d_{ij} = 0$ for only one pair of i and j, then the difference matrix would have rank equal to 2. The same also applies to the cases of $d_{12} = d_{21} = 0$ or $d_{11} = d_{22} = 0$. On the other hand, if $d_{11} = d_{21} = 0$ then it is equivalent to $\mathcal{E}_3$. Similarly, the case of $d_{11} = d_{12} = 0$ reduces to type-II.

Analyzing the probability of $\mathcal{E}_4$ might be the most troublesome as neither the weighted pairwise error probability technique used in analyzing type-II nor the conventional techniques [4] used for analyzing the DMT performance of CDA-based codes would work in this case. Furthermore, same as $\mathcal{E}_3$, this event again belongs to the situation when both users are in error but the difference matrix is singular, a situation violating the full NVD design criterion (1.8).

Nevertheless, following the same bounded distance argument as in type-I it can be shown that event $\mathcal{E}_4$ would occur if the noise matrix W has norm larger than half the minimum Euclidean distance $\min_{S\neq S':\mathcal{E}_4}\|H(S-S')\|$. Next, let $\underline{d}_1 = [d_{11}\ d_{21}]^\top$ and $\underline{d}_2 = [d_{12}\ d_{22}]^\top$; then note

$$\begin{aligned}\min_{S\neq S':\mathcal{E}_4}\|H(S-S')\|^2 &= \min_{\underline{d}_1,\underline{d}_2:\mathcal{E}_4}\left[\|\kappa H\underline{d}_1\|^2 + \|\kappa H\underline{d}_2\|^2\right]\\ &\geq \min_{\underline{d}_1,\underline{d}_2:\mathcal{E}_4}\|\kappa H\underline{d}_1\|^2 .\end{aligned}$$

Thus, we see

$$\begin{aligned}\Pr\{\mathcal{E}_4\} &\dot{\leq} \Pr\left\{H : \min_{\underline{d}_1,\underline{d}_2:\mathcal{E}_4}\left[\|\kappa H\underline{d}_1\|^2 + \|\kappa H\underline{d}_2\|^2\right] \dot{\leq}\text{SNR}^0\right\}\\ &\leq \Pr\left\{H : \min_{\underline{d}_1,\underline{d}_2:\mathcal{E}_4}\|\kappa H\underline{d}_1\|^2 \dot{\leq}\text{SNR}^0\right\}\\ &\doteq \Pr\{\mathcal{E}_{3,1}\} \ \dot{=}\!\leq\ \text{SNR}^{-d^*_{1,2}(r)}.\end{aligned}$$

Remark 12. *Another quick-and-dirty way to show the above is to note the relation between events $\mathcal{E}_3$ and $\mathcal{E}_4$, and it can be seen that*

$$\Pr\{\mathcal{E}_4\} \leq 1-(1-\Pr\{\mathcal{E}_{3,1}\})^2 \leq 2\Pr\{\mathcal{E}_{3,1}\} \dot{\leq} \text{SNR}^{-d^*_{1,2}(r)}.$$

The reason for this method being dirty is that the error probability calculation does not capture the fact that the channel remains static for two consecutive channel uses. It relies rather on the ergodicity of channel variation. □

8.5 Type-V Error Event

Finally, the last error event addresses the case when both users are in error but the difference matrix is of full rank, i.e. it is of the following form:

$$\mathcal{E}_5 := \left\{S-S' = \kappa\begin{bmatrix} d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix} : \text{rank}(S-S')=2\right\}. \tag{8.12}$$

Analyzing the probability of $\mathcal{E}_5$ is relatively easy since the matrix

$$D \ = \ \begin{bmatrix} d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix}$$

has determinant in $\mathbb{Z}[\imath]$. Therefore, the code satisfies the full NVD criterion in $\mathcal{E}_5$. It can be shown along similar lines as in [4] that

$$\begin{aligned}\Pr\{\mathcal{E}_5\} &\dot{\leq} \Pr\left\{\log\det\left(I_2 + \text{SNR}HH^\dagger\right) \leq 2r\log\text{SNR}\right\}\\ &\doteq \text{SNR}^{-d^*_{2,2}(2r)}.\end{aligned}$$

Overall, we have proved the following result.

Theorem 16. *The error probability of the simple code $\mathcal{S}$ is*

$$\Pr\{\mathcal{E}\} \ \leq \ \sum_{i=1}^{5}\Pr\{\mathcal{E}_i\} \doteq \max\left\{\text{SNR}^{-d^*_{1,2}(r)}, \text{SNR}^{-d^*_{2,2}(2r)}\right\},$$

and the diversity gain is

$$d(r) \ = \ \min\left\{d^*_{1,2}(r), d^*_{2,2}(2r)\right\}.$$

Hence the simple code $\mathcal{S}$ is MAC-DMT optimal. □

8.6 Further Extension

In this chapter, we have answered all the four open questions posed in Chapter 1.3. We showed it is possible to achieve the optimal MAC-DMT with $T < Kn_t + n_r - 1$, and previously known full NVD design criterion for MAC-DMT optimal codes [16] is only sufficient. A simple code not satisfying this full NVD criterion is provided, and we proved it is still MAC-DMT optimal. In view of this, we have provided an alternative, yet much more relaxed, criterion for constructing MAC-DMT optimal codes. This simple code is also modified for use in point-to-point MIMO channels. We showed the modified code is optimal in DMT in the sense that it achieves the same DMT performance as the Gaussian random coding schemes.

Below we state without proof a generalization of the results in this report.

Theorem 17. *Consider a MIMO-MAC channel with n users, each having $n_t = 1$ transmit antenna and transmitting at multiplexing gain r. Assume there are n receive antennas at receiver. Then the following overall code*

$$\mathcal{S}_n = \left\{ \kappa \begin{bmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{n1} & \cdots & s_{nn} \end{bmatrix} : s_{ij} \in \mathcal{A}(\mathrm{SNR}) \right\}$$

achieves the optimal MAC-DMT $d^*_{1,n,n}(r)$ with $T = n$ channel uses, where κ and $\mathcal{A}(\mathrm{SNR})$ are defined as before (cf. Chapter 8). Furthermore, the same result holds for the vector code $\mathcal{S}_{n,vec}$ obtained by taking the first column of code matrices in $\mathcal{S}_n$. Hence $d^*_{1,n,n}(r)$ holds for $T = n_t = 1$ as well. Finally, by setting the multiplexing gain at $\frac{r}{n}$ in $\mathcal{S}_{n,vec}$ the resulting code achieves DMT $d(r) = n - r$ in the point-to-point MIMO channel with $n_t = n_r = n$ and $T = 1$. It is the same DMT performance achieved by Gaussian random coding schemes. $\square$

Chapter 9

Inferences from DMT Analysis of Code $\mathcal{S}$

In this section we will take a closer look at the results presented in the previous section and then address the four open questions posed in Chapter 1.3.

9.1 Alternative Design Criterion for MAC-DMT Optimal Codes

Recall that among the five types of error events analyzed in the previous section, only $\mathcal{E}_4$ and $\mathcal{E}_5$ belong to the case when both users are in error. However, it was proved that $\mathcal{E}_4$ achieves diversity gain $d^*_{1,2}(r)$ rather than $d^*_{2,2}(2r)$, which was required by the design criterion (1.8). We also note that events $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, and $\mathcal{E}_4$ all correspond to the case when the difference matrix $(S - S')$ is of rank 1, and all achieve the same diversity order $d^*_{1,2}(r)$. Thus, below we summarize this observation and provide an alternative, yet much relaxed, design criterion for constructing MAC-DMT optimal codes.

Theorem 18 (Relaxed Design Criterion)**.** *In a MIMO-MAC channel with K users, each having n_t transmit antennas and transmitting at multiplexing gain r, let $\mathcal{S}_i$ be the space-time code of the ith user, and let $\mathcal{S} = \mathcal{S}_1 \times \cdots \mathcal{S}_K$ be the overall code obtained by vertically concatenating the code matrices from all users. Let $\mathcal{E}_{n,m}$ denote the error event that n users are in error but the difference matrix $(S - S')$ has only rank mn_t with $1 \leq m \leq n$. If for all m and n*

$$\Pr\{\mathcal{E}_{n,m}\} \ \leq \ \text{SNR}^{-d^*_{mn_t,n_r}(mr)}, \tag{9.1}$$

then $\mathcal{S}$ is optimal in MAC-DMT. □

The above design criterion is weaker than (1.8) since (1.8) excludes the possibility of having error event $\mathcal{E}_{n,m}$ when $m < n$, that is, (1.8) requires whenever n users are in error, the difference matrix must be of rank nn_t. Codes satisfying (9.1) must be MAC-DMT optimal since it follows from [11] that $\text{SNR}^{-d^*_{mn_t,n_r}(mr)} \leq \text{SNR}^{-d^*_{n_t,n_r}(r)}$ and $\text{SNR}^{-d^*_{mn_t,n_r}(mr)} \leq \text{SNR}^{-d^*_{Kn_t,n_r}(Kr)}$ for any $1 \leq m \leq K$. Hence the events $\mathcal{E}_{n,m}$ with $1 \leq m < n \leq K$ are not dominant in the union bound, and the requirement on the error performance of these error events can be much relaxed without worsening the overall DMT performance.

Thus, in this section we have answered the second and the third questions posed in Chapter I. We showed that criterion (1.8) is only sufficient, not necessary, and it is unnecessary to design codes to meet the full NVD criterion. Moreover, we have provided in Theorem 18 an alternative, yet much relaxed, code design criterion for constructing MAC-DMT optimal codes.

9.2 Requirement on Minimal Channel Coherence Time

In Theorem 1 it was shown that in the MIMO-MAC channel with $K = 2$ users, $n_t = 1$ and $n_r = 2$, the MAC-DMT $d^*_{n_t,n_r,K}(r)$ holds whenever the channel remains fixed for $T \geq Kn_t + n_r - 1 = 3$

channel uses. The requirement on T was improved by the simple code $\mathcal{S}$ analyzed in Chapter 8. We proved that $\mathcal{S}$ achieves the same MAC-DMT optimality with only $T = 2$ channel uses, and hence improves the result on minimal channel coherence time required by Theorem 1. In particular, we note that in this specific channel we actually have

$$d^*_{1,2}(r) \ \leq \ d^*_{2,2}(2r), \text{ for all } 0 \leq r \leq 1.$$

In other words, the single-user performance dominates the entire region of $r \in [0, 1]$, and there is no region of antenna-pooling [11] in this case.

From the analyses presented in the previous section, we can further strengthen the MAC-DMT result to the following. The vector code

$$\mathcal{S}_{\text{vec}} \ = \ \left\{ \kappa \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \ : \ s_i \in \mathcal{A}(\text{SNR}) \right\} \tag{9.2}$$

that is a subcode of $\mathcal{S}$ given in Chapter 8 and is obtained by taking only the first column of code matrices in $\mathcal{S}$, is in fact MAC-DMT optimal. To see this, from the error events $\mathcal{E}_1$ and $\mathcal{E}_3$, the error probability of $\mathcal{S}_{\text{vec}}$ is

$$\Pr\left\{\mathcal{E}\left(\mathcal{S}_{\text{vec}}\right)\right\} \ \leq \ \Pr\left\{\mathcal{E}_1\right\} + \Pr\left\{\mathcal{E}_3\right\} \dot{\leq} \text{SNR}^{-(2-2r)^+}. \tag{9.3}$$

It then implies that the MAC-DMT $d^*_{1,2,2}(r)$ holds even for fast fading channel, i.e. the case when $T = 1$. This answers the first question posed in Chapter 1.3.

9.3 Point-to-Point MIMO Channel

The vector code $\mathcal{S}_{\text{vec}}$ of (9.2) can be easily modified for use in a point-to-point MIMO channel. To this end, let

$$\mathcal{A}_{SU}(\text{SNR}) = \left\{ a + b\,\imath \ : \ |a|, |b| \leq \text{SNR}^{\frac{r}{4}}, \ a, b \text{ odd} \right\} \tag{9.4}$$

and set

$$\mathcal{S}_{\text{SU}} \ = \ \left\{ \kappa_{SU} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \ : \ s_i \in \mathcal{A}_{SU}(\text{SNR}) \right\} \tag{9.5}$$

where $\kappa^2_{SU} = \text{SNR}^{1-\frac{r}{2}}$. In other words, $\mathcal{S}_{\text{SU}}$ can be obtained from $\mathcal{S}_{\text{vec}}$ when both users transmit at multiplexing gain $\frac{r}{2}$ such that the overall multiplexing gain achieved by $\mathcal{S}_{\text{SU}}$ equals r. Because of this, the error probability of $\mathcal{S}_{\text{SU}}$ is upper bounded by

$$\Pr\left\{\mathcal{E}\left(\mathcal{S}_{\text{SU}}\right)\right\} \ \leq \ \text{SNR}^{-(2-r)^+}, \tag{9.6}$$

and the diversity gain equals $d(r) = 2 - r$ for $r \in [0, 2]$. The maximal multiplexing gain $r_{\max} = 2$, same as that indicated by the ergodic channel capacity [9, 23] of this channel. However, the resulting DMT is only $d(r) = 2-r$, much worse than the optimal DMT $d^*_{2,2}(r)$. It is understandable since the latter requires the channel to be fixed for at least two channel uses, while the former changes from one channel use to another, and there is no coding across independent channel uses.

The maximal diversity gain $d_{\max}$ achieves by $\mathcal{S}_{\text{SU}}$ is given by $d(0) = 2$, which is the same for any such vector codes. This can be easily seen from the pairwise error probability argument. Taking any fixed vector coding schemes that do not vary with SNR, the resulting multiplexing gain equals 0 and the maximal possible rank distance between any pairs of distinct code vectors equals 1. Hence the resulting diversity order is 2 since there are two receive antennas. Therefore, we conclude that for $T < n_t$ the maximal diversity order is $n_r T$, and the resulting DMT can never be the same as the optimal one $d^*_{n_t,n_r}(r)$, where the maximal diversity order equals $n_t n_r$. Furthermore, it means that the outage event does not dominate the error performance when $T < n_t$. These answer the fourth question posed in Chapter 1.3.

While the code $\mathcal{S}_{\text{SU}}$ is not optimal in terms of $d^*_{2,2}(r)$, in [9] Zheng and Tse proved the following result.

Theorem 19 ([9]). *For a point-to-point MIMO channel with n_t transmit antennas, n_r receive antennas, and $T < n_t + n_r - 1$, the Gaussian random coding scheme achieves the following DMT:*

$$d_G(r) = \inf_{\underline{\alpha} \in \mathcal{G}} \left\{ \left[\sum_{i=1}^{M} (2i - 1 + |n_t - n_r|)\, \alpha_i \right] + T \left(\sum_{i=1}^{M} (1 - \alpha_i) - r \right) \right\}, \tag{9.7}$$

where $M = \min\{n_t, n_r\}$, $\underline{\alpha} = [\alpha_1 \ \cdots \ \alpha_M]^\top$, and the constraint $\mathcal{G}$ is given by

$$\mathcal{G} := \left\{ \underline{\alpha} \in [0,1]^M : \alpha_1 \geq \cdots \geq \alpha_M, \sum_{i=1}^{M} (1 - \alpha_i) > r \right\}.$$

□

Substituting $n_t = n_r = 2$ and $T = 1$ into (9.7) gives

$$d_G(r) = \inf_{\underline{\alpha} \in \mathcal{G}} \{2 - r + 2\alpha_2\} = 2 - r.$$

Thus, we see that the DMT achieved by Gaussian random coding scheme is the same as that achieved by the deterministic code $\mathcal{S}_{\mathrm{SU}}$. Hence $\mathcal{S}_{\mathrm{SU}}$ is DMT optimal in the case of $T = 1$.

Finally, we remark that the well-known Alamouti scheme of orthogonal space-time codes [2] was shown to achieve DMT at $d_A(r) = 4(1 - r)^+$ for $r \in [0, 2]$ by Zheng and Tse [9]. Thus we see for multiplexing gain $r \geq \frac{2}{3}$ the Alamouti code would perform worse than the uncoded $\mathcal{S}_{\mathrm{SU}}$ in the DMT sense.

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