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A Note on Integrability in Matrix Models

M. H. Tu [1], J. C. Shaw[2], and H. C. Yen[1†]

[1] Department of Physics, National Tsing Hua University, Hsinchu, Taiwan 300, R.O.C.

[2] Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan 300, R.O.C.

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Based on the bilinear relation approach, we show that the hermitian one-matrix model at finite N can be embedded into 1d Toda chain and (modified) KP hierarchy respectively with the same Wronskian-type tau function. We also point out that which hierarchy to take depends on the choice of the wave function for constructing the bilinear relation.

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I. Introduction

Recently it has been noticed that the integrability of the matrix models [1] is maintained even at discrete level (finite N) before taking the double scaling limit. Progress in this direction is originally due to three independent papers, by Gerasimov et al. [2], by Martinec [3], and by Alvarez-Gaumé et al. [4]. In particular, the Lax pair and zero-curvature condition have been derived and the underlying integrable system has been identified with 1d Toda lattice hierarchy (TLH) for one-matrix model which becomes the KdV hierarchy in the scaling limit [6], 2d TLH and 2d Toda multi-component hierarchy [5] for the two-matrix model and for the general multi-matrix models, respectively. The partition functions are the r-functions of these integrable systems.

In Ref. [8], we have shown that the hermitian one-matrix model can also be embedded into KP hierarchy which is different from the previous result [2] for 1d TLH. However, the partition function turns out to be a particular Wronskian-type tau function for both hierarchies. Therefore, it seems that there are some relationships between these two hierarchies through matrix model. In the present letter, we will show that this is indeed the case. Inspired by previous studies on twomatrix model and 2d TLH [7], we show that these two hierarchies (1d Toda and (modified) KP) can be related to each other via a bilinear relation which is constructed from matrix model, thus connecting these two integrable systems in hermitian one-matrix model.

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II. Integrability in one-matrix model

Let us briefly summarize the integrability structures in hermitian one-matrix model. The main object of interest in the one-matrix model is the partition function, defined to be

$$Z_N^{(1)} \sim \int \mathrm{d}M \exp\left[\mathrm{Tr} \sum_{k=1}^{\infty} t_k M^k\right], \tag{1}$$

with M an N x N hermitian matrix, which can be reduced to [9-11]

$$Z_N^{(1)} = \frac{1}{N!} \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \exp\left[\sum_{i=1}^{N}\sum_{k=1}^{\infty} t_k \lambda_i^k\right] \prod_{i>j}^{N} (\lambda_i - \lambda_j)^2. \tag{2}$$

If we introduce a set of orthogonal polynomials $\{P_n(\lambda)\}$, defined by [10-12]

$$P_n(\lambda) = \lambda^n + O(\lambda^{n-1}), \tag{3}$$

$$\int \mathrm{d}\lambda e^{V(\lambda)} P_n(\lambda) P_m(\lambda) = h_n \delta_{nm}, \quad n, m = 0, 1, 2, \cdots \tag{4}$$

where $V(t,\lambda) \equiv \sum_{k=1}^{\infty} t_k \lambda^k$, then $Z_N^{(1)}$ in (2) can be evaluated to become

$$Z_N^{(1)} = h_{N-1} h_{N-2} \cdots h_0 \tag{5}$$

From (3) and (4), it can be shown [2] that

$$\lambda P_n(\lambda) = \sum_{m=0}^{\infty} \gamma_{nm} P_m(\lambda). \tag{6}$$

where the elements of the matrix γ are given by

$$\begin{aligned} &\gamma_{n,n+1} = 1, \quad \gamma_{nn} = \frac{\partial \ln h_n}{\partial t,}, \\ &\gamma_{n,n-1} = \frac{h_n}{h_{n-1}}, \quad \text{all other } \gamma_{nm} = 0. \end{aligned} \tag{7}$$

Thus,

$$P_{n+1}(\lambda) = (\lambda - \gamma_{nn}) P_n(\lambda) - \gamma_{n,n-1} P_{n-1}(\lambda) \tag{8}$$

is a characterizing recursion relation of the orthogonal polynomials. It is easy to show [8] that the orthogonal polynomials can be expressed as

$$P_n(\lambda) = \det[\lambda 1 - \gamma]_{n \times n}. \tag{9}$$

By differentiating (4) one can derive [2]

$$\frac{\partial \ln h_n}{\partial t_q} = (\gamma^q)_{nn}, \tag{10}$$

and

$$\frac{\partial P_n(\lambda)}{\partial t_q} = -\sum_{m=0}^{n-1} (\gamma^q)_{nm} P_m(\lambda). \tag{11}$$

Then combining (6) and (11), one obtains [2]

$$\frac{\partial \gamma}{\partial t_q} = [\beta_q, \gamma], \quad \beta_q \equiv (\gamma^q)_+. \tag{12}$$

where we define $A_+(A_-)$ is the upper (strictly lower) triangular part of A such that $A_+ + A_- = A$.

Eq. (12) is of the form of a standard Lax equation for integrable hierarchy systems [13,14], since it implies the Zakharov-Sabat equations (or the zero-curvature conditions):

$$\frac{\partial \beta_p}{\partial t_q} - \frac{\partial \beta_q}{\partial t_p} + [\beta_p, \beta_q] = 0. \tag{13}$$

Moreover, it seems that the hermitian one-matrix model at finite N has the same integrability structure as the 1d TLH, because it follows from (7) and (12) that

$$\frac{\partial^2 \ln h_n}{\partial t_1^2} = \frac{h_{n+1}}{h_n} - \frac{h_n}{h_{n-1}}, \tag{14}$$

and if we let $h_n(t) = e^{u_n(t)}$ then (14) can be rewritten as

$$\frac{\partial^2 u_n(t)}{\partial t_1^2} = e^{u_{n+1}(t)-u_n(t)} - e^{u_n(t)-u_{n-1}(t)} \tag{15}$$

which is nothing but the one-dimensional Toda lattice equation, which describes a nonlinear coupled oscillator with exponential type potential, and also the simplest equation among the 1d TLH.

III. Wronskian expressions for partition function

Using (5), the normalization constant h_n can be expressed in terms of the partition function by

$$h_n(t) = \frac{Z_{n+1}^{(1)}(t)}{Z_n^{(1)}(t)}, \tag{16}$$

then (14) can be reduced to

$$h_n^2 \left\{ Z_n^{(1)} \frac{\partial^2 Z_n^{(1)}}{\partial t_1^2} - \left(\frac{\partial Z_n^{(1)}}{\partial t_1} \right)^2 - Z_{n+1}^{(1)} Z_{n-1}^{(1)} \right\} = \left\{ Z_{n+1}^{(1)} \frac{\partial^2 Z_{n+1}^{(1)}}{\partial t_1^2} - \left(\frac{\partial Z_{n+1}^{(1)}}{\partial t_1} \right)^2 - Z_{n+2}^{(1)} Z_n^{(1)} \right\}. \quad (17)$$

If we impose the boundary conditions

$$Z_0^{(1)} = 1, \quad Z_n^{(1)} = 0 \text{ for } n < 0, \quad (18)$$

then by iteration, we get

$$\left\{ Z_n^{(1)} \frac{\partial^2 Z_n^{(1)}}{\partial t_1^2} - \left(\frac{\partial Z_n^{(1)}}{\partial t_1} \right)^2 - Z_{n+1}^{(1)} Z_{n-1}^{(1)} \right\} = 0, \quad (n \geq 1) \quad (19)$$

and we find that the solution of Eq. (19) turns out to be a Wronskian

$$\begin{aligned} Z_n^{(1)} &= W(Z_1^{(1)}, \partial_1 Z_1^{(1)}, \cdots, \partial_1^{(n-1)} Z_1^{(1)}) \\ &= \det \partial_1^{i+j} Z_1^{(1)}, \quad (0 \leq i, j \leq n-1). \end{aligned} \quad (20)$$

Therefore, the finite N matrix model can ultimately be expressed in terms of one single function, $Z_1^{(1)}$ (or $h_0(t)$). Note that (19) can be rewritten as

$$\frac{1}{2} D_1^2 Z_n^{(1)} \cdot Z_n^{(1)} = Z_{n+1}^{(1)} Z_{n-1}^{(1)} \quad (21)$$

where D_1 is the Hirota's bilinear differential operator defined by

$$\begin{aligned} P(D_x) f(x) \cdot g(x) &= P\left(\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_1'} \right), \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_2'} \right), \cdots \right) f(x) g(x')|_{x=x'} \\ &= P(\partial_z)(f(x+z) g(x-z)|_{z=0} \end{aligned} \quad (22)$$

It is well known that all the equations of the hierarchy share the same solution called r-function, and can be expressed in terms of the r-function through the Hirota bilinear differential operators [14]. So (21) indicates that the partition function may be a r-function of the truncated one-dimensional TLH. [1]

IV. The partition function as a tau-function

Now let us define

$$\Psi_n(t, \lambda | a) \equiv P_n(t, \lambda) e^{aV(t,\lambda)}, \quad 0 \leq a \leq 1. \quad (23)$$

[1] Because the discrete time variables t_n are semi-infinite.

which is different from [8] by introducing a parameter a which will play a crucial role later on. From (11) we have

$$\begin{aligned}\frac{\partial\Psi}{\partial t_q} &= [(a-1)\gamma^q+\beta_q]\Psi \\ &= [a\gamma_+^q + (a-1)\gamma_-^q]\Psi\end{aligned} \tag{24}$$

Furthermore, we have from (6)

$$\gamma\Psi = \lambda\Psi. \tag{25}$$

Then, the compatibility conditions for (24) and (25) give

$$\begin{aligned}\frac{\partial\gamma}{\partial t_q} &= [(a-1)\gamma^q+\beta_q,\gamma] \\ &= [\beta_q,\gamma]\end{aligned}$$

which is independent of a.

Now we define a sort of adjoint of $\Psi_n(\lambda)$ by (compare with (23)) [8]

$$\Psi_m^*(t',\lambda|a)\equiv P_m^*(t',\lambda)e^{-aV(t',\lambda)} \tag{26}$$

where $P_m^*(t,\lambda)\equiv\lambda^{-m}\det[(1-\gamma/\lambda)^{-1}]_{m\times m}$ (compare with (9)). Then the following bilinear relation holds for all t and $t', n=0,1,2,\cdots,m=1,2,3,\cdots$.

$$\begin{aligned}&\oint\frac{d\lambda}{2\pi i}\Psi_n(t,\lambda|a)\Psi_m^*(t',\lambda|a) \\ &= \frac{h_n(t)}{h_{m-1}(t')}\oint\frac{d\lambda}{2\pi i}\Psi_{n+1}^*(t,\lambda|a)\Psi_{m-1}(t',\lambda|a)e^{(2a-1)V(t-t',\lambda)},\end{aligned} \tag{27}$$

here we take an integration contour as a small circle around $\lambda=\infty$. A detailed calculation to arrived at (27) is shown in appendix A. The bilinear relation (27) is a generalization of the previous result [8]. We also note that (27) is an one-dimensional reduction of the bilinear relation for the 2d TLH [13] such that $\tau_n(x,y)$ only depend on $\{x_i-y_i\}\equiv\{t_i\}$ where $\{x_i\}$ and $\{y_i\}$ are two time flows in 2d TLH. In fact, the parameter a defined in (23) can be realized from the following identifications

$$\{x_i\}=\{at_i\},\quad \{y_i\}=\{(a-1)t_i\},\quad (0\le a\le 1). \tag{28}$$

Applying a theorem [13] in the theory of the 2d TLH, (27) ensures that there exists a τ-function, called $\tau_n(x,y)=\tau_n(x-y)=\tau_n(t)$, such that

$$\begin{aligned}P_n(t,\lambda) &= \lambda^n\frac{\tau_n(x-\epsilon(\lambda^{-1}),y)}{\tau_n(x,y)} = \lambda^n\frac{\tau_n(t-\epsilon(\lambda^{-1}))}{\tau_n(t)} \\ P_n^*(t,\lambda) &= \lambda^{-n}\frac{\tau_n(x+\epsilon(\lambda^{-1}),y)}{\tau_n(x,y)} = \lambda^{-n}\frac{\tau_n(t+\epsilon(\lambda^{-1}))}{\tau_n(t)} \\ \lambda h_n P_{n+1}^*(t,\lambda) &= \lambda^{-n}\frac{\tau_{n+1}(x,y-\epsilon(\lambda^{-1}))}{\tau_n(x,y)} = \lambda^{-n}\frac{\tau_{n+1}(t+\epsilon(\lambda^{-1}))}{\tau_n(t)} \\ \lambda h_{n-1}^{-1}P_{n-1}(t,\lambda) &= \lambda^n\frac{\tau_{n-1}(x,y+\epsilon(\lambda^{-1}))}{\tau_n(x,y)} = \lambda^n\frac{\tau_{n-1}(t-\epsilon(\lambda^{-1}))}{\tau_n(t)}\end{aligned} \tag{29}$$

where $\epsilon(\lambda^{-1}) \equiv (\frac{1}{\lambda}, \frac{1}{2\lambda^2}, \cdots)$. Then from (29), it is easy to show that

$$h_n(t) = \frac{\tau_{n+1}(t)}{\tau_n(t)} \tag{30}$$

or

$$\tau_n(t) = h_{n-1}h_{n-2}\cdots h_0 = Z_n^{(1)}(t). \tag{31}$$

Hence, the partition function is a r-function of the one-dimensional reduction of the 2d TLH.

V. Toda **and (modified)** KP hierarchy

Now let us derive the integrable hierarchies from (27). Substituting (29) into (27), the bilinear relation can be expressed as

$$\begin{aligned}&\oint \frac{d\lambda}{2\pi i}\lambda^{n-m}\tau_n(t-\epsilon(\lambda^{-1}))\tau_m(t'+\epsilon(\lambda^{-1}))e^{aV(t-t',\lambda)}\\ &\quad= \oint \frac{d\lambda}{2\pi i}\lambda^{m-n-2}\tau_{n+1}(t+\epsilon(\lambda^{-1}))\tau_{m-1}(t'-\epsilon(\lambda^{-1}))e^{(a-1)V(t-t',\lambda)}.\end{aligned} \tag{32}$$

Now we can derive the bilinear differential equations of the Hirota-type satisfied by $\tau_n(t)$ Let us consider the l.h.s. of (32) firstly.

Under a change of variables:

$$t \to x - y, \quad t' \to x + y, \tag{33}$$

the l.h.s. of (32) becomes

$$\text{Res}\left(\lambda^{n-m}e^{V(-2ay,\lambda)}e^{V(\tilde{\partial}_y,\lambda^{-1})}(\tau_n(x-y)\tau_m(x+y))\right). \tag{34}$$

Let $p_j(x)(j=0,1,\cdots)$ be a polynomial introduced through

$$e^{V(x,\lambda)} = \sum_{j=0}^{\infty} p_j(x)\lambda^j. \tag{35}$$

More explicitly,

$$p_j(x) = \sum_{||\alpha||=j} \frac{x^\alpha}{\alpha!} \tag{36}$$

where we use the abbreviations

$$||\alpha|| = \sum_i i\alpha(i), \; x^\alpha = \prod_i x_i^{\alpha(i)}, \; \alpha! = \prod_i \alpha(i)!. \tag{37}$$

Now Eq. (34) can be cast into the form

$$\text{Res}\left(\lambda^{n-m}(\sum_l \lambda^l p_l(-2ay))\sum_i \lambda^{-i}p_i(\tilde{\partial}_y)(\tau_n(x-y)\tau_m(x+Y))\right)$$
$$= \sum_\gamma y^\gamma \left(\sum_{\alpha+\beta=\gamma} \frac{(-2a)^{|\alpha|}}{\alpha!\beta!} p_{||\alpha||+n-m+1}(\tilde{D})D^\beta\right) \tau_m(x) . \tau_n(x) \tag{38}$$

where $\tilde{D}_x = (D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \cdots)$ and $|\alpha| = \Sigma_i \alpha(i)$.

Similarly, for the r.h.s. of Eq. (32), the final form read

$$\sum_\gamma y^\gamma \left(\sum_{\alpha+\beta=\gamma} \frac{(-2(a-1))^{|\alpha|}}{\alpha!\beta!} p_{||\alpha||+m-n-1}(-\tilde{D})D^\beta\right) \tau_{m-1}(x)\cdot\tau_{n+1}(x). \tag{39}$$

Thus, for each multi-index 7 = $(\gamma(1), \gamma(2), \cdots)$ andn'ides (m, n), there is a corresponding bilinear differential equation for r-function:

$$\left(\sum_{\alpha+\beta=\gamma} \frac{(-2a)^{|\alpha|}}{\alpha!\beta!} p_{||\alpha||+n-m+1}(\tilde{D})D^\beta\right) \tau_m(x)\cdot\tau_n(x)$$
$$= \left(\sum_{\alpha+\beta=\gamma} \frac{(-2(a-1))^{|\alpha|}}{\alpha!\beta!} p_{||\alpha||+m-n-1}(-\tilde{D})D^\beta\right) \tau_{m-1}(x)\cdot\tau_{n+1}(x). \tag{40}$$

There are two different cases to be discussed:

Case I: 0 < a < 1. In this case, the first few equations of (40) for the case m = n become

$$D_1\tau_n\cdot\tau_n = 0, \quad \gamma = (0,0,0,\cdots) \tag{41}$$

$$\frac{1}{2}D_1^2\tau_n \cdot\tau_n = \tau_{n-1}\tau_{n+1}, \quad 7 = (1,0,0,\cdots) \tag{42}$$

$$\frac{1}{2}D_1D_2\tau_n\cdot\tau_n = -D_1(\tau_{n-1}\cdot\tau_{n+1}), \; 7 = (0,1,0,\cdots) \tag{43}$$

where (41) is an identity and Eqs. (42) and (43) are just the Lax equations (12) for $q = 1$ and 2 respectively. In particular, Eq. (42) is the 1d Toda chain equation. This means that the bilinear relation (27) or (32), for 0 < a < 1, characterizes the 1d TLH. Thus, the hermitian one-matrix model can be embedded into 1d TLH [2-4].

Case II: a = 1 (or a = 0). In this case, the r.h.s. (or l.h.s.) of Eq. (32) is equal to zero for the case n $\geq m$, i.e.

$$\oint \frac{d\lambda}{2\pi i}\lambda^{n-m}\tau_n(t-\epsilon(\lambda^{-1}))\tau_m(t'+\epsilon(\lambda^{-1}))e^{V(t-t',\lambda)} = 0. \tag{44}$$

These are sometimes called the (n − m)-th modified KP hierarchy [17]. If we set n = m, then Eq. (44) reduce to

$$\left(\sum_{\alpha+\beta=\gamma}\frac{(-2)^{|\alpha|}}{\alpha!\beta!}p_{||\alpha||+1}(\tilde{D})D^{\beta}\right)\tau_n(x)\cdot\tau_n(x)=0. \tag{45}$$

The whole system of non-linear equations (45) is termed the original KP hierarchy (for n fixed). For instance the coefficient of y_1^4 in (45) gives an equation

$$(D_1^4-4D_1D_3+3D_2^2)\tau_n\cdot\tau_n=0,\quad \gamma=(4,0,0,\cdots) \tag{46}$$

or in terms of the dependent variable $u(t_1,t_2,t_3)=2\partial_1^2\ln\tau_n(t_1,\ t_2,\ t_3,0,0,\cdots)$

$$3\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial}{\partial x_1}\left(-4\frac{\partial u}{\partial x_3}+6u\frac{\partial u}{\partial x_1}+\frac{\partial^3 u}{\partial x_1^3}\right)=0 \tag{47}$$

which is a typical example of $2+1$ dimensional soliton equations, known as the Kadomtsev-Petviashvili (KP) equation [14] in the ordinary form. Thus, for each n, the partition function of the hermitian one-matrix model can be identified with a particular Wronskian-type τ function of the (modified) KP hierarchy [8].

VI. Conclusion and discussions

In conclusion, several remarks are in order:

(a) We have shown that the hermitian one-matrix model can be embedded into Toda and KP hierarchy respectively with the same Wronskian-type r-function with the properties

$$\tau_n=W(\tau_1,\partial_1\tau_1,\cdots,\partial_1^{(n-1)}\tau_1) \tag{48}$$

and

$$\partial_{t_i}\tau_1=(\partial_{t_1})^i\tau_1. \tag{49}$$

We also point out that which hierarchy to take depends on the choice of the wave function for constructing the bilinear relation. In fact, Hirota et al. [16] have shown that the tau function of the KP hierarchy and those of the 2d TLH admit common Wronskian structure. Therefore one-matrix model provids another example to illustrate this result.

(b) In our approach, the (modified) KP equation which is a differential equation emerges naturally from the bilinear relation but difficult to obtain from the formulation developed in sec. 2. That is a reason why original approach only obtain the Toda lattice hierarchy. On the other hand, we must point out that the size of the Wronskian (r-function) relates to the lattice site itself for case I (Toda lattice hierarchy) and to the number of solitons for case II (KP hierarchy).

(c) In the continuum limit, it has been known that the Virasoro constraints $L_n\tau=0(n\geq 0)$ are derived recursively from the string equation, which is equivalent to the lowest order constraint $L_{-1}\tau=0$ in addition to the KdV (2-reduced KP) flow equation [18,19]. But, there is no proof the same situation for the case of finite N. However a recent paper by Yoneya [20] has shown that this issues can be formulated by using the bilinear relation of the KP hierarchy in the continuum limit. In this situation, it seems advantageous to extend the framework to include the finite N case [21].

Appendix A: Proof of Eq. (27)

From (4) and (6), one can derive a general representation for $(\gamma^k)_{nn}$,

$$\left[\gamma(t')^k\right]_{nn} = \int dy e^{V(t',y)} P_n(t',y) y^k P_m(t',y)/h_m(t'). \tag{50}$$

By repeatedly using (7), we can verify that [8]

$$P_m^*(t',\lambda) = \sum_{k=0}^{\infty} \lambda^{-k-m} \int dy e^{V(t',y)} y^{k+m-1} P_{m-1}(t',y)/h_{m-1}(t'). \tag{51}$$

Using (23) and (51), the l.h.s. of (27) becomes

$$\sum_{k=0}^{\infty} \oint \frac{d\lambda}{2\pi i} P_n(t,\lambda)\lambda^{-k-m} e^{aV(t-t',\lambda)} \cdot \int dy e^{V(t',y)} y^{k+m-1} P_{m-1}(t',y)/h_{m-1}(t'). \tag{52}$$

Expanding $e^{aV(t-t',\lambda)}$ into the power series of λ, and picking out all λ^{-1} terms from each $P_n(t,\lambda)\lambda^{-k-m}e^{aV(t-t',\lambda)}$, we find (52) can eventually be transformed into

$$\begin{aligned} &\int dy e^{V(t',y)} \frac{P_{m-1}(t',y)}{h_{m-1}(t')} \left\{ P_n(t,y) e^{aV(t,y)-aV(t',y)} \right. \\ &\qquad \left. -(\text{polynomial in } y \text{ of order } m-2) \right\} \end{aligned} \tag{53}$$

which immediately reduces to

$$\int dy P_{m-1}(t',y) P_n(t,y) e^{aV(t,y)+(1-a)V(t',y)}/h_{m-1}(t'). \tag{54}$$

On the other hand, we have

$$\begin{aligned} \text{r.h.s of (27)} &= \frac{h_n(t)}{h_{m-1}(t')} \int \frac{d\lambda}{2\pi i} P_{n+1}^*(t,\lambda) P_{m-1}(t',\lambda) e^{(a-1)V(t-t',\lambda)} \\ &= \frac{h_n(t)}{h_{m-1}(t')} \int dy e^{V(t,y)} P_n(t,y) P_{m-1}(t',\lambda) e^{(a-1)V(t-t',\lambda)}/h_n(t) \\ &= \text{l.h.s.} \end{aligned} \tag{55}$$

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