行政院國家科學委員會補助專題研究計畫
■ 成果報告　　　　　　□ 期中進度報告

(組合設計及其應用 (3/3))

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計畫主持人：翁志文

共同主持人：無

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行政院國家科學委員會補助專題研究計畫結案報告

計畫編號：NSC 99-2115-M-009-005-MY3　計畫類別：個別型計畫
執行期間：　99 年　8 月 1 日至　102 年　7　月 31 日

計畫名稱: 具組合設計及其應用 (3/3)

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102 年 8 月 5 日

一、中文摘要

計畫提案人先前與黃大原教授於文獻 [T. Huang, C. Weng, Pooling spaces and non-adaptive pooling designs, Discrete Math. 282 (2004) 163–169] 中提出一種稱為池空間的有秩偏序結構, 並舉許多例子. 在此成果報告中, 我們提出池半格的結構, 也舉出相當多對應的例子, 每一個例子都能用來建構一類與群試設計有關的池設計.

關鍵詞: 池設計, 半格, 池半格, 正規池半格, 池空間

二、英文摘要

In [T. Huang, C. Weng, Pooling spaces and non-adaptive pooling designs, Discrete Math. 282 (2004) 163–169], Huang and Weng introduced pooling spaces, and constructed pooling designs from a pooling space. In this report, we introduce the concept of pooling semilattices and prove that a pooling semilattice is a pooling space, then show how to construct pooling designs from a pooling semilattice. Moreover, we give many examples of pooling semilattices and thus obtain the corresponding pooling designs.

Keywords. Pooling design, semilattice, pooling semilattice, regular pooling semilattice, pooling space

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三、緣由與目的

The basic problem of group testing is to identify the set of defective items in a large population of items. A group testing algorithm is *non-adaptive* if all tests must be specified without knowing the outcomes of other tests. A group test is applicable to an arbitrary subset of items with two possible outcomes: a negative outcome indicates that all items in the subset are negative, and a positive outcome indicates otherwise. A pooling design is a specification of all tests such that they can be performed simultaneously with the goal being to identify all positive items with a small number of tests [3]. A non-adaptive pooling design is usually represented by a binary matrix with columns indexed with items and rows indexed with pools. A cell (i, j) contains a 1-entry if and only if the ith pool contains the jth item. By treating a column as a set of row indices intersecting the column with a 1-entry, we can talk about the union of several columns. A binary matrix is s^e-*disjunct* if every column has at least $e+1$ 1-entries not contained in the union of any other s columns [15]. An s^0-disjunct matrix is also called s-disjunct. An s^e-disjunct matrix is called *fully s^e-disjunct* if it is neither $(s+1)^e$-disjunct nor s^{e+1}-disjunct. An s^e-disjunct matrix is $\lfloor e/2 \rfloor$-error-correcting [5, 11].

Macula [14] proposed a novel way of constructing disjunct matrices by means of the containment relation of subsets in a finite set. D'yachkov et al. [5] discussed the error-correcting capability of Macula's designs. Ngo and Du [17] constructed a family of disjunct matrices by means of the containment relation of subspaces in a finite vector space. D'yachkov et al. [4] discussed the error-tolerance capability of Ngo-Du's designs. In [7, 8], Jun Guo and Kaishun Wang proposed a new model for pooling designs–the intersection type incidence construction, and generalized Macula's and Ngo-Du's designs. Under this model, the pooling designs have surprisingly high degree of error correction. Huang and Weng [11] generalized the containment matrix construction of pooling designs to pooling spaces.

Let $(X, \preceq)$ be a finite partially ordered set (poset) with the least element 0. For $x, y \in X$, if $x \preceq y$, we say that y *contains* x. Moreover, if there does not exist element z such that $x \prec z \prec y$, we say that y *covers* x. An *atom* in X is an element in X that covers 0. The poset X is *ranked* and has *rank function*, if there is a function ℓ from X to the integer set such that $\ell(0) = 0$ and $\ell(y) = \ell(x) + 1$ if y covers x. The maximum value of $\ell(x)$ is called the *rank* of X, denoted by N. The *fibers* (or *levels*) $X_0, X_1, \ldots, X_N$ of the poset are the subsets of X given by $X_i = \{x \in X \mid \ell(x) = i\}$. Pick any $x, y \in X$ such that $x \preceq y$. By the *interval* $[x, y]$, we mean the subposet $[x, y] := \{z \in X \mid x \preceq z \preceq y\}$ of X. A ranked poset X is called *atomic* whenever each element $x \in X \setminus \{0\}$ is the least upper bound of the set $[0, x] \cap X_1$. A *pooling space* is a finite poset $(X, \preceq)$ such that the subposet induced on $w^+ = \{w \preceq y \mid y \in X\}$ is atomic for each $w \in X$. Huang and Weng [11] showed that how to construct pooling designs from pooling spaces.

Theorem 1. ([11]) *Let X be a pooling space with rank $N \geq 1$. For $1 \leq d \leq k \leq N$, let $M(k, N)$ be the binary matrix with rows indexed with X_k and columns indexed with X_N such that $M(x, y) = 1$ if and only if $x \preceq y$. Then $M(k, N)$ is d^e-disjunct, where*

$$e = \min |\cup([y, x] \cap X_k)| - 1,$$

the minimum is taken over all pairs (x, T) with $T \subseteq X_N, |T| \leq d$ and $x \in X_N \setminus T$; the union is taken over all $y \in [0, x] \cap X_d$ such that $y \not\preceq z$ for all $z \in T$.

Let $(X, \preceq)$ be a finite poset with with the rank function ℓ and fibers $X_0, \ldots, X_N$. We call X a *semilattice*, if any two elements x and y of X have the greatest lower bound, denoted by $x \wedge y$. As usual, we denote by $x \vee y$ the least upper bound of x and y if it exists. Note that if X is a semilattice and $x, y \in X$ have a common upper bound, then $x \vee y$ exists; indeed $x \vee y$ is the greatest lower bound of the set of upper bounds of x and y. X is a *lattice* if $x \vee y$ exists for any $x, y \in X$.

Let X denote a semilattice with the rank function ℓ and fibers $X_0, \ldots, X_N$. We are concerned with the following axioms:

(A1) For $u \in X_r$ and $z \in X_t$ with $u \prec z$, the number $|[u, z] \cap X_{r+1}|$ is a constant $\mu(r, r+1, t)$, where $0 \leq r < t \leq N$. Moreover, the function $\mu(0, 1, t)$ is strictly increasing about t, i.e. $1 = \mu(0, 1, 1) < \mu(0, 1, 2) < \cdots < \mu(0, 1, N)$.

(A2) For $x, y \in X$, if $x \vee y$ exists, then $\ell(x \vee y) \leq \ell(x) + \ell(y) - \ell(x \wedge y)$.

(A3) For $x, y \in X$, if $x \vee y$ exists, then $\ell(x \vee y) = \ell(x) + \ell(y) - \ell(x \wedge y)$.

We call X a *pooling semilattice*, if it satisfies (A1) and (A2). We call X a *regular pooling semilattice*, if it satisfies (A1) and (A3). Note that (A3) implies (A2) and thus a regular pooling semilattice is a pooling semilattice. In addition if X is a lattice, we use lattice to replace the above semilattice. We call X a *geometric lattice* if X is a finite atomic lattice and satisfies (A2).

四、結果與討論

In this project, we mainly focus on the construction of pooling designs from a pooling semilattice. We first discuss some properties of pooling semilattices, then show how to construct pooling designs from a pooling semilattice. We give many families of examples of pooling semilattices. They fall into three categories: (i) regular pooling semilattices from sets, vector spaces and maps; (ii) non-regular pooling semilattices from affine spaces; and (iii) pooling semilattices from distance-regular graphs. We generalize the intersection type incidence construction to pooling lattices and give four families of examples of pooling lattices. All these results will be appeared in an article coauthored with Jun Guo and Kaishun Wang. See 附錄 below for details.

五、附錄

1 Pooling semilattices

In this section, we always assume that X denotes a pooling semilattice with the rank function ℓ and fibers $X_0, \ldots, X_N$.

A poset can be described by a diagram in the plane in which y covers x if and only if there is a line moving upwards from x to y.

Lemma 2. *A pooling semilattice X is atomic.*

Proof. Pick any element $w \in X \setminus \{0\}$. Suppose that u is the least upper bound of the set $[0, w] \cap X_1$. Then $u \preceq w$ and $\mu(0, 1, \ell(w)) \leq \mu(0, 1, \ell(u))$. By (A1), one gets $\ell(w) = \ell(u)$ and $u = w$, as desired. □ □

The usage of the term "pooling semilattice" is justified by the following proposition.

Proposition 3. *Let X be a pooling semilattice with rank N. Then X is a pooling space. In particular, for each $1 \leq r < N$, the function $\mu(r, r+1, t)$ is strictly increasing about t where $r + 1 \leq t \leq N$.*

Proof. Let $w \in X_r$ be given. We shall prove that the subposet w^+ is atomic. Pick any $x \in w^+$. Then $[0, x]$ is a semilattice. By Lemma 2, $[0, x]$ is a atomic and hence is a geometric lattice. It is well-known that an interval in a geometric lattice is a geometric lattice [12, Page 307], [9, Lemma 5.2]. Hence $[w, x]$ is geometric. Theorem 5.4 in [9] tells us that a geometric lattice is a pooling space, which implies that $[w, x]$ is a pooling space. In particular x is the least upper bound of $[w, x] \cap X_{r+1}$. This proves the first statement. Fix $u_1 \in X_{t-1}$ and $u_2 \in X_t$ with $w \prec u_1 \prec u_2$, Since u_1 (resp. u_2) is the least bound of the $\ell(r, r+1, t-1)$ (resp. $\ell(r, r+1, t)$) elements in $[w, u_1] \cap X_{r+1}$ (resp. $[w, u_2] \cap X_{r+1}$), we have $\ell(r, r+1, t-1) < \ell(r, r+1, t)$ to conclude the second statement. □ □

The following lemma says that the local assumption of (A1) can imply a global property on X.

Lemma 4. *Let X be a pooling semilattice and $0 \le r \le s \le t \le N$. Then for $u \in X_r$ and $z \in X_t$ with $u \preceq z$, the number $\mu(r, s, t) := |[u, z] \cap X_s|$ is a constant. Moreover for given r, s with $0 \le r \le s \le N$, the function $\mu(r, s, t)$ is strictly increasing about t, where $s \le t \le N$.*

Proof. Note that $\mu(r, r, t) = \mu(r, t, t) = 1$, $\mu(r, r+1, t)$ is a constant by (A1) and $\mu(r, r+1, t-1) < \mu(r, r+1, t)$ by Proposition 3, where $r + 1 \le t \le N$. We prove the lemma by induction on $t - r$, and assume in the nontrivial situation $r + 2 \le s \le t - 1$. Fix $u \in X_r$ and $z \in X_t$ with $u \prec z$. Counting pairs $(v, w) \in X_{r+1} \times X_s$ with $u \prec v \prec w \prec z$ in two ways yields a constant

$$\mu(r, s, t) = \mu(r, r + 1, t)\mu(r + 1, s, t)/\mu(r, r + 1, s) \tag{1.1}$$

by induction. Also $\mu(r, s, t-1) = \mu(r, r+1, t-1)\mu(r+1, s, t-1)/\mu(r, r+1, s) < \mu(r, r+1, t)\mu(r+1, s, t)/\mu(r, r+1, s) = \mu(r, s, t)$ since $\mu(r, r + 1, t - 1) < \mu(r, r + 1, t)$ by Proposition 3, and $\mu(r + 1, s, t - 1) < \mu(r + 1, s, t)$ by induction. □ □

Lemma 5. *Let X be a pooling semilattice and $1 \le s < t \le N$. Then the function $\mu(r, s, t)$ is strictly decreasing about r, where $1 \le r < s$, i.e. $\mu(1, s, t) > \mu(2, s, t) > \cdots > \mu(s - 1, s, t) > \mu(s, s, t) = 1$.*

Proof. As the above two-way counting argument,

$$\frac{\mu(r, s, t)}{\mu(r + 1, s, t)} = \frac{\mu(r, r + 1, t)}{\mu(r, r + 1, s)} > 1$$

by Lemma 4. □ □

Definition 1. Suppose that X is a pooling semilattice. For positive integers $1 \le d < k < N$, let $M(d, k; N)$ be the binary matrix with rows indexed with X_d and columns indexed with X_k such that $M(x, y) = 1$ if and only if $x \preceq y$.

Theorem 6. *Let X be a pooling semilattice. Then the following results hold.*

(i) *If $1 \le s \le d$, then $M(d, k; N)$ is an s^e-disjunct matrix, where $e = \mu(s, d, k) - 1$.*

(ii) *If $1 \le s < \mu(0, d, k)/\mu(0, d, k - 1)$, then $M(d, k; N)$ is an s^e-disjunct matrix, where $e = \mu(0, d, k) - s\mu(0, d, k - 1) - 1$.*

Proof. (i) Let $y_0, y_1, \ldots, y_s$ be any $s + 1$ distinct columns of $M(d, k; N)$. Note that $\ell(y_0 \wedge y_j) \le k - 1$ for each $j \in \{1, 2, \ldots, s\}$. By (A1) we have $\mu(0, 1, k) > \mu(0, 1, \ell(y_0 \wedge y_j))$, which implies that there exists some $a_j \in X_1$ such that $a_j \preceq y_0$ but $a_j \not\preceq y_j$ for each $j \in \{1, 2, \ldots, s\}$. Since y_0 is a common upper bound of $a_1, a_2, \ldots, a_s$, the least upper bound of these elements exists. Suppose that $x_0 = a_1 \vee a_2 \vee \cdots \vee a_s$. Then $x_0 \preceq y_0$ and $x_0 \not\preceq y_j$ for each $j \in \{1, 2, \ldots, s\}$. By (A2) we have $1 \le \ell(x_0) \le s$. By Lemma 4, the size of $X_d \cap [x_0, y_0]$ is $\mu(\ell(x_0), d, k)$. From Lemma 5, we deduce that $\mu(\ell(x_0), d, k)$ is decreasing for $1 \le \ell(x_0) \le s$ and gets its minimum at $\ell(x_0) = s$, which implies that the size of $X_d \cap [x_0, y_0]$ is at least $\mu(s, d, k)$, as desired.

(ii) Let $y_0, y_1, \ldots, y_s$ be any $s + 1$ distinct columns of $M(d, k; N)$. Note that y_0 contains $\mu(0, d, k)$ many elements in X_d and $\ell(y_0 \wedge y_j) \le k - 1$ for each $j \in \{1, 2, \ldots, s\}$. By Lemma 4, each $y_0 \wedge y_j$ contains at most

$\mu(0,d,k-1)$ elements in X_d. Thus, the number of elements in X_d contained in y_0 but not in y_j for each $j \in \{1,2,\dots,s\}$ is at least $\mu(0,d,k) - s\mu(0,d,k-1)$, as desired. □ □

Theorem 7. *Let X be a regular pooling semilattice. If $k-d \geq 2$ and $1 \leq s \leq (\mu(0,d,k)-\mu(0,d,k-1))/(\mu(0,d,k-1)-\mu(0,d,k-2))$, then $M(d,k;N)$ is an s^e-disjunct matrix, where $e = \mu(0,d,k) - s\mu(0,d,k-1) + (s-1)\mu(0,d,k-2) - 1$. In particular, if $s \leq \min\{\mu(k-2,k-1,k), (\mu(0,d,k)-\mu(0,d,k-1))/(\mu(0,d,k-1)-\mu(0,d,k-2))\}$ and $|x^+ \cap X_k| > 1$ for any $x \in X_{k-1}$, then $M(d,k;N)$ is fully s^e-disjunct.*

Proof. Let $y_0, y_1, \dots, y_s$ be any $s+1$ distinct columns of $M(d,k;N)$. Note that y_0 contains $\mu(0,d,k)$ many elements in X_d and $\ell(y_0 \wedge y_j) \leq k-1$ for each $j \in \{1,2,\dots,s\}$. To obtain the maximum elements with rank d in $\bigcup_{j=1}^{s}\{x \in X_d \mid x \preceq (y_0 \wedge y_j)\}$, by Lemma 4 we may assume that $y_0 \wedge y_1, \dots, y_0 \wedge y_s$ are s distinct elements in X_{k-1}. Then the element $y_0 \wedge y_1$ contains at most $\mu(0,d,k-1)$ elements in X_d. Since $(y_0 \wedge y_1) \vee (y_0 \wedge y_j) \preceq y_0$ and $\ell(y_0 \wedge y_1 \wedge y_j) \leq k-2$ for each $j \in \{2,\dots,s\}$, by (A3) $\ell((y_0 \wedge y_1) \vee (y_0 \wedge y_j)) = k$ and $\ell(y_0 \wedge y_1 \wedge y_j) = k-2$. By Lemma 4, each of $y_0 \wedge y_2, \dots, y_0 \wedge y_s$ can contain at most $\mu(0,d,k-1)-\mu(0,d,k-2)$ elements in X_d not contained in $y_0 \wedge y_1$. Thus, the number of elements in X_d contained in y_0 but not in y_j for each $j \in \{1,2,\dots,s\}$ is at least $\mu(0,d,k) - \mu(0,d,k-1) - (s-1)(\mu(0,d,k-1) - \mu(0,d,k-2))$. Hence $M(d,k;N)$ is s^e-disjunct.

Let $s \leq \min\{\mu(k-2,k-1,k), (\mu(0,d,k)-\mu(0,d,k-1))/(\mu(0,d,k-1)-\mu(0,d,k-2))\}$ and $|x^+ \cap X_k| > 1$ for any $x \in X_{k-1}$. We show that $M(d,k;N)$ is fully s^e-disjunct. Let $u \in X_{k-2}$ with $u \preceq y_0$. By Lemma 4 the number of elements $x \in X_{k-1}$ such that $u \preceq x \preceq y_0$ is $\mu(k-2,k-1,k)$, and so we can choose s distinct ones among them, say x_j $(1 \leq j \leq s)$. Since $|x_j^+ \cap X_k| > 1$ we can choose pairwise distinct y_j in $X_k \setminus \{y_0\}$ such that $x_j = y_0 \wedge y_j$. Then the number of elements in X_d contained in y_0 but not in y_j for each $j \in \{1,2,\dots,s\}$ is $e+1$. Therefore $M(d,k;N)$ is not s^{e+1}-disjunct. View the function $e = e(s) = \mu(0,d,k) - s\mu(0,d,k-1) + (s-1)\mu(0,d,k-2) - 1$ as a function of s, and notice that $e(s+1) - e(s) = \mu(0,d,k-2) - \mu(0,d,k-1) < 0$. The above argument with $s+1$ to replace s implies that $M(d,k;N)$ is not $(s+1)^e$-disjunct. □ □

2 Examples

In this section we give many examples of pooling semilattices, and give their parameters. By Theorems 6 and 7, we can construct pooling designs from these pooling semilattices.

Let q be a positive integer. Fix a positive integer n. The *Gaussian binomial coefficients with basis q* is defined by

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = \begin{cases} \prod_{j=0}^{i-1} \frac{n-j}{i-j} & \text{if } q = 1, \\ \prod_{j=0}^{i-1} \frac{q^n - q^j}{q^i - q^j} & \text{if } q \neq 1. \end{cases}$$

In the case $q = 1$, for convenience, we write $\binom{n}{i}$ instead of $\begin{bmatrix} n \\ i \end{bmatrix}_1$.

2.1 Regular pooling semilattices from sets, vector spaces and maps

In this subsection we give thirteen families of regular pooling semilattices with rank N.

Example 1. ([5, 14] The Boolean Algebra). Let X be the collection of all subsets of $[N] := \{1, 2, \ldots, N\}$. Ordered by inclusion, X is a regular pooling semilattice with the rank function $\ell(x) = |x|$ and the parameters

$$|X_r| = \binom{N}{r}, \quad \mu(r, s, t) = \binom{t-r}{s-r}.$$

Example 2. ([4, 17] The Projective Geometry). Let $\mathbb{F}_q^N$ be the N-dimensional vector space over the finite field $\mathbb{F}_q$ and X be the collection of all subspaces of $\mathbb{F}_q^N$. Ordered by inclusion, X is a regular pooling semilattice with the rank function $\ell(x) = \dim x$ and the parameters

$$|X_r| = \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \mu(r, s, t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

Example 3. ([11] The Attenuated Space). For fixed positive integers n and N, let w be a fixed n-dimensional subspace of $\mathbb{F}_q^{n+N}$. Let X be the collection of all subspaces x of $\mathbb{F}_q^{n+N}$ with $x \cap w = \{0\}$. Ordered by inclusion, X is a regular pooling semilattice with the rank function $\ell(x) = \dim x$ and the parameters

$$|X_r| = q^{rn} \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \mu(r, s, t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

Example 4. ([10] The Classical Polar Space). Classical finite *polar spaces* are incidence structures, consisting of all the totally isotropic subspaces of $\mathbb{F}_q^n$ with respect to a certain non-degenerate sesquilinear or quadratic form f. The *rank* of the polar space is the algebraic dimension of the maximal totally isotropic subspaces, denoted by N. The summary is given in the following table:

name	n	form	$\lvert X_r\rvert$
$[C_N(q)]$	$2N$	symplectic	$\left[{N \atop r}\right]_q \prod_{i=0}^{r-1}(q^{N-i}+1)$
$[B_N(q)]$	$2N+1$	quadratic	$\left[{N \atop r}\right]_q \prod_{i=0}^{r-1}(q^{N-i}+1)$
$[D_N(q)]$	$2N$	quadratic (with rank N)	$\left[{N \atop r}\right]_q \prod_{i=0}^{r-1}(q^{N-i-1}+1)$
$[^2D_{N+1}(q)]$	$2N+2$	quadratic (with rank N)	$\left[{N \atop r}\right]_q \prod_{i=0}^{r-1}(q^{N-i+1}+1)$
$[^2A_{2N}(r)]$	$2N+1$	Hermitian ($q = r^2$)	$\left[{N \atop r}\right]_q \prod_{i=0}^{r-1}(q^{N-i+1/2}+1)$
$[^2A_{2N-1}(r)]$	$2N$	Hermitian ($q = r^2$)	$\left[{N \atop r}\right]_q \prod_{i=0}^{r-1}(q^{N-i-1/2}+1)$

Let X be the collection of all totally isotropic subspaces of $\mathbb{F}_q^n$. Ordered by inclusion, X is a regular pooling semilattice with the rank function $\ell(x) = \dim x$ and the parameters

$$|X_r| = \begin{bmatrix} N \\ r \end{bmatrix}_q \prod_{i=0}^{r-1}(q^{N+e-i-1}+1), \quad \mu(r, s, t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

where $e = 1, 1, 0, 2, 3/2, 1/2$ according to $[C_N(q)], [B_N(q)], [D_N(q)], [^2D_{N+1}(q)], [^2A_{2N}(r)]$, $[^2A_{2N-1}(r)]$, respectively.

For fixed positive integers n and m, let w be an l-dimensional subspace of $\mathbb{F}_q^{n+m}$, denote also by w an $l \times (n+m)$ matrix of rank l whose rows span the subspace w and call the matrix w a matrix representation of the subspace w.

Example 5. (The Attenuated Classical Polar Space). For fixed positive integers n and m, let $\mathbb{F}_q^n$ be the classical polar space with rank N as in Example 4 and $w = (0^{(m,n)}\, I^{(m)})$. Then the quotient space $\mathbb{F}_q^{n+m}/w$ is isomorphic to $\mathbb{F}_q^n$. Let X be the collection of all subspaces $x = (x_1\, x_2)$ of $\mathbb{F}_q^{n+m}$ with $x \cap w = \{0\}$, where x_1 is a totally isotropic subspace of $\mathbb{F}_q^n$ and x_2 is a matrix. Ordered by inclusion, X is a regular pooling semilattice with the rank function $\ell(x) = \dim x$ and the parameters

$$|X_r| = q^{rm} \begin{bmatrix} N \\ r \end{bmatrix}_q \prod_{i=0}^{r-1} (q^{N+e-i-1} + 1), \quad \mu(r,s,t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

where e as in Example 4.

Example 6. (The Map). Let X be the collection of all pairs (w, f), where w is a subset of $[N] := \{1, 2, \ldots, N\}$ and $f : w \to [N]$ is a map. Ordered by inclusion, that is $(w, f) \preceq (u, g)$ if $w \subseteq u$ and $g|_w = f$, X is a regular pooling semilattice with the rank function $\ell(w, f) = |w|$ and the parameters

$$|X_r| = N^r \binom{N}{r}, \quad \mu(r,s,t) = \binom{t-r}{s-r}.$$

Example 7. (The Injective Map). Let X be the collection of all pairs (w, f), where w is a subset of $[N]$ and $f : w \to [N]$ is an injective map. Ordered by inclusion, X is a regular pooling semilattice with the rank function $\ell(w, f) = |w|$ and the parameters

$$|X_r| = \binom{N}{r} N(N-1)\cdots(N-r+1), \quad \mu(r,s,t) = \binom{t-r}{s-r}.$$

Example 8. (The Bilinear Form). Let X be the collection of all pair (w, f), where w is a subspace of $\mathbb{F}_q^N$ and $f : w \to \mathbb{F}_q^N$ is a linear map. Ordered by inclusion, X is a strongly pooling semilattice with the rank function $\ell(w, f) = \dim w$ and the parameters

$$|X_r| = q^{rN} \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \mu(r,s,t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

Example 9. (The Injective Linear Map). Let X be the collection of all pair (w, f), where w is a subspace of $\mathbb{F}_q^N$ and $f : w \to \mathbb{F}_q^N$ is an injective linear map. Ordered by inclusion, X is a regular pooling semilattice with the rank function $\ell(w, f) = \dim w$ and the parameters

$$|X_r| = q^{r(r-1)/2} \prod_{i=N-r+1}^{N} (q^i - 1) \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \mu(r,s,t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

Example 10. (The Square Bilinear Form). Let X be the collection of all pair (w, f), where w is a subspace of $\mathbb{F}_q^N$ and $f : w \to w$ is a bilinear form on w. Ordered by inclusion, X is a regular pooling semilattice with the rank function $\ell(w, f) = \dim w$ and the parameters

$$|X_r| = q^{r^2} \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \mu(r,s,t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

Example 11. (The Alternating Form). Let X be the collection of all pair (w, f), where w is a subspace of $\mathbb{F}_q^N$ and $f : w \to w$ is an alternating bilinear form on w. Ordered by inclusion, X is a regular pooling semilattice with the rank function $\ell(w, f) = \dim w$ and the parameters

$$|X_r| = q^{r(r-1)/2} \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \mu(r,s,t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

Example 12. (The Hermitian Form). Let X be the collection of all pair (w, f), where w is a subspace of $\mathbb{F}_q^N$ and $f : w \to w$ is a Hermitian form on w, where $q = r^2$ is square. Ordered by inclusion, X is a regular pooling semilattice with the rank function $\ell(w, f) = \dim w$ and the parameters

$$|X_r| = q^{r^2/2} \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \mu(r, s, t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

Example 13. (The Symmetric Bilinear Form). Let X be the collection of all pair (w, f), where w is a subspace of $\mathbb{F}_q^N$ and $f : w \to w$ is a symmetric bilinear form on w. Ordered by inclusion, X is a regular pooling semilattice with the rank function $\ell(w, f) = \dim w$ and the parameters

$$|X_r| = q^{r(r+1)/2} \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \mu(r, s, t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

2.2 Pooling semilattices from affine spaces

In this subsection we give four families of examples of non-regular pooling semilattices with rank $N + 1$. These examples are from an affine space.

Example 14. ([9, 10] The Affine Geometry). Let $\mathbb{F}_q^N$ and X be as in Example 2. Let X' be the collection of all cosets of subspaces in X together with the empty set $\emptyset$. We define $\ell(\emptyset) = 0$. Ordered by inclusion, X' is a pooling semilattice with the rank function $\ell(x) = \dim x + 1$ and the parameters

$$|X'_{r+1}| = q^{N-r} \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \mu(r+1, s+1, t+1) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

Example 15. (The Affine Attenuated Space). Let $\mathbb{F}_q^{n+N}$ and X be as in Example 3. Let X' be the collection of all cosets of subspaces in X together with the empty set $\emptyset$. Ordered by inclusion, X' is a pooling semilattice with the rank function $\ell(x) = \dim x + 1$ and the parameters

$$|X'_{r+1}| = q^{n+N+rn-r} \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \mu(r+1, s+1, t+1) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

Example 16. ([10] The Affine Classical Polar Space). Let $\mathbb{F}_q^n$ and X be as in Example 4. Let X' be the collection of all cosets of subspaces in X together with the empty set $\emptyset$. Ordered by inclusion, X' is a pooling semilattice with the rank function $\ell(x) = \dim x + 1$ and the parameters

$$|X'_{r+1}| = q^{2N+\delta-r} \begin{bmatrix} N \\ r \end{bmatrix}_q \prod_{i=0}^{r-1} (q^{N+e-i-1} + 1), \quad \mu(r+1, s+1, t+1) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q,$$

where $\delta = 0, 1, 0, 2, 1, 0$ according to $[C_N(q)], [B_N(q)], [D_N(q)], [^2D_{N+1}(q)], [^2A_{2N}(r)], [^2A_{2N-1}(r)]$, respectively, and e is as in Example 4.

Example 17. (The Affine Attenuated Classical Polar Space). Let $\mathbb{F}_q^{n+m}$ and X be as in Example 5. Let X' be the collection of all cosets of subspaces in X together with the empty set $\emptyset$. Ordered by inclusion, X' is a pooling semilattice with the rank function $\ell(x) = \dim x + 1$ and the parameters

$$|X'_{r+1}| = q^{2N+\delta+m+rm-r} \begin{bmatrix} N \\ r \end{bmatrix}_q \prod_{i=0}^{r-1} (q^{N+e-i-1} + 1), \quad \mu(r+1, s+1, t+1) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

2.3 Pooling semilattices from distance-regular graphs

In this subsection, we give four families of examples of pooling semilattices with rank N. These examples are from distance-regular graphs.

Let Γ be a connected regular graph. We identify Γ with the set of vertices. For two vertices u and v, let $\partial(u,v)$ denote the usual distance between u and v. The maximum value of the distance function in Γ is called the *diameter* of Γ, denoted by $D(\Gamma)$. For vertices u and v at distance i, define

$$\begin{aligned} C(u,v) &= C_i(u,v) = \{w \mid \partial(u,w) = i-1, \partial(w,v) = 1\}, \\ A(u,v) &= A_i(u,v) = \{w \mid \partial(u,w) = i, \partial(w,v) = 1\}. \end{aligned}$$

For the cardinalities of these sets we use lower case letters $c_i(u,v)$ and $a_i(u,v)$. A connected regular graph Γ with diameter D is called *distance-regular* if $c_i(u,v)$ and $a_i(u,v)$ depend only on i for all $1 \le i \le D$. The reader is referred to [2] for general theory of distance-regular graphs.

Let Γ be a distance-regular graph. A r-subset $\{x_1, x_2, \ldots, x_r\} \subseteq \Gamma$ is said to be a *t-clique* of Γ with size r if any two distinct vertices in $\{x_1, x_2, \ldots, x_r\}$ are at distance t.

Example 18. ([1, 20] The Johnson Graph). Let $N = \lfloor n/t \rfloor$ and X be the collection of all t-cliques of the Johnson graph $J(n,t)$ together with the empty set $\emptyset$. Ordered by inclusion, X is a regular pooling semilattice with the rank function $\ell(x) = |x|$ and the parameters

$$|X_r| = \binom{n}{rt}(rt)!/(t!)^r r!, \quad \mu(r,s,t) = \binom{t-r}{s-r}.$$

A distance-regular graph Γ with diameter $D \ge 2$ is said to be *antipodal*, if $\partial(x,y) = \partial(x,z) = D$ and $y \ne z$ implies $\partial(y,z) = D$. For $u \in \Gamma$, the size of the set $\{v \in \Gamma \mid \partial(u,v) = D\}$ depends only on D, denoted by k_D.

Example 19. ([1] The Antipodal Distance-Regular Graph). Suppose that Γ is an antipodal distance-regular graph with diameter D. Let $N = k_D + 1$ and X be the collection of all D-cliques of Γ together with the empty set $\emptyset$. Ordered by inclusion, X is a regular pooling semilattice with the rank function $\ell(x) = |x|$ and the parameters

$$|X_r| = \binom{k_D+1}{r}|\Gamma|/(k_D+1), \quad \mu(r,s,t) = \binom{t-r}{s-r}.$$

A distance-regular graph Γ is said to be of *order* (l,k) if, for each vertex $x \in \Gamma$, the induced subgraph on $\Gamma(x)$ is a disjoint union of $k+1$ cliques with size l. Then each maximal clique is of size $l+1$, and each vertex is contained in $k+1$ maximal cliques.

Example 20. ([1] The Distance-Regular Graph of Order (l,k)). Suppose that Γ is a distance-regular graph of order (l,k). Let $N = l+1$ and X be the collection of all cliques of Γ together with the empty set $\emptyset$. Ordered by inclusion, X is a regular pooling semilattice with the rank function $\ell(x) = |x|$ and the parameters

$$|X_r| = \binom{l+1}{r}n(k+1)/(l+1), \quad \mu(r,s,t) = \binom{t-r}{s-r}.$$

Recall that a subgraph induced on a subset Δ of Γ is called *strongly closed* if $C(u,v) \cup A(u,v) \subseteq \Delta$ for every pair of vertices $u, v \in \Delta$. A distance-regular graph Γ with diameter D is called *D-bounded*, if every strongly closed subgraph of Γ is regular, and any two vertices x and y are contained in a common strongly closed subgraph with diameter $\partial(x, y)$. A regular strongly closed subgraph of Γ is called a *subspace* of Γ. For any two subspaces Δ_1 and Δ_2 of Γ, $\Delta_1 + \Delta_2$ denotes the minimum subspace containing Δ_1 and Δ_2.

Proposition 8. ([6, Lemma 2.1]). *Let Γ be a D-bounded distance-regular graph with diameter $D \geq 2$. For $1 \leq i+1 \leq i+s \leq i+s+t \leq D$, suppose that Δ and Δ' are two subspaces satisfying $\Delta \subseteq \Delta'$, $D(\Delta) = i$ and $D(\Delta') = i+s+t$. Then the number of the subspaces with diameter $i+s$ containing Δ and contained in Δ', denoted by $N(i, i+s, i+s+t)$, is*

$$\frac{(b_i - b_{i+s+t})(b_{i+1} - b_{i+s+t}) \cdots (b_{i+s-1} - b_{i+s+t})}{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}.$$

Example 21. ([19] The D-Bounded Distance-Regular Graph). Let Γ be a D-bounded distance-regular graph with $D = N$. For $x \in \Gamma$, let X be the collection of all subspaces Δ containing x in Γ. Ordered by inclusion, X is a pooling semilattice with the rank function $\ell(\Delta) = D(\Delta)$ and the parameters

$$|X_r| = N(0, r, D), \quad \mu(r, s, t) = N(r, s, t).$$

In particular, if $D(\Delta_1)+D(\Delta_2) = D(\Delta_1+\Delta_2)+D(\Delta_1 \cap \Delta_2)$ for any $\Delta_1, \Delta_2 \in X$, then the pooling semilattice X is a regular pooling semilattice.

3 Pooling lattices

In this section, we show how to construct pooling designs from the pooling lattices by the intersection type incidence method.

Lemma 9. *Let X be a pooling lattice with rank N and $0 \leq r \leq s, t \leq N$. For $u \in X_r$, $x \in X_s$ with $u \preceq x$, the number of elements $z \in X_t$ such that $x \wedge z = u$ is a constant $\pi(r, s, t)$. Moreover, for given r and t, the function $\pi(r, s, t)$ is decreasing about s and is indeed strictly decreasing until its value is zero, i.e. $\pi(r, r, t) > \pi(r, r+1, t) > \cdots > \pi(r, p, t) > \pi(r, p+j, t) = 0$ for some $p \geq r$ and any $1 \leq j \leq N-p$.*

Proof. We prove the first statement by induction on $s - r$. The case $s - r = 0$ follows from Lemma 4 with $\pi(r, r, t) = |[u, 1] \cap X_t| = \mu(r, t, N)$, where the element 1 is the greatest element of X. Suppose $s - r \geq 1$. Choose any $u \in X_r$ and $x \in X_s$ with $u \preceq x$. Note that the set $u^+ \cap X_t$ is partitioned into $U_i = \{z \mid z \in u^+ \cap X_t, \ell(x \wedge z) = i\}$ for $r \leq i \leq s$. Since each element $z \in U_i$ has the greatest lower bound $x \wedge z \in [u, x] \cap X_i$, $|U_i| = \mu(r, i, s)\pi(i, s, t)$ by induction for $r < i$. Hence

$$\pi(r, s, t) = \mu(r, t, N) - \sum_{i=r+1}^{s} \mu(r, i, s)\pi(i, s, t) \tag{3.1}$$

is a constant, where $\pi(i, s, t) = 0$ if $i > t$. The first statement follows. Let $x_1 \in X_s$ and $x_2 \in X_{s+1}$ with $u \preceq x_1 \prec x_2$, where $r \leq s \leq N-1$. Then $\{z \in X_t \mid x_1 \wedge z = u\} \supseteq \{z \in X_t \mid x_2 \wedge z = u\}$, which implies that $\pi(r, s, t) \geq \pi(r, s+1, t)$. Choose the largest $p \leq N$ such that $\pi(r, p, t) > 0$, and restrict to $s \leq p-1$ in the above proof. Pick $y \in X_t$ with $x_1 \wedge y = u$, $a \in [u, y] \cap X_{r+1}$ and let $x_2 = x_1 \vee a$. Then $x_1 \wedge a = u$ and $a \preceq x_2 \wedge y$, which implies that $\ell(x_2) = s+1$. Hence $y \notin \{z \in X_t \mid x_2 \wedge z = u\}$. The second statement follows. $\square$ $\square$

Lemma 10. *Let X be a pooling lattice with rank N and $1 \leq r \leq s, t \leq N$. For $x \in X_s$, the number of elements $z \in X_t$ such that $\ell(x \wedge z) = r$ is $\mu(0, r, s)\pi(r, s, t)$.*

Proof. This is clear by Lemma 9, since for $x \in X_s$ the number $\mu(0, r, s)\pi(r, s, t) = |[0, x] \cap X_r|\pi(r, s, t)$ counts the desired z. □ □

Definition 2. Suppose that X is a pooling semilattice. For positive integers i, d, k, N with $1 \leq i \leq d < k < N$, let $M(i; d, k; N)$ be the binary matrix with rows indexed with X_d and columns indexed with X_k such that $M(x, y) = 1$ if and only if $\ell(x \wedge y) = i$.

Theorem 11. *Suppose that X is a pooling lattice and $1 \leq i \leq d < k < N$. Then the following results hold.*

(i) *Let s satisfies $1 \leq s \leq i$ and $N - (s + 1)k \geq d - i$. Then $M(i; d, k; N)$ is an s^e-disjunct matrix, where $e = \mu(s, i, k)\pi(i, (s + 1)k, d) - 1$.*

(ii) *Let s satisfies $1 \leq s < \mu(0, i, k)/\mu(0, i, k - 1)$ and $N - (s + 1)k \geq d - i$. Then $M(i; d, k; N)$ is an s^e-disjunct matrix, where $e = (\mu(0, i, k) - s\mu(0, i, k - 1))\pi(i, (s + 1)k, d) - 1$.*

Proof. (i) Let $y_0, y_1, \ldots, y_s$ be any $s+1$ distinct columns of $M(i; d, k; N)$. Similar to the proof of Theorem 6 (i), there exists an $a_j \in X_1$ such that $a_j \preceq y_0$ but $a_j \not\preceq y_j$ for each $j \in \{1, 2, \ldots, s\}$. Suppose $a_0 = a_1 \vee a_2 \vee \cdots \vee a_s$. By the proof of Theorem 6 (i), the size of $X_i \cap [a_0, y_0]$ is at least $\mu(s, i, k)$. Let $x_0 \in [a_0, y_0] \cap X_i$ and $x \in X_d$ satisfy $x \wedge (y_0 \vee y_1 \vee \cdots \vee y_s) = x_0$. Then $x \wedge y_0 = x_0$ and $x \wedge y_j \preceq x_0$ for each $j \in \{1, 2, \ldots, s\}$, which implies that $\ell(x \wedge y_j) < i$ by $a_0 \not\preceq y_j$. Since $x \wedge (y_0 \vee y_1 \vee \cdots \vee y_s) = x_0$, by (A2) $\ell(y_0 \vee y_1 \vee \cdots \vee y_s) \leq (s+1)k$ and $\ell(x \vee y_0 \vee y_1 \vee \cdots \vee y_s) \leq d + (s+1)k - i \leq N$. By Lemma 9, the number of elements $x \in X_d$ satisfying $x \wedge (y_0 \vee y_1 \vee \cdots \vee y_s) = x_0$ is at least $\pi(i, (s+1)k, d)$. Therefore, the number of elements $x \in X_d$ satisfying $\ell(x \wedge y_0) = i$ and $\ell(x \wedge y_j) \neq i$ for each $j \in \{1, 2, \ldots, s\}$ is at least $\mu(s, i, k)\pi(i, (s + 1)k, d)$, as desired.

(ii) Let $y_0, y_1, \ldots, y_s$ be any $s + 1$ distinct columns of $M(i; d, k; N)$. By Theorem 6 (ii), the number of elements in X_i contained in y_0 but not in y_j for each $j \in \{1, 2, \ldots, s\}$ is at least $\mu(0, i, k) - s\mu(0, i, k - 1)$. Given $x_0 \in X_i$ with $x_0 \preceq y_0$ but $x_0 \not\preceq y_j$ for each $j \in \{1, 2, \ldots, s\}$. By the proof of (i), the number of elements $x \in X_d$ satisfying $x \wedge y_0 = x_0$ and $\ell(x \wedge y_j) < i$ for each $j \in \{1, 2, \ldots, s\}$ is at least $\pi(i, (s+1)k, d)$. Therefore, the desired result follows. □ □

Theorem 12. *Suppose that X is a regular pooling lattice. Let s, i, d, k and N satisfy $k - i \geq 2$, $1 \leq s \leq (\mu(0, i, k) - \mu(0, i, k - 1))/(\mu(0, i, k - 1) - \mu(0, i, k - 2))$ and $N - k - s(k - \max\{2i - d, 0\}) \geq d - i$. Then $M(i; d, k; N)$ is an s^e-disjunct matrix, where $e = (\mu(0, i, k) - \mu(0, i, k - 1) - (s - 1)(\mu(0, i, k - 1) - \mu(0, i, k - 2)))\pi(i, k + s(k - \max\{2i - d, 0\}), d) - 1$.*

Proof. Let $y_0, y_1, \ldots, y_s$ be any $s+1$ distinct columns of $M(i; d, k; N)$. By Theorem 7 the number of elements of X_i contained in y_0 but not in y_j for each $1 \leq j \leq s$ is at least $\mu(0, i, k) - \mu(0, i, k - 1) - (s - 1)(\mu(0, i, k - 1) - \mu(0, i, k - 2))$. Let $x \in X_d$ satisfy $\ell(x \wedge y_0) = i$. If there exists $j \in \{1, 2, \ldots, s\}$ such that $\ell(x \wedge y_j) = i$, by $(x \wedge y_0) \vee (x \wedge y_j) \preceq x$ and (A3), we have

$$\begin{aligned}
\ell(y_0 \wedge y_j) &\geq \ell(x \wedge y_0 \wedge y_j) \\
&= \ell(x \wedge y_0) + \ell(x \wedge y_j) - \ell((x \wedge y_0) \vee (x \wedge y_j)) \\
&\geq \max\{2i - d, 0\}.
\end{aligned}$$

Suppose $\ell(y_0 \wedge y_j) \geq \max\{2i - d, 0\}$ for each $j \in \{1, 2, \ldots, s\}$. By (A3) we have

$$\begin{aligned}
& \ell(y_0 \vee y_1 \vee \cdots \vee y_s) \\
= \; & \ell(y_0 \vee y_1 \vee \cdots \vee y_{s-1}) + \ell(y_s) - \ell((y_0 \vee y_1 \vee \cdots \vee y_{s-1}) \wedge y_s) \\
\leq \; & \ell(y_0 \vee y_1 \vee \cdots \vee y_{s-1}) + \ell(y_s) - \ell(y_0 \wedge y_s) \\
\leq \; & \ell(y_0 \vee y_1 \vee \cdots \vee y_{s-1}) + k - \max\{2i - d, 0\} \\
\leq \; & \ell(y_0) + s(k - \max\{2i - d, 0\}) \\
= \; & k + s(k - \max\{2i - d, 0\}).
\end{aligned}$$

Given $x_0 \in X_i$ with $x_0 \preceq y_0$ but $x_0 \not\preceq y_j$ for each $j \in \{1, 2, \ldots, s\}$. By the proof of Theorem 11, the number of elements $x \in X_d$ satisfying $x \wedge y_0 = x_0$ and $\ell(x \wedge y_j) < i$ for each $j \in \{1, 2, \ldots, s\}$ is at least $\pi(i, k + s(k - \max\{2i - d, 0\}), d)$. Therefore, the desired result follows. □ □

Now we give four families of pooling lattices. By Theorems 11 and 12, we can construct pooling designs from these lattices.

Example 22. ([7] The Boolean Algebra). Let X be as in Example 1. Then X is a regular pooling lattice with the parameters

$$|X_r| = \binom{N}{r}, \quad \mu(r, s, t) = \binom{t - r}{s - r}, \quad \pi(r, s, t) = \binom{N - s}{t - r}.$$

Example 23. ([8] The Projective Geometry). Let X be as in Example 2. Then X is a regular pooling lattice with the parameters

$$|X_r| = \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \mu(r, s, t) = \begin{bmatrix} t - r \\ s - r \end{bmatrix}_q, \quad \pi(r, s, t) = q^{(s-r)(t-r)} \begin{bmatrix} N - s \\ t - r \end{bmatrix}_q.$$

Example 24. (The Affine Geometry). Let X' be as in Example 14. Then X' is a pooling lattice with the parameters

$$|X'_{r+1}| = q^{N-r} \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \mu(r+1, s+1, t+1) = \begin{bmatrix} t - r \\ s - r \end{bmatrix}_q, \quad \pi(r+1, s+1, t+1) = q^{(s-r)(t-r)+s-r} \begin{bmatrix} N - s \\ t - r \end{bmatrix}_q.$$

Example 25. (The D-Bounded Distance-Regular Graph). Let X be as in Example 21. Then X is a pooling lattice with the parameters

$$|X_r| = N(0, r, D), \quad \mu(r, s, t) = N(r, s, t), \quad \pi(r, s, t),$$

where $\pi(r, s, t)$ can be computed using (3.1). In particular, if $D(\Delta_1) + D(\Delta_2) = D(\Delta_1 + \Delta_2) + D(\Delta_1 \cap \Delta_2)$ for any $\Delta_1, \Delta_2 \in X$, then the pooling lattice X is a regular pooling lattice.

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