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MATHEMATICAL MODEL AND UNDERCUTTING ANALYSIS OF ELLIPTICAL GEARS GENERATED BY RACK CUTTERS

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Abstract—The theory of gearing and geometry of the straight-sided rack cutter are employed to develop a mathematical model of elliptical gears that takes backlash into consideration. The mathematical model also includes the bottom land of the elliptical gear, which is equidistant from the pitch ellipse. A computer program is developed to generate the tooth profile of elliptical gears. The profile of elliptical gears obtained by applying the proposed method is compared with that obtained by the evolute method. Undercutting of elliptical gears is also investigated. The results of this research should be helpful in the design, manufacture, and measurement of elliptical gears. Copyright © 1996 Elsevier Science Ltd

INTRODUCTION

Cycloidal cranks, cyclic three-gear drives, and drag-links are typical irregular rotation mechanisms that meet various purposes. The elliptical gear is a type of noncircular gear whose pitch curve is an ellipse. This type of gear also works well when irregular motion without full stop is required. Therefore, the elliptical gear, which is kinematically equivalent to the crossed link, can also be used to produce irregular motion [1]. The elliptical gear is well known for providing excellent characteristics such as accurate transmission, compact size, and ease of dynamic balance. Hence elliptical gears have been successfully used in various types of automatic machinery, packaging machines, quick-return mechanisms, flying shears, pumps, flow meters, and a wide array of instruments [2, 3].

In the past, elliptical gears have not been widely used in industry because of difficulties in their design and manufacture. To date, little research has been devoted to this topic. Most research has focused on computer-aided design (CAD) and kinematic analysis of elliptical pitch curves [1–4]. Litvin [5, 6] proposed extending tooth evolute curves to form the tooth profile, and derived the tooth evolute of an ellipse. Chang [7] developed a mathematical model of elliptical gears, based on the tooth evolute curve, whose rotation shaft coincides with one of its foci. A computer program for generating the tooth profile of elliptical gears was also developed by Chang. However, a mathematical model for the complete tooth profile of elliptical gears including fillets and bottom land curves is not yet available in the literature.

Several different methods have been used to manufacture elliptical gears [6]. All of these methods may be considered theoretically to consist of a cutter performing a pure rolling motion on the pitch ellipse. Recently, the development of more advanced CAD and CNC gear-cutting machines has made the design and manufacture of elliptical gears more efficient and economical than ever. In this paper, the cutting mechanism is considered in a manner such that the driving ellipse with its rotation shaft coincides with one of its foci, and the driven rack cutter translates along two perpendicular directions and performs pure rolling without sliding on the pitch ellipse. A complete mathematical model of elliptical gears is developed based on the theory of gearing and the geometry of the straight-sided rack cutter, which includes fillets, working regions and top lands. In addition, backlashes of the elliptical gear are considered in the mathematical model. The elliptical gear tooth profiles obtained by applying the proposed method are then compared with those obtained by the evolute method [7]. The comparison shows that the working parts of the tooth profiles are exactly the same, and the method proposed here can also be used to obtain the fillet, bottom land, and backlash of elliptical gears. In order to avoid undercutting in the manufacturing process, the parameters of the rack cutter must be properly limited. In this paper, a general equation for the conditions under which undercutting occurs in the cutting process proposed here is derived by examining when a singular point appears on the generated elliptical gear tooth surface. Therefore, the relative velocity and equation of meshing [5] between the rack cutter and elliptical gear must be considered. The mathematical model and undercutting analysis proposed here are very helpful in the design and production of high-precision elliptical gears.

GEOMETRIC PROPERTIES OF THE ELLIPSE

The geometric equations and properties of ellipses have been described in detail in the literature [4, 8]. Several important equations will be derived here for convenience and to provide a more complete characterization of elliptical gears. Figure 1 shows a pitch curve of the elliptical gear; the pitch curve can be represented in polar coordinates by

$$r_1 = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\phi_1} = \frac{b^2}{a(1+\epsilon\cos\phi_1)}, \tag{1}$$

where $\epsilon = e/a = \sqrt{a^2 - b^2}/a$ is the eccentricity, a is the major semi-axis, and b is the minor

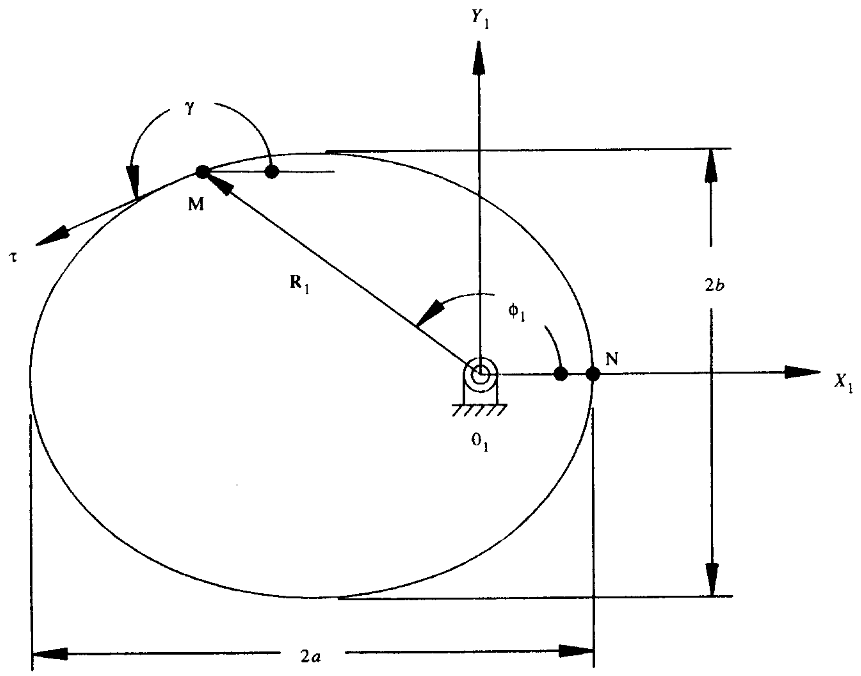


Fig. 1. Basic properties of pitch ellipse for elliptical gears.

semi-axis. The position vector of the pitch ellipse may also be represented in the Cartesian coordinate system as follows:

$$x_1 = \frac{b^2\cos\phi_1}{a(1+\epsilon\cos\phi_1)}$$
$$y_1 = \frac{b^2\sin\phi_1}{a(1+\epsilon\cos\phi_1)} \tag{2}$$

The unit tangent vector at any point M on the pitch ellipse can be obtained by differentiating and znormalizing equation (2). Therefore,

$$\begin{aligned}\boldsymbol{\tau}_1 &= \tau_x\,\mathbf{i}_1 + \tau_y\,\mathbf{j}_1\\ &= \frac{-\sin\phi_1}{\sqrt{\epsilon^2+2\epsilon\cos\phi_1+1}}\,\mathbf{i}_1 + \frac{\epsilon+\cos\phi_1}{\sqrt{\epsilon^2+2\epsilon\cos\phi_1+1}}\,\mathbf{j}_1\end{aligned} \tag{3}$$

From the geometry of the pitch ellipse shown in Fig. 1, it can be found that the tangent vector at any point on the pitch ellipse is

$$\boldsymbol{\tau}_1 = \cos\gamma\,\mathbf{i}_1 + \sin\gamma\,\mathbf{j}_1 \tag{4}$$

From equations (3) and (4), the relationship between ϕ_1 and γ can be expressed as follows:

$$\cos\gamma = \frac{-\sin\phi_1}{\sqrt{\epsilon^2+2\epsilon\cos\phi_1+1}}$$

or

$$\sin\gamma = \frac{\epsilon+\cos\phi_1}{\sqrt{\epsilon^2+2\epsilon\cos\phi_1+1}} \tag{5}$$

The unit normal vector $\mathbf{n}_1$ may also be obtained by

$$\begin{aligned}\mathbf{n}_1 &= \boldsymbol{\tau}_1 \times \mathbf{k}_1\\ &= \sin\gamma\,\mathbf{i}_1 - \cos\gamma\,\mathbf{j}_1\\ &= n_x\,\mathbf{i}_1 + n_y\,\mathbf{j}_1\,,\end{aligned} \tag{6}$$

where $\mathbf{k}_1$ is the unit vector along the Z_1-axis of the Cartesian coordinate system. The arc length on the pitch ellipse, measured from initial point N to point M, can be calculated by applying the equation

$$\begin{aligned}S_{MN} &= \int_0^{\phi_M}\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\phi_1}\right)^2}\,\mathrm{d}\phi_1\\ &= \int_0^{\phi_M}\frac{a(1-\epsilon^2)\sqrt{\epsilon^2+2\epsilon\cos\phi_1+1}}{(1+\epsilon\cos\phi_1)^2}\,\mathrm{d}\phi_1\end{aligned} \tag{7}$$

MATHEMATICAL MODEL OF THE RACK CUTTER

For simplicity, the generation of elliptical gears can be considered a two-dimensional problem. All methods for manufacturing of elliptical gears can be kinematically considered to consist of a rack cutter that performs pure rolling on the pitch ellipse in the generating process, as shown in Fig. 2(a). In this paper, a standard rack cutter [9] with a complete cutter surface, as shown in Fig. 2(b), is chosen. The shape of the rack cutter consists of two straight lines that form a pressure angle ψ_c with respect to the X_c-axis. The circular arcs of radius r with centers at C and D generate the fillet surfaces of elliptical gears, while the straight line $\overline{M_0^{(i)}M_1^{(i)}}$ ($i = 3$, 4 indicates regions 3 and 4 of the rack cutter, respectively) generates the working tooth surfaces of the elliptical gears. Backlash is also considered in the mathematical model, when a negative value of Δb is chosen. In the process of tooth surface generation, axis Y_c must coincide with the tangent direction of the pitch

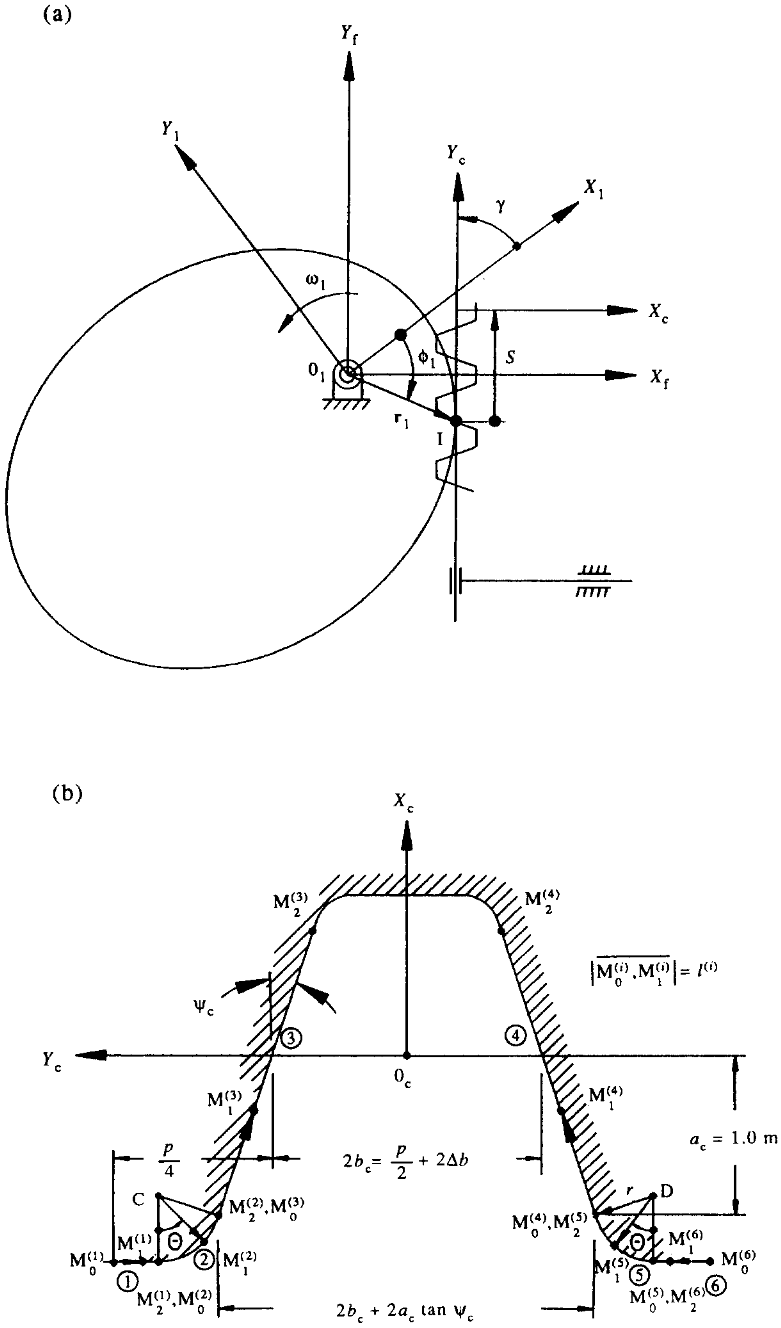


Fig. 2. Generating mechanism and geometry of rack cutter.

ellipse, as shown in Fig. 2(a). The contact point of the pitch ellipse and rack cutter pitch line is located at the Y_c-axis of coordinate system $S_c(X_c, Y_c)$, and the rack cutter translates along the X_c and Y_c axes. Because each tooth of an elliptical gear may have a different shape, both sides of the rack cutter must be considered. Therefore, the rack cutter may be divided into six regions, as shown in Fig. 2(b). Regions 1 and 6 of the rack cutter surface can be considered to generate the bottom land of elliptical gears, regions 2 and 5 the fillet surface, and regions 3 and 4 the working tooth surface. The addendum curve, which may be considered to be the shape of the gear blank, is a curve equidistant from the pitch ellipse [4]. The equations of the rack cutter, represented in the coordinate system $S_c(X_c, Y_c)$, can be obtained as described in Sections 3.1–3.3.

Regions 1 *and* 6 *of rack cutter surfaces*

Recall that regions 1 and 6 (the top land) of the rack cutter surface are used to generate the bottom land of elliptical gears. As shown in Fig. 2, $\ell^{(i)}$ is the design parameter of the rack cutter surface that determines the location of points on the rack cutter surface. From the rack cutter geometry, equations for regions 1 and 6 of the rack cutter surface can be expressed as follows:

$$\mathbf{R}_c^{(i)} = \begin{Bmatrix} x_c^{(i)} \\ y_c^{(i)} \end{Bmatrix} = \begin{Bmatrix} -a_c + r\sin\psi_c - r \\ \pm\dfrac{\pi m}{2} \mp \ell^{(i)} \end{Bmatrix}, \tag{8}$$

and

$$0 \le \ell^{(i)} \le \frac{\pi m}{2} - b_c - a_c\tan\psi_c - r\cos\psi_c,\ i = 1,6,$$

where $\ell^{(i)} = |\overline{M_0^{(i)}M_1^{(i)}}|$ $(i = 1, 6)$ represents the distance measured from the initial point $M_0^{(i)}$, along the straight line $\overline{M_0^{(i)}M_2^{(i)}}$ to any point $M_1^{(i)}$ on the top land of rack cutter surface. Parameter m is the module, r is the radius of the circular arc, ψ_c is the pressure angle, and a_c and b_c are design parameters shown in Fig. 2. The upper sign of equation (8) indicates region 1 of the rack cutter surface while the lower sign indicates region 6. The unit normals and normals to regions 1 and 6 of the rack cutter surface are represented by the equation

$$\mathbf{n}_c^{(i)} \frac{\mathbf{N}_c^{(i)}}{|\mathbf{N}_c^{(i)}|},\ i = 1,6\,, \tag{9}$$

and

$$\mathbf{N}_c^{(i)} = \frac{\partial \mathbf{R}_c^{(i)}}{\partial \ell^{(i)}} \times \mathbf{k}_c,$$

where R~$_c^{(i)}$ indicates the position vector of the rack cutter surface represented in coordinate system S_c, as expressed in equation (8). Equations (8) and (9) yield the following for the unit normals to regions 1 and 6 of the rack cutter surface:

$$\mathbf{n}_c^{(i)} = \begin{Bmatrix} \mp 1 \\ 0 \end{Bmatrix} \tag{10}$$

Regions 2 *and* 5 *of rack cutter surfaces*

Regions 2 and 5 of the rack cutter surface are used to generate different sides of the fillet surface of elliptical gears. Θ is the design parameter of the rack cutter surface which determines the location of points on the fillet. Here we have $\ell^{(i)} = \Theta$ (Fig. 2(b)) in this case. The position vectors for regions 2 and 5 of the rack cutter surface can be expressed as follows:

$$\mathbf{R}_c^{(i)} = \begin{Bmatrix} x_c^{(i)} \\ y_c^{(i)} \end{Bmatrix} = \begin{Bmatrix} -a_c + r\sin\psi_c - r\cos\theta \\ \pm b_c \pm a_c\tan\psi_c \pm r\cos\psi_c \mp r\sin\theta \end{Bmatrix}, \tag{11}$$

and

$$0 \le \theta \le 90^\circ - \psi_c,\ i = 2,5$$

The upper sign of equation (11) indicates region 2 of the rack cutter surface, while the lower sign represents region 5. The unit normals to regions 2 and 5 of the rack cutter surfaces can be obtained by applying equation (9), which results in the following expression:

$$\mathbf{n}_c^{(i)} = \begin{Bmatrix} \mp\cos\theta \\ -\sin\theta \end{Bmatrix} \tag{12}$$

Regions 3 *and* 4 *of rack cutter surfaces*

Regions 3 and 4 of the rack cutter surface are used to generate different sides of the working tooth surface of elliptical gears. $\ell^{(i)}$ is the design parameter of the rack cutter surface which determines the location of points on the working surface. The position vector for regions 3 and 4 of the rack cutter surface can be obtained as follows:

$$\mathbf{R}_c^{(i)} = \begin{Bmatrix} x_c^{(i)} \\ y_c^{(i)} \end{Bmatrix} = \begin{Bmatrix} -a_c + \ell^{(i)}\cos\psi_c \\ \pm b_c \pm a_c\tan\psi_c \mp \ell^{(i)}\sin\psi_c \end{Bmatrix}, \tag{13}$$

and

$$\ell^{(i)} = |\overline{M_0^{(i)}M_1^{(i)}}|,\ i = 3,4$$

The upper sign of equation (13) indicates region 3 of the rack cutter surface, while the lower sign indicates region 4 of the rack cutter surface. The unit normals to regions 3 and 4 of the rack cutter surfaces can be obtained as follows:

$$\mathbf{n}_c^{(i)} = \begin{Bmatrix} \mp\sin\psi_c \\ -\cos\psi_c \end{Bmatrix} \tag{14}$$

GENERATED ELLIPTICAL GEAR TOOTH SURFACES

To derive the mathematical model for the complete tooth profile of elliptical gears, coordinate systems $S_c(X_c, Y_c)$, $S_1(X_1, Y_1)$, and $S_f(X_f, Y_f)$ must be set up. The coordinate systems S_c, S_1, and S_f are attached to the rack cutter, elliptical gear, and gear housing, respectively, as shown in Fig. 2(a). The contact lines of the gear blank and rack cutter, represented in coordinate system S_c, can be obtained by simultaneously considering the following equations [5]:

$$\mathbf{R}_c^{(i)} = \mathbf{R}_c^{(i)}(\ell^{(i)}),\ i = 1,\ldots,6 \tag{15}$$

and

$$\frac{X_c^{(i)} - x_c^{(i)}}{n_{xc}^{(i)}} = \frac{Y_c^{(i)} - y_c^{(i)}}{n_{yc}^{(i)}}, \tag{16}$$

where i indicates regions 1 to 6 of the corresponding rack cutter surface; $X_c^{(i)}$ and $Y_c^{(i)}$ are coordinates, represented in coordinate system S_c, of the instantaneous center of rotation for the generation mechanism; $x_c^{(i)}$ and $y_c^{(i)}$ are the surface coordinates of the rack cutter; and $n_{xc}^{(i)}$ and $n_{yc}^{(i)}$ are the direction cosines of the rack cutter surface unit normal $\boldsymbol{n}_c^{(i)}$. The relation shown in equation (16) is the so-called equation of meshing. It relates the surface coordinates $\ell^{(i)}$ of the rack cutter to the motion parameter ϕ_1 of the elliptical gear. The generated elliptical gear tooth surfaces can be considered a set of contact lines, represented in coordinate system S_1, of rack cutter surface Σ_c and gear blank surface Σ_1. The homogeneous coordinate transform matrix equation can be applied to transform the contact lines from coordinate system S_c to S_1. Therefore, the equation of the generated elliptical gear tooth surface is expressed by

$$\mathbf{R}_l^{(i)} = [M_{lc}]\mathbf{R}_c^{(i)},\ i = 1,\ldots,6 \tag{17}$$

$$\frac{X_c^{(i)} - x_c^{(i)}}{n_{xc}^{(i)}} = \frac{Y_c^{(i)} - y_c^{(i)}}{n_{yc}^{(i)}} \tag{18}$$

and

$$[M_{1c}] = \begin{bmatrix} \sin\gamma & \cos\gamma & r_1\cos\phi_1 + S\cdot\cos\gamma \\ -\cos\gamma & \sin\gamma & -r_1\sin\phi_1 + S\cdot\sin\gamma \\ 0 & 0 & 1 \end{bmatrix},$$

where $X_c^{(i)} = 0$ and $Y_c^{(i)} = -S$; S is the translation distance of the rack cutter and can be obtained from the equation of meshing (equation (18)); and r_1 is the distance measured from the rotation axis Z_1 to the instantaneous center of rotation I. According to equations (8)–(14), (17) and (18), a complete tooth profile of an elliptical gear generated by regions 1 to 6 of the rack cutter can be obtained as described in Sections 4.1–4.3.

Bottom lands of the elliptical gear tooth surfaces

Recall that the bottom lands of the elliptical gear tooth surfaces are generated by the top lands (regions 1 and 6) of the rack cutter. According to equations (8)–(10), (17) and (18), the bottom lands of the elliptical gear tooth surfaces can be represented by

$$x_1^{(i)} = (-a_c + r\sin\psi_c - r)\sin\gamma + \left(\pm\frac{\pi m}{2} \mp \ell^{(i)}\right)\cos\gamma + r_1\cos\phi_1 + S\cos\gamma$$

$$y_1^{(i)} = -(-a_c + r\sin\psi_c - r)\cos\gamma + \left(\pm\frac{\pi m}{2} \mp \ell^{(i)}\right)\sin\gamma - r_1\sin\phi_1 + S\sin\gamma$$

$$S = \mp\frac{\pi m}{2} \pm \ell^{(i)}, \tag{19}$$

where $0 \le \ell^{(i)} \le (\pi m/2 - b_c - a_c\tan\psi_c - r\cos\psi_c)$ and $i = 1, 6$. The upper sign indicates the bottom land generated by region 1 of the rack cutter and the lower sign indicates the bottom land generated by region 6 of the rack cutter. Equation (19) may be simplified as follows:

$$\begin{Bmatrix} x_1^{(i)} \\ y_1^{(i)} \end{Bmatrix} = \begin{Bmatrix} r_1\cos\phi_1 \\ -r_1\sin\phi_1 \end{Bmatrix} + (-a_c + r\sin\psi_c - r)\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}. \tag{20}$$

where matrix $\{n_x, n_y\}^T$ represents the unit normal at any point on the pitch ellipse, as expressed by equation (6). Equation (20) shows that the bottom land of an elliptical gear is a curve equidistant from the pitch ellipse, and this equidistance is $(a_c - r\sin\psi_c + r)$ in the negative direction of the unit normal.

Fillets of the elliptical gear tooth surfaces

Fillets of the elliptical gear tooth surfaces are generated by regions 2 and 5 of the rack cutter surface, as shown in Fig. 2. According to equations (11), (12), (17) and (18), the fillet of elliptical gear tooth surfaces can be represented by

$$\begin{aligned} x_1^{(i)} &= (-a_c + r\sin\psi_c - r\cos\theta)\sin\gamma \\ &\quad + (\pm b_c \pm a_c\tan\psi_c \pm r\cos\psi_c \mp r\sin\theta)\cos\gamma + r_1\cos\phi_1 + S\cos\gamma \\ y_1^{(i)} &= -(-a_c + r\sin\psi_c - r\cos\theta)\cos\gamma \\ &\quad + (\pm b_c + a_c\tan\psi_c \pm r\cos\psi_c \mp r\sin\theta)\sin\gamma - r_1\sin\phi_1 + S\sin\gamma \\ S &= \pm(-a_c + r\sin\psi_c)\tan\theta \mp (b_c + a_c\tan\psi_c + r\cos\psi_c), \end{aligned} \tag{21}$$

where $0 \le \theta \le 90^\circ - \psi_c$ and $i = 2, 5$. The upper sign indicates the fillet of elliptical gears generated by region 2 of the rack cutter, while the lower sign represents the fillet of elliptical gears generated by region 5 of the rack cutter.

Working surfaces of the elliptical gear

Working surfaces of the elliptical gear are generated by regions 3 and 4 of the rack cutter surface, as shown in Fig. 2. According to Equations (13), (14), (17) and (18), the working surface of an elliptical gear can be represented by

$$x_1^{(i)} = (-a_c + \ell^{(i)}\cos\psi_c)\sin\gamma + (\pm b_c \pm a_c\tan\psi_c \mp \ell^{(i)}\sin\psi_c)\cos\gamma + r_1\cos\phi_1 + S\cos\gamma$$

$$y_1^{(i)} = -(-a_c + \ell^{(i)}\cos\psi_c)\cos\gamma + (\pm b_c \pm a_c\tan\psi_c \mp \ell^{(i)}\sin\psi_c)\sin\gamma - r_1\sin\phi_1 + S\sin\gamma$$

$$\ell^{(i)} = \frac{a_c}{\cos\psi_c} + b_c\sin\psi_c \pm S\sin\psi_c, \tag{22}$$

where $\ell^{(i)} = |\overline{M_0^{(i)}M_1^{(i)}}|$. The upper sign indicates the working surface of the elliptical gear generated by region 3 of the rack cutter, and the lower sign represents the working surface of the elliptical gear generated by region 4 of the rack cutter.

The mathematical model represented in equations (20)–(22) was used to develop a computer program to generate the complete tooth profile of elliptical gears. In the design of elliptical gears, one constraint that must be satisfied is that the circumference of the pitch ellipse must equal the product of the number of teeth n and the circular pitch p. Otherwise, the generated elliptical gear will have an incomplete tooth [7]. An example is given below that illustrates how to determine the tooth profile of an elliptical gear. The tooth profile of an elliptical gears obtained by applying the proposed method is compared with that obtained by applying the evolute method.

Example 1. The standard rack cutter shown in Fig. 2(b) is chosen to generate an elliptical gear with module $m = 5.0$ mm, number of teeth $n = 45$, pressure angle $\psi_c = 20°$, radius of circular arc $r = 0.3$ module, $\Delta b = 0$ mm for non-backlash, and major semi-axis $a = 125$ mm. The short semi-axis b is calculated by solving for the pitch ellipse circumference $S = \pi mn$ and equation (7). It is found that $b = 99.261$ mm and $\epsilon = 0.608$.

The computer program developed here and computer graphics were applied to obtain the complete tooth profile of the elliptical gear shown in Fig. 3(a). Figure 3(b) compares the tooth profile obtained by applying the proposed method with that obtained from the evolute method [7]. It is clear that the proposed mathematical model and generation method can be used to generate the tooth profile not only for working surfaces but also for fillets and bottom lands, whereas the evolute method can only generate the working surfaces. The addendum and dedendum curves are generated so that they are equidistant from the pitch ellipse. The results also show that the profiles of the working surfaces generated by these two methods are exactly the same.

Example 2. The standard rack cutter shown in Fig. 2(b) is also chosen to generate an elliptical gear with major axis $2a = 41.465$ mm, short axis $2b = 38.103$ mm, number of teeth $n = 21$, pressure angle $\psi_c = 20°$, radius of circular arc $r = 0.3$ module and $\Delta b = -0.03$ mm for backlash consideration. The module m is then calculated by solving for the pitch ellipse circumference $S = \pi mn$ and equation (7). It is found that $m = 1.895$ mm and $\epsilon = 0.394$.

By substituting the above design and calculated parameters into the developed mathematical model and computer program, the profile of the elliptical gear can be obtained. Figure 4 shows the meshing elliptical gear pair manufactured by the proposed mathematical model and CNC wire cut. It is one of the elliptical gear pairs resulting from application of the developed computer program (which was used to design the gear for Professor Tshen-Chan Lin, in Department of Agricultural Machinery Engineering, National Chung Hsing University) and manufactured by CNC wire cut. This elliptical gear pair has been used successfully in the agricultural machine design by Professor Lin.

UNDERCUTTING ANALYSIS

Undercutting is an important problem in gear design and manufacturing. When undercutting occurs, the thickness near the gear fillets will be decreased. Therefore, both the load capacity of the tooth and the length of the line of action are also reduced. Mathematically, the problem of preventing undercutting is the problem of avoiding the appearance of singular points on the generated tooth shape. A method proposed by Litvin [5], which considers the relative velocity and equation of meshing between the gear blank and rack cutter, is applied here to determine the limit of the rack cutter parameters. Figure 2 shows the relative motion between the gear blank and rack cutter. We shall consider under what circumstances a singular point appears on the working surface of elliptical gears generated by region 3 of the rack cutter.

The relative velocity between the gear blank and rack cutter, represented in coordinate system S_c, can be obtained as follows:

$$\mathbf{V}_c^{(12)} = \omega_1(b_c + a_c\tan\psi_c - \ell^{(3)}\sin\psi_c + S)\,\mathbf{i}_c + \omega_1(a_c - \ell^{(3)}\cos\psi_c)\,\mathbf{j}_c \tag{23}$$

Recall that the equation of meshing for the working surface and rack cutter surface is expressed by the third part of equation (22). It is rewritten here for convenience:

$$f(\ell^{(3)}) = \ell^{(3)} - \frac{a_c}{\cos\psi_c} - S\left(\phi_1\right)\sin\psi_c - b_c\sin\psi_c = 0 \tag{24}$$

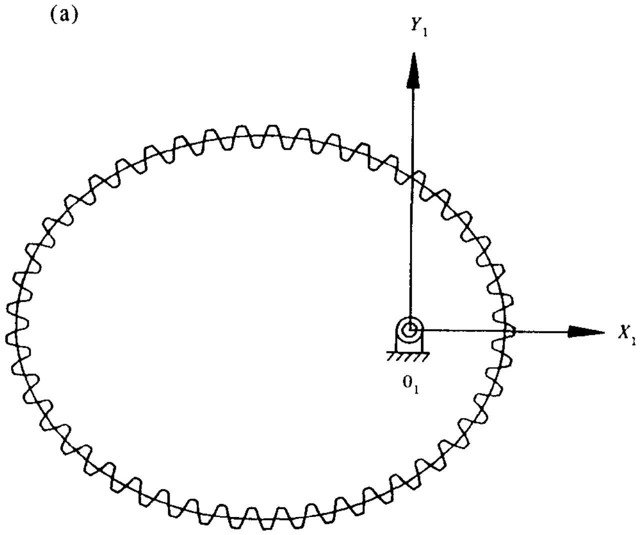
The relative velocity of the generated elliptical gear can be obtained by applying the equation

$$\begin{aligned} \mathbf{V}_r^{(2)} &= \mathbf{V}_r^{(1)} + (\mathbf{V}_{tr}^{(1)} - \mathbf{V}_{tr}^{(2)}) \\ &= \mathbf{V}_r^{(1)} + \mathbf{V}^{(12)} \end{aligned} \tag{25}$$

The subscript r represents the relative motion and the subscript tr represents the transfer motion. At a regular point of the generated tooth surface, a tangent vector **T** to the surface exists, that is, $\mathbf{T} \neq 0$. When undercutting occurs, singular points appear on the generated elliptical gear surface and the tangent vector $\mathbf{T} = 0$ at these points. Therefore,

$$\mathbf{V}_r^{(1)} + \mathbf{V}^{(12)} = 0 \tag{26}$$

(a)



(b)

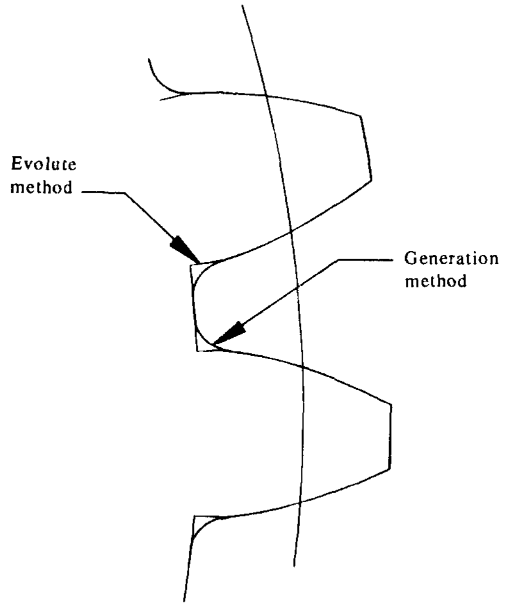


Fig. 3. (a) Complete tooth profile of an elliptical gear. (b) Comparison of the profile obtained by the generation method and the evolute method.

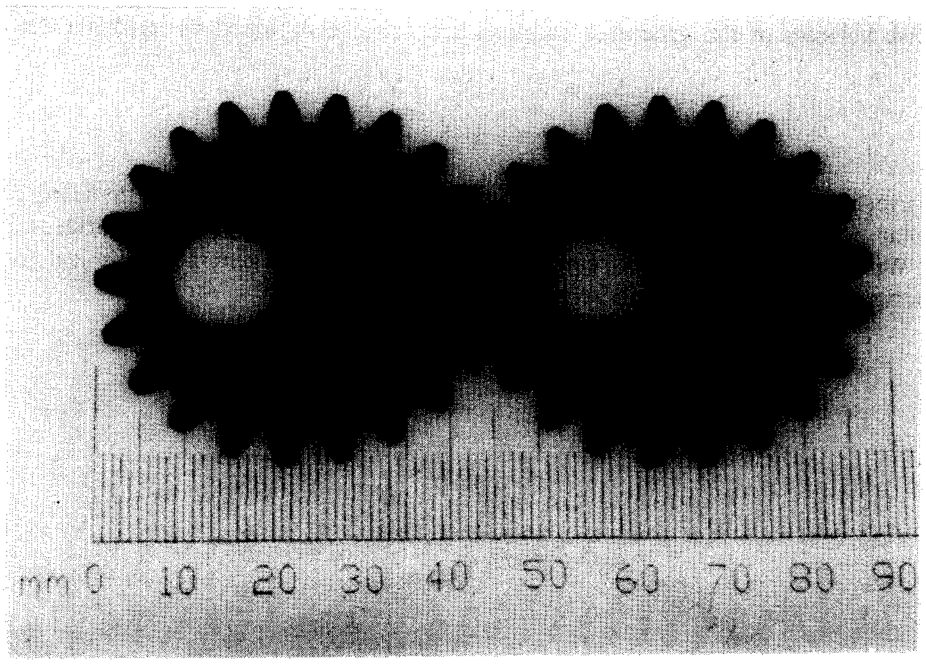


Fig. 4. Meshing elliptical gear pair manufactured by the proposed mathematical model and CNC wire cut.

The relative velocity $\mathbf{V}^{(12)}$ expressed in equation (26) may be rewritten in X_c and Y_c components as follows:

$$\frac{\mathrm{d}x_c}{\mathrm{d}\ell^{(3)}}\frac{\mathrm{d}\ell^{(3)}}{\mathrm{d}t} = -V_{xc}^{(12)}$$

$$\frac{\mathrm{d}y_c}{\mathrm{d}\ell^{(3)}}\frac{\mathrm{d}\ell^{(3)}}{\mathrm{d}t} = -V_{yc}^{(12)} \tag{27}$$

Differentiation of the equation of meshing (equation (24)) yields

$$\frac{\partial f}{\partial \ell^{(3)}}\frac{\mathrm{d}\ell^{(3)}}{\mathrm{d}t} = -\frac{\partial f}{\partial \phi_1}\frac{\mathrm{d}\phi_1}{\mathrm{d}t} \tag{28}$$

Equations (27) and (28) form a system of three linear equations with one unknown. The system of equations possesses a unique solution if and only if the following equations are satisfied:

$$\begin{vmatrix} \dfrac{\mathrm{d}x_c}{\mathrm{d}\ell^{(3)}} & -V_{xc}^{(12)} \\ \dfrac{\partial f}{\partial \ell^{(3)}} & -\dfrac{\partial f}{\partial \phi_1}\dfrac{\mathrm{d}\phi_1}{\mathrm{d}t} \end{vmatrix} = 0 \tag{29}$$

and

$$\begin{vmatrix} \dfrac{\mathrm{d}y_c}{\mathrm{d}\ell^{(3)}} & -V_{yc}^{(12)} \\ \dfrac{\partial f}{\partial \ell^{(3)}} & -\dfrac{\partial f}{\partial \phi_1}\dfrac{\mathrm{d}\phi_1}{\mathrm{d}t} \end{vmatrix} = 0 \tag{30}$$

It is noted that for noncircular gears $\omega_1 = \mathrm{d}\gamma/\mathrm{d}t$ [6]. Substituting equations (23) and (24) into equation (29) or (30), we can obtain the conditions of undercutting as follows:

$$\frac{\partial S}{\partial \gamma}\cdot\sin^2\psi_c - \left(a_c - \ell^{(3)}\cos\psi_c\right) = 0 \tag{31}$$

When $a_c = 1.0m$, at the lowest point on the straight-sided rack cutter, i.e. $\ell^{(3)} = 0$, equation (31) entails that the limiting value of the gear module m is

$$m = \frac{\partial S}{\partial \gamma} \cdot \sin^2\psi_c$$

$$= \rho \cdot \sin^2\psi_c \tag{32}$$

where ρ is the radius of the ellipse curvature. Note that $\rho_{\min}$ could be obtained at both sides of the major axis of the ellipse and $\rho_{\min} = b^2/a$. The result is the same as that obtained by Wu *et al.* [10].

Example 3. When the same parameters given in the previous examples are used, the module of the rack cutter is 5.0 mm, which is smaller than the limiting value of the gear module calculated from equation (32), i.e. $m = 9.22$ mm. Therefore, undercutting of the generated elliptical gear tooth will not occur. However, suppose that another rack cutter with module $m = 15$ mm is used to generate an elliptical gear of the same pitch ellipse, and that the number of teeth of the elliptical gear is 15. Then the proposed computer program and computer graphics yield the tooth profile of an elliptical gear shown in Fig. 5. It is clear that undercutting has occurred in this case. This verifies the equations developed above.

For the special case where the major semi-axis a and the minor semi-axis b are equal, and are equal to R, the elliptical gear will regress to a spur gear, and $m = 2R/n$ and $\rho = R$ can thus be obtained. Substituting these values into equation (32) yields the following expression for the minimum number of teeth:

$$n = \frac{2}{\sin^2\psi_c} \tag{33}$$

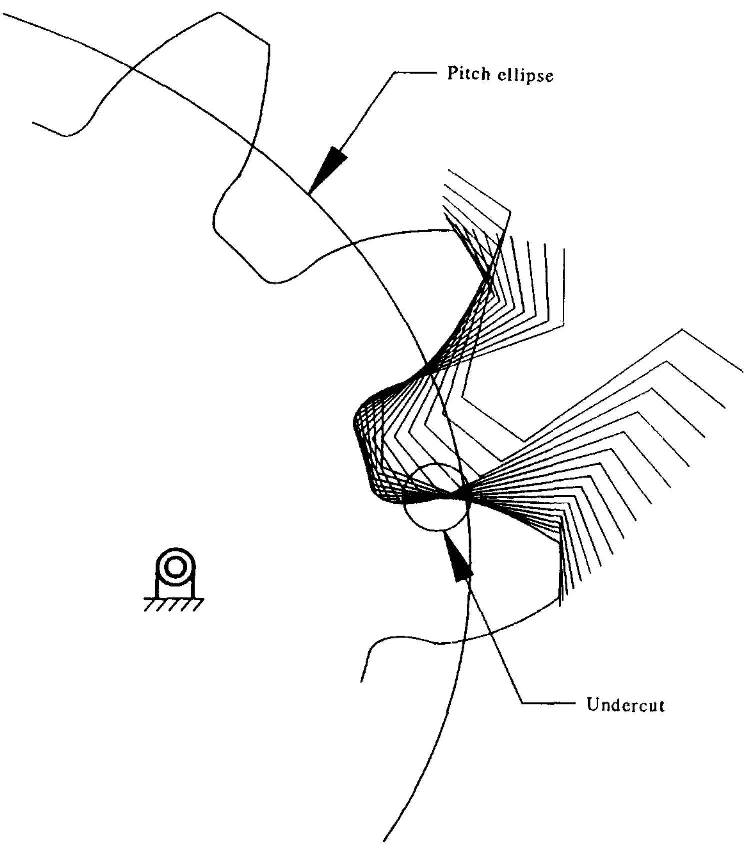


Fig. 5. Undercut in an elliptical gear.

SUMMARY

A complete mathematical model of elliptical gears, including fillets, bottom lands, and working surfaces of the tooth profile, has been developed in this paper. The model shows that the bottom land of an elliptical gear is equidistant from the pitch ellipse. A computer program has also been developed to generate elliptical gear tooth surfaces. The conditions under which undercutting occurs have been analyzed so as to enable designers to avoid undercutting elliptical gears in the cutting process, and the constraints on the surface parameters of rack cutters have been investigated. The mathematical model and undercutting analysis proposed here for elliptical gears should be helpful in the design and production of high-precision elliptical gears. They should also be helpful in the measurement and finite element stress analysis of this type of gearing.

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