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Edge-bipancyclicity of conditional faulty hypercubes ☆

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Abstract

Xu et al. showed that for any set of faulty edges F of an n-dimensional hypercube Q_n with $|F| \leqslant n-1$, each edge of $Q_n - F$ lies on a cycle of every even length from 6 to 2^n, $n \geqslant 4$, provided not all edges in F are incident with the same vertex. In this paper, we find that under similar condition, the number of faulty edges can be much greater and the same result still holds. More precisely, we show that, for up to $|F| = 2n-5$ faulty edges, each edge of the faulty hypercube $Q_n - F$ lies on a cycle of every even length from 6 to 2^n with each vertex having at least two healthy edges adjacent to it, for $n \geqslant 3$. Moreover, this result is optimal in the sense that there is a set F of $2n-4$ conditional faulty edges in Q_n such that $Q_n - F$ contains no hamiltonian cycle.

Keywords: Cycles; Pancyclic; Conditional fault; Hypercube; Fault-tolerant; Interconnection networks

1. Introduction

The ring embedding problem, which deals with all the possible lengths of the cycles in a given graph, is investigated in a lot of interconnection networks [2–4]. If a graph contains cycles of all lengths, it is called pancyclic [7]. Bipancyclicity is essentially a restriction of the concept of pancyclicity to cycles of even lengths. A bipartite graph is *vertex-bipancyclic* [6] if every vertex lies on a cycle of every even length from 4 to $|V(G)|$. Similarly, a bipartite graph is *edge-bipancyclic* if every edge lies on a cycle of every even length from 4 to $|V(G)|$. A bipartite graph is *k-edge-fault-tolerant edge-bipancyclic* if $G - F$ remains edge-bipancyclic for any set of faulty edges $F \subset E(G)$ with $|F| \leqslant k$. A path P is a sequence of adjacent vertices, written as $\langle v_0, v_1, \ldots, v_m \rangle$. The *length* of a path P, denoted by $l(P)$, is the number of edges in P. A *hamiltonian cycle* is a cycle which includes every vertex of G. In addition, we call e a *healthy edge* when e is fault-free in a graph.

Chan and Lee [1] considered an injured n-dimensional hypercube where each vertex is incident with at least two healthy edges, and proved that it still contains a hamiltonian cycle even it has $(2n-5)$ edge faults. Tsai [8] proved that such injured hypercube Q_n contains a cycle of every even length from 4 to 2^n, even if it has up to $(2n-5)$ edge faults. Recently, Xu et al. [9] showed that for any set of faulty edges F of Q_n with $|F| \leqslant n-1$, each edge of $Q_n - F$ lies on a cycle of every even length from 6 to 2^n, $n \geqslant 4$, provided not

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all faulty edges are incident with the same vertex. We observe that not all faulty edges are incident with the same vertex is equivalent to stating that each vertex has at least two healthy edges adjacent to it, if $|F| \leqslant n-1$. In this paper, we consider a set of faulty edges satisfying the condition that each vertex of $Q_n - F$ is incident with at least two healthy edges. Such a set of faulty edges F is called a set of conditional faulty edges and $Q_n - F$ is called a conditional faulty hypercube. We find that under this condition, the number of faulty edges can be much greater and the same result still holds. We show that, for up to $|F| = 2n-5$ conditional faulty edges, each edge of a faulty hypercube $Q_n - F$ lies on a cycle of every even length from 6 to 2^n, for $n \geqslant 3$. We observe that, if $|F| < 2n-5$, we may arbitrarily delete some more edges to make a faulty edge set $F' \supseteq F$ and $|F'| = 2n-5$. If our result holds for F', it holds for F. From now on, we shall assume $|F| = 2n-5$.

The above result is optimal in the sense that the result cannot be guaranteed, if there are $2n-4$ conditional faulty edges. For example, take a cycle of length four in Q_n, let $\langle u_1, u_2, u_3, u_4 \rangle$ be the consecutive vertices on this cycle. Suppose that all the $(n-2)$ edges incident with vertex u_1 (respectively, vertex u_3) are faulty except those two edges on the four cycle are healthy. There are $2(n-2)$ conditional faulty edges. Then there does not exist a hamiltonian cycle in this faulty Q_n, for $n \geqslant 3$.

We now give a formal definition of a hypercube. An n-dimensional hypercube is denoted by Q_n with the vertex set $V(Q_n)$ and the edge set $E(Q_n)$. Each vertex u of Q_n can be distinctly labeled by a n-bit binary strings, $u = u_{n-1}u_{n-2}\ldots u_1u_0$. There is an edge between two vertices if and only if their binary labels differ in exactly one bit position. Let u and v be two adjacent vertices. If the binary labels of u and v differ in ith position, then the edge between them is said to be in ith dimension and the edge (u, v) is called an ith dimension edge. Let i be a fixed position, we use Q^0_{n-1} to denote the subgraph of Q_n induced by $\{u \in V(Q_n) \mid u_i = 0\}$ and Q^1_{n-1} to denote the subgraph of Q_n induced by $\{u \in V(Q_n) \mid u_i = 1\}$. We say that Q_n is decomposed into Q^0_{n-1} and Q^1_{n-1} by dimension i, and Q^0_{n-1} and Q^1_{n-1} are $(n-1)$-dimensional subcube of Q_n induced by the vertices with the ith bit position being 0 and 1, respectively. Q^0_{n-1} and Q^1_{n-1} are all isomorphic to Q_{n-1}. For each vertex $u \in V(Q^0_{n-1})$, there is exactly one vertex in Q^1_{n-1}, denoted by $u^{(1)}$, such that $(u, u^{(1)}) \in E(Q_n)$. Conversely, for each $u \in V(Q^1_{n-1})$, there is one vertex in Q^0_{n-1}, denoted by $u^{(0)}$, such that $(u, u^{(0)}) \in E(Q_n)$. Let D_i be the set of all edges with one end in Q^0_{n-1} and the other in Q^1_{n-1}. These edges are called crossing edges in the ith dimension between Q^0_{n-1} and Q^1_{n-1}. We also call D_i the set of all ith dimension edges. Consequently, $|D_i| = 2^{n-1}$ for all $0 \leqslant i \leqslant n-1$.

2. Some preliminaries

To prove our main theorem, we need some preliminary results.

Lemma 1. *(See [5].) Q_n is edge-bipancyclic, and is $(n-2)$-edge-fault-tolerant edge-bipancyclic, for $n \geqslant 3$.*

Lemma 2. *(See [9].) Each edge of $Q_4 - F$ lies on a cycle of every even length from 6 to $2^n = 16$ for any $F \subset E(Q_4)$ with $|F| = 3$, provided not all the faulty edges in F are incident with the same vertex.*

Lemma 3. *(See [9].) Any two edges in Q_n are included in a hamiltonian cycle, for $n \geqslant 2$.*

The above lemma can be improved; In addition, we have the following lemmas to simplify our proof.

Lemma 4. *Let $C_0 = \langle u, P_0, v, u \rangle$ be a cycle in Q^0_{n-1} with its even length from l_0 to 2^{n-1}, and $C_1 = \langle u^{(1)}, P_1, v^{(1)}, u^{(1)} \rangle$ be a cycle in Q^1_{n-1} with its even length from l_1 to 2^{n-1}. Then $C = \langle u, P_0, v, v^{(1)}, P_1, u^{(1)}, u \rangle$ is a cycle in Q_n with its even length from $l_0 + l_1$ to 2^n.*

Proof. The proof of this lemma is omitted. □

Lemma 5. *Let Q_n be an n-dimensional hypercube, $n \geqslant 2$, and let e_1 and e_2 be two edges in the same dimension i. Then there exists another dimension $j \neq i$ such that decomposing Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j, we have* (1) *neither e_1 nor e_2 is a crossing edge,* (2) *not e_1 and e_2 are in the same subcube.*

Proof. Let $e_1 = (a, b)$ and $e_2 = (s, t)$ be two edges in the same dimension i. Let $a = a_n \ldots a_i \ldots a_1$ and $s = s_n \ldots s_i \ldots s_1$. Then $b = a_n \ldots \overline{a_i} \ldots a_1$ and $t = s_n \ldots \overline{s_i} \ldots s_1$. Since $e_1 \neq e_2$ and $n \geqslant 2$, there exists another dimension $j \neq i$, such that $a_j \neq s_j$. We decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j. Then, e_1 and e_2 are not crossing edges and are in the different subcubes. □

Lemma 6. *Consider an n-dimensional hypercube Q_n, for $n \geqslant 4$. Let e_0, e_1 and e_2 be any three edges in Q_n, there is a cycle C containing e_1 and e_2 in $Q_n - \{e_0\}$ with the length $l(C) = 2^n$, $2^n - 2$ and $2^n - 4$.*

Proof. To prove this lemma, we consider the following two cases:

Case 1: *Both e_1 and e_2 are in the same dimension, say dimension i.* By Lemma 5, we can choose a dimension j such that e_1 and e_2 are in different subcubes. Without loss of generality, we assume that e_1 is in Q_{n-1}^0 and e_2 is in Q_{n-1}^1. We then consider two cases:

1.1: *e_0 is not a crossing edge.* We assume, without loss of generality, that e_0 is in Q_{n-1}^0. By Lemma 1, in $Q_{n-1}^0 - \{e_0\}$, there exists a cycle C_0 of every even length $4 \leqslant l(C_0) \leqslant 2^{n-1}$ going through e_1. Since $n \geqslant 4$, we can choose an edge (u, v) on cycle C_0 such that $(u, v) \neq e_1$, and $(u^{(1)}, v^{(1)}) \neq e_2$. By Lemma 3, in Q_{n-1}^1, there exists a cycle C_1 of length 2^{n-1} going through e_2 and $(u^{(1)}, v^{(1)})$. Thus, the conclusion follows according to Lemma 4.

1.2: *e_0 is a crossing edge.* By Lemma 1, there exists a C_0 of every even length $4 \leqslant l(C_0) \leqslant 2^{n-1}$ going through e_1 in Q_{n-1}^0. We can choose an edge (u, v) on cycle C_0 such that (u, v) is not adjacent to e_0 and $(u, v) \neq e_1$ and $(u^{(1)}, v^{(1)}) \neq e_2$, since $n \geqslant 4$. By definition, $(u^{(1)}, v^{(1)})$ is an edge in Q_{n-1}^1. By Lemma 3, there exists a hamiltonian cycle C_1 going through e_2 and $(u^{(1)}, v^{(1)})$ in Q_{n-1}^1. So the conclusion follows according to Lemma 4.

Case 2: *e_1 and e_2 are in different dimensions.* Suppose that e_0 is in the ith dimension. We decompose Q_n into Q_{n-1}^0 and Q_{n-1}^1 by dimension i. Then, e_0 is a crossing edge. Next, we consider two further cases:

2.1: *Either e_1 or e_2 is a crossing edge.* Without loss of generality, we assume that e_1 is a crossing edge, and e_2 is in Q_{n-1}^0. Let $e_1 = (u, u^{(1)})$, where $u \in V(Q_{n-1}^0)$ and $u^{(1)} \in V(Q_{n-1}^1)$. Since $n \geqslant 4$, there is a neighbor of u, say v, such that $(u, v) \neq e_2$ and (u, v) is not adjacent to e_0. By Lemma 3, there exists a hamiltonian cycle C_0 going through (u, v) and e_2 in Q_{n-1}^0. By Lemma 1, in Q_{n-1}^1, there exists a cycle C_1 of every even length $4 \leqslant l(C_1) \leqslant 2^{n-1}$ going through $(u^{(1)}, v^{(1)})$. By Lemma 4, the conclusion follows.

2.2: *Both e_1 and e_2 are not crossing edge.* If e_1 and e_2 are in different subcubes, this subcase is similar to case 1.2, and the proof is omitted. Otherwise, both e_1 and e_2 are in the same subcube. We assume, without loss of generality, that e_1 and e_2 are in Q_{n-1}^0. By Lemma 3, there exists a hamiltonian cycle C_0 going through e_1 and e_2 in Q_{n-1}^0, and $l(C_0) = 2^{n-1}$. Since $n \geqslant 4$, there is a third edge (u, v) other than e_1 and e_2 on cycle C_0, and (u, v) is not adjacent to e_0. By Lemma 1, there exists a cycle C_1 of every even length $4 \leqslant l(C_1) \leqslant 2^{n-1}$ going through $(u^{(1)}, v^{(1)})$ in Q_{n-1}^1. By Lemma 4, the conclusion follows. □

Let F be a set of faulty edges of Q_n. Suppose that we decompose Q_n into Q_{n-1}^0 and Q_{n-1}^1 by dimension j, and let $F_L = F \cap E(Q_{n-1}^0)$, $F_R = F \cap E(Q_{n-1}^1)$. Suppose that F is a set of conditional faulty edges of Q_n. If we arbitrarily decompose Q_n into Q_{n-1}^0 and Q_{n-1}^1 by a dimension, F_L and F_R may not be conditional faulty edges in Q_{n-1}^0 and Q_{n-1}^1, respectively. However, we will show that it is always possible to find some suitable dimension such that decomposing by this dimension, both F_L and F_R are conditional faulty sets in Q_{n-1}^0 and Q_{n-1}^1, respectively.

Lemma 7. *Consider an n-dimensional hypercube Q_n, for $n \geqslant 4$. Let F be a set of conditional faulty edges with $|F| = 2n - 5$. There are at most two vertices in Q_n incident with $(n-2)$ faulty edges.*

Proof. If there are three vertices in Q_n incident with $(n-2)$ faulty edges, the number of faulty edge F is at least $3n - 8$. However, $(3n - 8) > (2n - 5)$ for all $n \geqslant 4$ which is a contradiction. □

Lemma 8. *Consider an n-dimensional hypercube Q_n, $n \geqslant 4$. Let F be a set of conditional faulty edges with $|F| = 2n - 5$. If there are two vertices x and y both incident with $n - 2$ faulty edges, then x and y are adjacent in Q_n and the edge (x, y) is a faulty edge. Suppose that (x, y) is in dimension j. Then decomposing Q_n into Q_{n-1}^0 and Q_{n-1}^1 by dimension j, both F_L and F_R are sets of conditional faulty edges in Q_{n-1}^0 and Q_{n-1}^1, respectively. Moreover, $|F_L| \leqslant 2n - 6$ and $|F_R| \leqslant 2n - 6$.*

Proof. If there are two vertices x and y in Q_n incident with $(n-2)$ faulty edges, then these two vertices are connected by a faulty edge. Otherwise, $|F| = 2(n-2) = 2n - 4 > 2n - 5$ which is a contradiction. Suppose the edge (x, y) is in dimension j, we decompose Q_n into two subcubes. It is clearly that each vertex in Q_{n-1}^0 and Q_{n-1}^1 is still incident with at least two healthy edges, and both F_L and F_R are conditional faulty edges in Q_{n-1}^0 and Q_{n-1}^1, respectively. Then, $|F_L| = |F_R| = n - 3 \leqslant 2n - 6$, for $n \geqslant 4$. □

Lemma 9. *Consider an n-dimensional hypercube Q_n, for $n \geqslant 4$. Let F be a set of conditional faulty edges with $|F| = 2n - 5$. Suppose that there exists exactly one vertex x having $(n-2)$ faulty edges incident with it. Since $n - 2 \geqslant 2$, let e_1 and e_2 be two faulty edges incident with x, and let e_1 and e_2 be jth and kth dimension edges, respectively. Then decomposing Q_n into Q_{n-1}^0 and Q_{n-1}^1 by either one of these two dimensions j and* *k, F_L and F_R are still sets of conditional faulty edges in Q^0_{n-1} and Q^1_{n-1}, respectively. Moreover, $|F_L| \leqslant 2n-6$ and $|F_R| \leqslant 2n-6$.*

Proof. If there exists only one vertex x having $(n-2)$ faulty edges incident with it, there are at least two faulty edges e_1 and e_2 incident with it, since $n \geqslant 4$. Obviously, these two faulty edges are in different dimensions. Without loss of generality, we may assume that e_1 is in dimension j and e_2 is in dimension k, for $j \neq k$. We can decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by either jth or kth dimension, and either e_1 or e_2 is a crossing edge. Therefore, each vertex in these two subcubes is incident with at least two healthy edges and $|F_L| \leqslant 2n-6$ and $|F_R| \leqslant 2n-6$. □

Lemma 10. *Let Q_n be an n-dimensional hypercube, F be a set of faulty edges with $|F| \geqslant 2$, and e be a healthy edge, $n \geqslant 2$. Then there exists a dimension j, decomposing Q_n into Q^0_{n-1} and Q^1_{n-1} by this dimension, such that e is not a crossing edge and not all the faulty edges are in the same subcube.*

Proof. Suppose that $e = (u, v)$ is in dimension i. If there is a faulty edge f not in dimension i, say in dimension j. We decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j. Then f is a crossing edge but e is not, and all the faulty edges are not in the same subcube. Otherwise, all the faulty edges are in the same dimension i as e is in. We now choose any two faulty edges f_1 and f_2 in F. By Lemma 5, Q_n can be decomposed into Q^0_{n-1} and Q^1_{n-1} by some dimension $j \neq i$ such that edges f_1 and f_2 are not in the same subcube, and e is not a crossing edge. □

3. Main theorem

We now prove our main result.

Theorem 1. *Let Q_n be an n-dimensional hypercube, and F be a set of conditional faulty edges with $|F| \leqslant 2n-5$. Then each edge of the conditional faulty hypercube $Q_n - F$ lies on a cycle of every even length from 6 to 2^n, for $n \geqslant 3$.*

Proof. We prove this theorem by induction on n. For $n = 3$, since $2n-5 = n-2$, by Lemma 1, the result is true. For $n = 4$, $2n-5 = n-1$, by Lemma 2, the result holds. Assume the theorem holds for $n-1$, for some $n \geqslant 5$, we shall show that it is true for n.

As we mentioned before, we may assume $|F| = 2n-5$. Let $e = (u, v)$ be an edge in $Q_n - F$. We shall find a cycle of every even length from 6 to 2^n passing through e in $Q_n - F$. Assume that e is an ith dimension edge, $e \in D_i$, for some $i \in \{1, 2, \ldots, n\}$. The proof is divided into three major cases:

Case 1: *There are two vertices x and y in Q_n incident with $(n-2)$ faulty edges.* By Lemma 8, (x, y) is an edge in Q_n and is a faulty edge. We denote this edge by e_f. Suppose that e_f is a jth dimension edge. We decompose Q_n into Q^0_{n-1} and Q^1_{n-1} by dimension j. We then consider two further cases:

1.1: *$e_f = (x, y)$ and $e = (u, v)$ are in the same dimension.* Thus, $j = i$ and $e_f \in D_i$ (Fig. 1(a)). In this case, e is an edge crossing Q^0_{n-1} and Q^1_{n-1}. Without loss of generality, assume that $u \in V(Q^0_{n-1})$ and $v \in V(Q^1_{n-1})$. Since $n \geqslant 5$, u has a neighboring vertex $w \in V(Q^0_{n-1})$, by the definition of hypercube, $w^{(1)}$ is a neighbor of v such that the edge $(w, w^{(1)})$ is a healthy edge and $(w, w^{(1)})$ is a crossing edge between

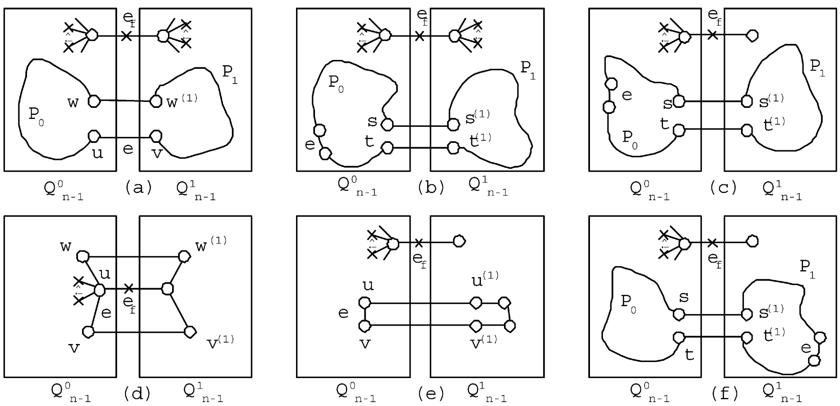

Fig. 1. Illustration for theorem.

Q^0_{n-1} and Q^1_{n-1}. By Lemma 1, there exists a cycle C_0 in $Q^0_{n-1} - F_L$ passing through (u, w) of every even length $4 \leqslant l(C_0) \leqslant 2^{n-1}$ and a cycle C_1 in $Q^1_{n-1} - F_R$ going through $(v, w^{(1)})$ of every even length $4 \leqslant l(C_1) \leqslant 2^{n-1}$. We write C_0 as $\langle u, P_0, w, u\rangle$, and C_1 as $\langle v, P_1, w^{(1)}, v\rangle$. Thus, $\langle u, P_0, w, w^{(1)}, v, u\rangle$ is a cycle of length 6 with $l(P_0) = 3$. By Lemma 4, $\langle u, P_0, w, w^{(1)}, P_1, v, u\rangle$ can form a cycle of every even length from 8 to 2^n through e in $Q_n - F$.

1.2: *e_f and e are in different dimensions*. Thus, $j \neq i$ and $e_f \notin D_i$ (Fig. 1(b)). In this case, e is in Q^0_{n-1} or Q^1_{n-1}. Without loss of generality, we may assume that $e \in E(Q^0_{n-1})$. By Lemma 1, there exists a cycle C in $Q^0_{n-1} - F_L$ going through the edge e of every even length l, $6 \leqslant l \leqslant 2^{n-1}$. Let C_0 be a cycle of length $2^{n-1} - 2$ or 2^{n-1} passing through e in $Q^0_{n-1} - F_L$. Since $n \geqslant 5$, there exists an edge (s, t) on C_0 such that neither s nor t is adjacent to e_f and $(s, t) \neq e$. By definition, $(s^{(1)}, t^{(1)})$ is an edge in Q^1_{n-1}, and $(s, s^{(1)})$, $(t, t^{(1)})$ are healthy edges. By Lemma 1, there exists a cycle C_1 in $Q^1_{n-1} - F_R$ through $(s^{(1)}, t^{(1)})$ of every even length $4 \leqslant l(C_1) \leqslant 2^{n-1}$. Thus, the conclusion follows according to Lemma 4.

Case 2: *There is exactly one vertex in Q_n incident with $(n - 2)$ faulty edges.* Let x be the vertex having $(n - 2)$ faulty edges incident with it. Let f_1 and f_2 be two faulty edges incident with x, so f_1 and f_2 are in different dimensions j and k. By Lemma 9, decomposing Q_n into Q^0_{n-1} and Q^1_{n-1} by either jth or kth dimension, both $F_L = F \cap E(Q^0_{n-1})$ and $F_R = F \cap E(Q^1_{n-1})$ are sets of conditional faulty edges in Q^0_{n-1} and Q^1_{n-1}, respectively. Between dimension j and k, we choose one to decompose Q_n into Q^0_{n-1} and Q^1_{n-1}, say dimension j, such that the required edge e is not a crossing edge. Therefore, there is an faulty edge crossing Q^0_{n-1} and Q^1_{n-1}, we denote this edge by e_f, and $e_f \in F \cap D_j$ is incident with x. Without loss of generality, we may assume that $x \in V(Q^0_{n-1})$.

2.1: *Suppose $|F_L| \leqslant 2n - 7$ and $|F_R| \leqslant 2n - 7$* (Fig. 1(c)). Without loss of generality, we further assume that $e \in E(Q^0_{n-1})$. By induction hypothesis, there exists a cycle C in $Q^0_{n-1} - F_L$ of every even length $6 \leqslant l(C) \leqslant 2^{n-1}$ passing through e. Let C_0 be a cycle of length $2^{n-1} - 4 \leqslant l(C_0) \leqslant 2^{n-1}$ through e in $Q^0_{n-1} - F_L$. Since $|C_0 - e| \geqslant 2^{n-1} - 4 - 1 > 2(2n - 5) \geqslant 2|F \cap D_j|$, for all $n \geqslant 5$. There exists an edge (s, t) on C_0 such that (s, t) is not e, and both $(s, s^{(1)})$ and $(t, t^{(1)})$ are healthy edges. By induction hypothesis, there exists a cycle C_1 in $Q^1_{n-1} - F_R$ of every even length $6 \leqslant l(C_1) \leqslant 2^{n-1}$ passing through $(s^{(1)}, t^{(1)})$. By Lemma 4, the conclusion follows.

2.2: $|F_L| = 2n - 6$. In this case, $|F \cap D_j| = 1$ and $|F \cap E(Q^1_{n-1})| = |F_R| = 0$.

2.2.1: *e is in subcube Q^0_{n-1}*. To find a cycle of length 6 passing through $e = (u, v)$, we discuss the case that whether e is incident with x or not. If e is incident with x, without loss of generality, we assume that $u = x$ (Fig. 1(d)). Thus, $(v, v^{(1)})$ is a healthy edge. Since F_L is a set of conditional faulty edges in Q^0_{n-1}, vertex $u = x$ has two healthy edges incident with it. Let w be a neighbor of u in Q^0_{n-1} such that (w, u) and $(w, w^{(1)})$ are healthy edges and $w \neq v$. Thus, $\langle u, v, v^{(1)}, u^{(1)}, w^{(1)}, w, u\rangle$ is a cycle of length 6 in $Q_n - F$. Otherwise, e is not incident with x, then $(u, u^{(1)})$ and $(v, v^{(1)})$ are healthy edges (Fig. 1(e)). By Lemma 1, there exists a cycle $C_1 = \langle u^{(1)}, P_1, v^{(1)}, u^{(1)}\rangle$ of length four in Q^1_{n-1} through the edge $(u^{(1)}, v^{(1)})$. Thus, $\langle u, u^{(1)}, P_1, v^{(1)}, v, u\rangle$ is a cycle of length 6 in $Q_n - F$, where $l(P_1) = 3$.

Let e_1 be a faulty edge in Q^0_{n-1} that is not adjacent to e_f. Though e_1 is a faulty edge, we treat it as a healthy edge temporarily, then the total number of faulty edge in Q^0_{n-1} is $2n - 7$. By induction hypothesis, there exists a cycle C_0 of every even length $6 \leqslant l(C_0) \leqslant 2^{n-1}$ going through e in $Q^0_{n-1} - \{F_L - \{e_1\}\}$. If C_0 passes e_1, we choose e_1, or else, we choose any one edge other then e on C_0 which is not adjacent to e_f. Let the chosen edge be denoted by (s, t). We write cycle C_0 as $\langle s, P_0, t, s\rangle$. Since $|F \cap D_j| = 1$ and $|F_R| = 0$, $(s, s^{(1)})$, $(t, t^{(1)})$ and $(s^{(1)}, t^{(1)})$ are all healthy edges. Thus, $\langle s, P_0, t, t^{(1)}, s^{(1)}, s\rangle$ is a cycle of length 8 in $Q_n - F$ if $l(P_0) = 5$. Suppose that $10 \leqslant l \leqslant 2^n$ and l is even. By Lemma 1, in Q^1_{n-1}, there exists a cycle C_3 of length $4 \leqslant l(C_3) \leqslant 2^{n-1}$ passing through $(s^{(1)}, t^{(1)})$. We write C_3 as $\langle s^{(1)}, P_3, t^{(1)}, s^{(1)}\rangle$. By Lemma 4, $\langle s, P_0, t, t^{(1)}, P_3, s^{(1)}, s, t\rangle$ is a cycle of length l through e in $Q_n - F$.

2.2.2: *e is in subcube Q^1_{n-1}* (Fig. 1(f)). By Lemma 1, there exists a cycle C of every even length $4 \leqslant l \leqslant 2^{n-1}$ passing through e in Q^1_{n-1}. Suppose that $2^{n-1} + 2 \leqslant l \leqslant 2^n$ and l is even. Since F_L is a set of conditional faulty edges, there are at most $(n - 3)$ faulty edges adjacent to e_f in Q^0_{n-1}. For $n \geqslant 5$, $n - 3 \geqslant 2$, we can choose a faulty edge $e_2 = (s, t)$ in Q^0_{n-1} such that e_2 is not adjacent to e_f and $(s^{(1)}, t^{(1)})$ is not e. Treating the edge e_2 as a healthy edge, by induction hypothesis, there exists a cycle C_0 of length $6 \leqslant l(C_0) \leqslant 2^{n-1}$ going through e_2 in $Q^0_{n-1} - \{F_L - \{e_2\}\}$. We observe that $(s, s^{(1)})$ and $(t, t^{(1)})$ are healthy edges. By Lemma 6, there exists a cycle C_1 of every length $2^{n-1}-4$, $2^{n-1}-2$, or 2^{n-1} through $(s^{(1)},t^{(1)})$ and e in Q^1_{n-1}. By Lemma 4, the conclusion follows.

Case 3: *Every vertex in Q_n is incident with at most $(n-3)$ faulty edges.* In this case, suppose that $e=(u,v)$ is in dimension i. By Lemma 10, Q_n can be decomposed into Q^0_{n-1} and Q^1_{n-1} by a dimension j different from i such that e is not a crossing edge and not all the faulty edges are in the same subcube. Then $|F_L|\leqslant 2n-6$ and $|F_R|\leqslant 2n-6$. Next, we consider two further cases:

3.1: *At least one faulty edge is a jth dimension edge.* Thus, $|F\cap D_j|\neq 0$.

We then consider two cases: (a) $|F_L|\leqslant 2n-7$ and $|F_R|\leqslant 2n-7$, and (b) $|F_L|=2n-6$ or $|F_R|=2n-6$. The proof of this subcase is exactly the same as that of case 2.

3.2: *None of the faulty edges is a jth dimension edge.* Thus, $|F\cap D_j|=0$.

3.2.1: $|F_L|\leqslant 2n-7$ and $|F_R|\leqslant 2n-7$. Without loss of generality, we may assume that $e\in E(Q^0_{n-1})$. By induction hypothesis, there exists a cycle C of every even length $6\leqslant l(C)\leqslant 2^{n-1}$ in $Q^0_{n-1}-F_L$ passing through e. Let C_0 be a cycle of every even length $2^{n-1}-4\leqslant l(C_0)\leqslant 2^{n-1}$ going through e in $Q^0_{n-1}-F_L$. There exists an edge (s,t) other than e in C_0. Since $|F\cap D_j|=0$, $(s,s^{(1)})$ and $(t,t^{(1)})$ are healthy edges. We write C_0 as $\langle s,P_0,t,s\rangle$. By induction hypothesis, there exists a cycle C_1 of every even length $6\leqslant l(C_1)\leqslant 2^{n-1}$ in $Q^1_{n-1}-\{F_R-(s^{(1)},t^{(1)})\}$ through $(s^{(1)},t^{(1)})$. Thus, the conclusion follows according to Lemma 4.

3.2.2: *Suppose $|F_L|=2n-6$ or $|F_R|=2n-6$, say the former case.* In this case, $|F_R|=1$. We then consider two cases: (a) e is in subcube Q^0_{n-1}, and (b) e is in subcube Q^1_{n-1}.

(a) *$e=(u,v)$ is in subcube Q^0_{n-1}.* Since $|F\cap D_j|=0$, both $(u,u^{(1)})$ and $(v,v^{(1)})$ are healthy edges. Let l be an even number with $6\leqslant l\leqslant 2^{n-1}$. By Lemma 1, there exists a cycle C_1 of every even length from 4 to 2^{n-1} passing through $(u^{(1)},v^{(1)})$ in $Q^1_{n-1}-\{F_R-(u^{(1)},v^{(1)})\}$. We write C_1 as $\langle u^{(1)},P_1,v^{(1)},u^{(1)}\rangle$. No matter $(u^{(1)},v^{(1)})$ is healthy or not, $\langle u,u^{(1)},P_1,v^{(1)},v,u\rangle$ forms a cycle of length l through e in Q_n-F. Suppose that $2^{n-1}+2\leqslant l\leqslant 2^n$. Let e_1 be a faulty edge in Q^0_{n-1}. We may treat e_1 as a healthy edges temporarily. By induction hypothesis, there exists a cycle C_0 of length $6\leqslant l(C_0)\leqslant 2^{n-1}$ going through e in $Q^0_{n-1}-\{F_L-\{e_1\}\}$. If C_0 passes the edge e_1, we choose e_1 to be deleted. Otherwise, we choose another edge other than e on cycle C_0. Let the chosen edge be denoted by (s,t). We write the cycle C_0 as $\langle s,P_0,t,s\rangle$. Treating $(s^{(1)},t^{(1)})$ as a healthy edge, by Lemma 1, there exists a cycle C_3 of every even length from 4 to 2^{n-1} passing through $(s^{(1)},t^{(1)})$ in $Q^1_{n-1}-\{F_R-(s^{(1)},t^{(1)})\}$. By Lemma 4, the conclusion follows.

(b) *e is in subcube Q^1_{n-1}.* Let e_1 be the only faulty edge in Q^1_{n-1}. By Lemma 1, there exists a cycle C of every even length from 6 to 2^{n-1} through e in $Q^1_{n-1}-\{e_1\}$. Suppose that $2^{n-1}+2\leqslant l\leqslant 2^n$, and l is even. Let $e_0=(s,t)$ be a faulty edge in Q^0_{n-1} such that $(s^{(1)},t^{(1)})\neq e$ and $(s^{(1)},t^{(1)})\neq e_1$. By induction hypothesis, there exists a cycle C_0 of length $6\leqslant l(C_0)\leqslant 2^{n-1}$ in $Q^0_{n-1}-\{F_L-\{e_0\}\}$ going through e_0. If $(s^{(1)},t^{(1)})=e_1$, treat e_1 as a healthy edge temporarily, by Lemma 6, there exists a cycle C_1 of length $2^{n-1}-4$, $2^{n-1}-2$, or 2^{n-1}, respectively, going through both $(s^{(1)},t^{(1)})$ and e in Q^1_{n-1}. By Lemma 4, the conclusion follows. Otherwise, if $(s^{(1)},t^{(1)})\neq e_1$, by Lemma 6, there exists a cycle C_3 of length 2^{n-1}, $2^{n-1}-2$, or $2^{n-1}-4$, respectively, going through both e and $(s^{(1)},t^{(1)})$ in $Q^1_{n-1}-\{e_1\}$. Thus, the conclusion follows according to Lemma 4.

This completes the proof. □

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References

[1] M.Y. Chan, S.J. Lee, On the existence of hamiltonian circuits in faulty hypercubes, SIAM J. Discrete Math. 4 (1991) 511–527.
[2] K. Day, A. Tripathi, Embedding of cycles in arrangement graph, IEEE Trans. Comput. 12 (1993) 1002–1006.
[3] A. Germa, M.C. Heydemann, D. Sotteau, Cycles in the cube connected cycles graph, Discrete Appl. Math. 83 (1998) 135–155.
[4] S.C. Hwang, G.H. Chen, Cycle in butterfly graphs, Networks 35 (2000) 161–171.
[5] T.K. Li, C.H. Tsai, J.J.M. Tan, L.H. Hsu, Bipanconnectivity and edge-fault-tolerant bipancyclicity of hypercubes, Inform. Process. Lett. 87 (2003) 107–110.
[6] J. Mitchem, E. Schmeichel, Pancyclic and bipancyclic graphs—a survey, in: Proc. First Colorado Symp. on Graphs and Applications, Boulder, CO, Wiley-Interscience, New York, 1985, pp. 271–278.
[7] J.A. Bondy, Pancyclic graphs, I, J. Combin. Theory (1971) 80–84.
[8] C.H. Tsai, Linear array and ring embeddings in conditional faulty hypercubes, Theoret. Comput. Sci. 314 (2004) 431–443.
[9] J.M. Xu, Z.Z. Du, M. Xu, Edge-fault-tolerant edge-bipancyclicity of hypercubes, Inform. Process. Lett. 96 (2005) 146–150.