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A new solution for a partially penetrating constant-rate pumping well with a finite-thickness skin

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SUMMARY

A mathematical model describing the constant pumping is developed for a partially penetrating well in a heterogeneous aquifer system. The Laplace-domain solution for the model is derived by applying the Laplace transforms with respect to time and the finite Fourier cosine transforms with respect to vertical co-ordinates. This solution is used to produce the curves of dimensionless drawdown *versus* dimensionless time to investigate the influences of the patch zone and well partial penetration on the drawdown distributions. The results show that the dimensionless drawdown depends on the hydraulic properties of the patch and formation zones. The effect of a partially penetrating well on the drawdown with a negative patch zone is larger than that with a positive patch zone. For a single-zone aquifer case, neglecting the effect of a well radius will give significant error in estimating dimensionless drawdown, especially when dimensionless distance is small. The dimensionless drawdown curves for cases with and without considering the well radius approach the Hantush equation (*Advances in Hydroscience*. Academic Press: New York, 1964) at large time and/or large distance away from a test well. Copyright © 2007 John Wiley & Sons, Ltd.

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KEY WORDS: groundwater; analytical solution; constant pumping test; heterogeneous aquifer; finite Fourier cosine transforms; Laplace transforms; partially penetrating well

1. INTRODUCTION

The pumping test with a constant pumping rate is a popular technique for high-transmissivity aquifers. The aquifer parameters such as transmissivity and storage coefficient can be determined from a test-data analysis with the measured drawdowns. These aquifer parameters have been widely

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utilized for estimating the water resources in the ground-water hydraulics. Most of the data analysis methods used in the ground-water field are developed for a fully penetrating well and the well diameter is considered as infinitesimally small. In addition, the aquifer is assumed as homogenous, isotropic, infinite-extent and with a constant thickness. For a small-diameter well, the aquifer parameters are usually estimated according to the Theis formula [1] in which the pumping well was considered as a line source. On the other hand, the aquifer parameters for a large-diameter well may be founded using an approach suggested by Papadopulos and Cooper [2] in which the wellbore storage was also considered. Several researchers had provided the analytical models or results for the various types of problems in engineering applications, e.g. [3–5]. Hantush [3] developed an analytical model of a constant pumping test in a partially penetrating well. His solution in the Laplace domain was obtained *via* the Laplace transform and the finite Fourier sine transform, and the time-domain solution was derived using the inverse Laplace transforms.

An aquifer having a small region of anomalous hydrogeological properties may be called a cylindrical inhomogeneity (patchy aquifer). A patchy aquifer may have the radius of heterogeneous cylinder (patch region) up to 60 m [6] and can be considered as a composite aquifer system. The patch zone may affect the pumping drawdown; consequently, the aquifer-drawdown distribution depends on the thickness and properties of the patch and formation zones.

In the well constructions, the well drilling induces the invasion of drilling mud into aquifer and may produce a positive patch zone that has a lower permeability than that of the original formation. In contrast, the extensive well development and substantial spalling and fracturing of borehole wall may increase the permeability of the adjacent formation around the wellbore and form a negative patch zone. In any case, the thickness of a patch zone may range from a few millimetres to several meters and thus must be considered in the pumping-test data analyses [7]. For wells in a heterogeneous aquifer, Novakowski [7] presented a composite analytical solution by using the Laplace transforms. He provided some type curves generated from the Laplace-domain solution and used them to explore the effects of the wellbore storage and patch zone on the head distributions. Using the Laplace transforms and Bromwich integral method, Yeh *et al.* [8] obtained the time-domain solution for a radial two-layer drawdown equation for an aquifer under constant-flux pumping in a finite-radius well. They also proposed a numerical method to efficiently evaluate the solution with accuracy to five decimal places. The existing solutions, addressing the problems of the partially penetrating well in a heterogeneous aquifer system, had been developed mostly under some simplified conditions. Examples for the solutions in a petroleum industry are Bixel and van Poolen [9] and Jargon [10] and in a groundwater hydraulic are Barker and Herbert [6] and Butler [11]. Cassiani and Kabala [12] mentioned that those articles simplified the problems by assuming a uniform point flux along a screened portion of the wellbore. Considering the wellbore storage and skin effect, Park and Zhan [13] provided a solution of groundwater flow for a finite-diameter horizontal well in an anisotropic leaky aquifer. Their solution was derived based on the separation of the source and geometric functions. Zhan and Bian [14] had derived the closed-form solutions of the steady-state leakage rates and volumes for both the constant-rate and constant-drawdown pumping wells. With the scale-invariant relationship, those solutions of the total leakage rate and volume can be generalized to finite size aquifers with impermeable boundaries.

Markle *et al.* [15] developed an analytical model for a constant-head test conducted in a vertically fractured media. Their solution was presented in the Laplace domain and numerically inverted to obtain the values of the time-domain solution. They provided small- and large-time approximations to analyse the test data. More recently, Moench [16] presented a Laplace-domain solution for flow to a finite-diameter well in a homogeneous, anisotropic unconfined aquifer. His solution accounts for the effects of the wellbore storage and patch zone and allows for the non-instantaneous release of water from an unsaturated zone. Cassiani *et al.* [17] developed a semi-analytical solution for a pumping test on the partially penetrating wells in a confined aquifer that accounting not only for the wellbore storage, infinitesimal skin, and anisotropic aquifer, but also for a mixed-type boundary condition at a well face. Their solution was obtained *via* the method of the dual integral equation. Inversion of the Laplace transform and Fourier transform were handled numerically *via* the Stehfest algorithm and the fast Fourier transform.

The purpose of this article is to present a new mathematical model describing a constant pumping test in a partially penetrating well that has a patch zone in a radial confined aquifer system. The Laplace-domain solution is derived by applying the Laplace transforms with respect to time and the finite Fourier cosine transforms with respect to vertical co-ordinates. Then the time-domain results are evaluated when applying the modified Crump algorithm [18, 19] to invert the Laplace-domain solution. Simplified solutions obtained with the flux–flux discontinuous boundary conditions (e.g. [3, 8]) are compared with our solution.

2. MATHEMATICAL MODEL

A partially penetrating well in a heterogeneous aquifer system is illustrated in Figure 1. Several assumptions made for the solutions in terms of drawdowns are:

(1) The aquifer is anisotropic, infinite-extent and with a constant thickness.
(2) The well is partially penetrated with a finite radius.
(3) The pumping flow rate is maintained at a constant value throughout the whole test period.
(4) The patch zone has uniform thickness in the z-direction within the top and bottom impermeable layers.

A term representing the vertical flow is included in the equations of a radial confined aquifer system to account for the effect of well partial penetration. According to the assumptions mentioned above, the governing equations of drawdowns, $s(r, z, t)$, can be expressed within the patch and formation zones, respectively, as

$$K_{r1}\frac{\partial^2 s_1(r,z,t)}{\partial r^2}+\frac{K_{r1}}{r}\frac{\partial s_1(r,z,t)}{\partial r}+K_{z1}\frac{\partial^2 s_1(r,z,t)}{\partial z^2}=S_{s1}\frac{\partial s_1(r,z,t)}{\partial t},\quad r_w\leqslant r\leqslant r_1 \tag{1}$$

and

$$K_{r2}\frac{\partial^2 s_2(r,z,t)}{\partial r^2}+\frac{K_{r2}}{r}\frac{\partial s_2(r,z,t)}{\partial r}+K_{z2}\frac{\partial^2 s_2(r,z,t)}{\partial z^2}=S_{s2}\frac{\partial s_2(r,z,t)}{\partial t},\quad r_1\leqslant r<\infty \tag{2}$$

where the subscript 1 denotes the patch zone, the subscript 2 denotes the formation zone, K_r is the hydraulic conductivity in the radial direction, K_z is the hydraulic conductivity in the vertical direction, S_s is the specific storage, r is the radial distance from the centreline of pumping well, r_w is the radius of pumping well, r_1 is the outer radius of patch zone and t is the time from the star of pumping.

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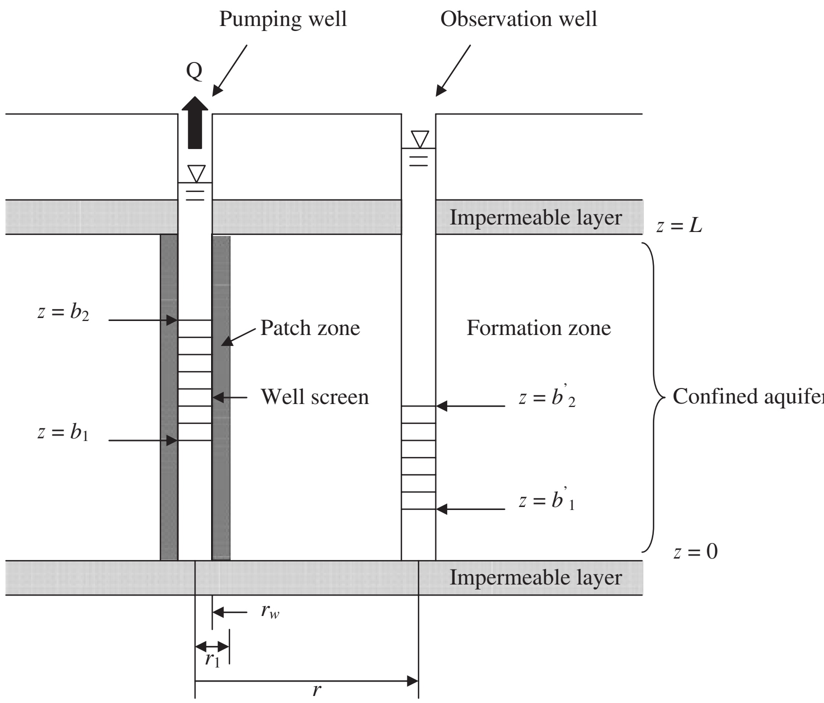

Figure 1. Schematic diagram of the well and aquifer configurations.

The drawdowns are initially assumed to be zero within both the patch and formation zones, these are

$$s_1(r, z, 0) = s_2(r, z, 0) = 0 \tag{3}$$

The drawdown tends to be zero when r approaches infinity. Therefore, the outer boundary condition for the formation zone is given by

$$s_2(\infty, z, t) = 0 \tag{4}$$

The continuities of drawdown and flux between the patch and formation zones, respectively, require

$$s_1(r_1, z, t) = s_2(r_1, z, t) \tag{5}$$

and

$$K_{r1} \frac{\partial s_1(r_1, z, t)}{\partial r} = K_{r2} \frac{\partial s_2(r_1, z, t)}{\partial r} \tag{6}$$

The lower and upper boundary conditions in a z-direction are, respectively,

$$\frac{\partial s_1(r, 0, t)}{\partial z} = \frac{\partial s_2(r, 0, t)}{\partial z} = 0 \tag{7}$$

and

$$\frac{\partial s_1(r, L, t)}{\partial z} = \frac{\partial s_2(r, L, t)}{\partial z} = 0 \tag{8}$$

where L is the thickness of the confined aquifer.

According to Darcy's law, the boundary condition for maintaining a constant flux across the screen is assumed as

$$\frac{\partial s_1(r_{\mathrm{w}}, z, t)}{\partial r} = -\frac{Q}{2\pi r_{\mathrm{w}} K_{\mathrm{r1}}(b_2 - b_1)}[U(z - b_1) - U(z - b_2)], \quad 0 \leqslant z \leqslant L \tag{9}$$

where Q is the pumping rate, b_1 and b_2 are, respectively, the lower and upper vertical co-ordinates of well screen, and $U(\bullet)$ is a unit step function defining that $U(z - b_i)$ equals one when $b_i \leqslant z$ but equals zero otherwise for $i = 1$ or 2. Equation (9) assumes that the flow rate along the well screen is uniform. Such an assumption is similar to that made in Reference [20, p. 304, (15) and (16)].

2.1. Analytical solutions

To solve the boundary value problem, the Laplace transform and finite Fourier cosine transform are applied to the governing equations and boundary conditions. The Laplace transform is taken with respect to time and the finite Fourier cosine transform is taken with respect to the z-co-ordinate. The inverse finite Fourier transform is analytically performed to obtain the solutions for drawdowns within the patch and formation zones. Detailed derivations for the Laplace-domain solutions are given in Appendix A and the results are

$$\begin{aligned}\overline{s_1}(r, z, p) = {} & \frac{Q}{4\pi T_2}\left[\frac{1}{p}\frac{2T_2}{r_{\mathrm{w}}T_1}\frac{\phi_3 I_0(q_3 r) + \phi_4 K_0(q_3 r)}{\phi_\infty}\right] + \frac{Q}{4\pi T_2}\frac{1}{p}\frac{4T_2}{(b_2 - b_1) r_{\mathrm{w}} T_1} \\ & \times \sum_{n=1}^{\infty}\left[\frac{\phi_1}{\phi_0} I_0(q_1 r) + \frac{\phi_2}{\phi_0} K_0(q_1 r)\right] F(b_1, b_2)\cos(w_n z)\end{aligned} \tag{10}$$

and

$$\begin{aligned}\overline{s_2}(r, z, p) = {} & \frac{Q}{4\pi T_2}\left[\frac{1}{p}\frac{2T_2}{r_{\mathrm{w}}T_1}\frac{\phi_5 K_0(q_4 r)}{\phi_\infty K_0(q_4 r_1)}\right] \\ & + \frac{Q}{4\pi T_2}\frac{1}{p}\frac{4T_2}{(b_2 - b_1) r_{\mathrm{w}} T_1}\sum_{n=1}^{\infty}\left[\frac{\phi}{\phi_0}\frac{K_0(q_2 r)}{K_0(q_2 r_1)}\right] F(b_1, b_2)\cos(w_n z)\end{aligned} \tag{11}$$

where p is the Laplace variable, $F(b_1, b_2) = [\sin(w_n b_2) - \sin(w_n b_1)]/w_n$, $I_0(\bullet)$ and $K_0(\bullet)$ are the modified Bessel functions of the first and second kinds of order zero, and $I_1(\bullet)$ and $K_1(\bullet)$ are the modified Bessel functions of the first and second kinds of order one. Notice that the right-hand side of (10) and (11) have two terms; the first term represents the solution for a confined radial flow, and the second term contains a summation term accounting for the effect of a partially penetrating well.

2.2. *Average drawdown to the observation well*

The water level in an observation well as shown in Figure 1 represents the average drawdown in an aquifer that is in contact with the well screen (or perforated section) of an observation well. The average drawdown in an observation well that is screened between the depths of b_1' and b_2' can be obtained by integrating the drawdown equations with respect to z between the limits of b_1' and b_2', and then dividing the result by $(b_2' - b_1')$. Thus, the average drawdown within the patch and formation zones can be expressed as

$$\overline{s_1}(r,p) = \frac{Q}{4\pi T_2}\left[\frac{1}{p}\frac{2T_2}{r_\mathrm{w}T_1}\frac{\phi_3 I_0(q_3 r) + \phi_4 K_0(q_3 r)}{\phi_\infty}\right] + \frac{Q}{4\pi T_2}\frac{1}{p}\frac{4T_2}{(b_2-b_1)(b_2'-b_1')r_\mathrm{w}T_1} \times \sum_{n=1}^{\infty}\left[\frac{\phi_1}{\phi_0}I_0(q_1 r) + \frac{\phi_2}{\phi_0}K_0(q_1 r)\right]F(b_1,b_2)F(b_1',b_2') \tag{12}$$

and

$$\overline{s_2}(r,p) = \frac{Q}{4\pi T_2}\left[\frac{1}{p}\frac{2T_2}{r_\mathrm{w}T_1}\frac{\phi_5 K_0(q_4 r)}{\phi_\infty K_0(q_4 r_1)}\right] + \frac{Q}{4\pi T_2}\frac{1}{p}\frac{4T_2}{(b_2-b_1)(b_2'-b_1')r_\mathrm{w}T_1} \times \sum_{n=1}^{\infty}\left[\frac{\phi K_0(q_2 r)}{\phi_0 K_0(q_2 r_1)}\right]F(b_1,b_2)F(b_1',b_2') \tag{13}$$

2.3. *Full penetration with a patch zone*

For the case of a fully penetrating well with a patch zone, b_1 equals zero and b_2 equals the thickness of confined aquifer, L. The Laplace-domain solutions of drawdown distributions within the patch and formation zones can, respectively, reduce to

$$\overline{s_1}(r,p) = \frac{Q}{4\pi T_2}\left[\frac{1}{p}\frac{2T_2}{r_\mathrm{w}T_1}\frac{\phi_3 I_0(q_3 r) + \phi_4 K_0(q_3 r)}{\phi_\infty}\right] \tag{14}$$

and

$$\overline{s_2}(r,p) = \frac{Q}{4\pi T_2}\left[\frac{1}{p}\frac{2T_2}{r_\mathrm{w}T_1}\frac{\phi_5 K_0(q_4 r)}{\phi_\infty K_0(q_4 r_1)}\right] \tag{15}$$

These two solutions can also be obtained by solving (1)–(6) when neglecting the second derivative of the drawdown with respect to z in (1) and (2) [8]. Thus, (9), the average flow rate across the wellbore, turns out to be

$$\frac{\partial s_1(r_\mathrm{w},t)}{\partial r} = -\frac{Q}{2\pi r_\mathrm{w} K_{r1} L} \tag{16}$$

2.4. *Solution of radial single-zone flow*

If the patch zone is absent, the formation becomes a single-zone aquifer system. For the case of the isotropic aquifer and partially penetrating well, the Laplace-domain solutions of (10) and (11)

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reduce to

$$\overline{s}(r,z,p)=\frac{Q}{4\pi T}\frac{2}{r_{\mathrm{w}}}\frac{K_0(q_6r)}{pq_6K_1(q_6r_{\mathrm{w}})}+\frac{Q}{4\pi T}\frac{4}{(b_2-b_1)r_{\mathrm{w}}}\times\sum_{n=1}^{\infty}\left[\frac{K_0(q_5r)}{pq_5K_1(q_5r_{\mathrm{w}})}\right]F(b_1,b_2)\cos(w_nz) \tag{17}$$

The limit of (17) as r_{w} approaches zero can be written as

$$\overline{s}(r,z,p)=\frac{Q}{4\pi T}\lim_{r_{\mathrm{w}}\to 0}\frac{2}{r_{\mathrm{w}}}\frac{K_0(q_6r)}{pq_6K_1(q_6r_{\mathrm{w}})}+\frac{Q}{4\pi T}\lim_{r_{\mathrm{w}}\to 0}\frac{4}{(b_2-b_1)r_{\mathrm{w}}}\times\sum_{n=1}^{\infty}\left[\frac{K_0(q_5r)}{pq_5K_1(q_5r_{\mathrm{w}})}\right]F(b_1,b_2)\cos(w_nz) \tag{18}$$

Carslaw and Jaeger [21] gave a formula

$$I_0(x)K_1(x)+K_0(x)I_1(x)=\frac{1}{x} \tag{19}$$

According to $I_0(0)=1$ and $I_1(0)=0$, the limit of (19) as $x\to 0$ gets

$$\lim_{x\to 0}[xK_1(x)]=1 \tag{20}$$

Accordingly, (18) reduces to

$$\overline{s}(r,z,p)=\frac{Q}{4\pi T}\frac{2K_0(q_6r)}{p}+\frac{Q}{4\pi T}\frac{4}{(b_2-b_1)}\times\sum_{n=1}^{\infty}\left[\frac{K_0(q_5r)}{p}\right]F(b_1,b_2)\cos(w_nz) \tag{21}$$

which is the Laplace-domain solution for a line source case presented in Hantush [3].

2.5. Dimensionless solutions

The dimensionless variables are defined as $\alpha=K_z/K_{\mathrm{r}}$, $\beta=S_{\mathrm{s}}/K_{\mathrm{r}}$, $\gamma=K_{\mathrm{r2}}/K_{\mathrm{r1}}$, $\tau=K_{\mathrm{r}}t/S_{\mathrm{s}}r_{\mathrm{w}}^2$, $\rho=r/r_{\mathrm{w}}$, $L_{\mathrm{D}}=L/r_{\mathrm{w}}$, $B_1=b_1/r_{\mathrm{w}}$, $B_2=b_2/r_{\mathrm{w}}$, $w_{n\mathrm{D}}=n\pi/L_{\mathrm{D}}$, $\overline{\sigma}=\overline{s}(4\pi T)/Q$ and $\sigma=s(4\pi T)/Q$ where α represents the ratio of vertical hydraulic conductivity to horizontal hydraulic conductivity, β represents the ratio of specific storage to horizontal hydraulic conductivity, γ represents the ratio of formation horizontal hydraulic conductivity to patch horizontal hydraulic conductivity, τ represents the dimensionless time during the pumping, ρ represents the dimensionless distance from the centreline of well, L_{D} represents the dimensionless thickness of confined aquifer, $\overline{\sigma}$ represents the dimensionless drawdown in the Laplace domain and σ represents the dimensionless drawdown in the time domain.

The Laplace-domain solutions for dimensionless average drawdowns of (12) and (13) are

$$\overline{\sigma_1}(\rho,p)=\left[\frac{2\gamma}{p}\frac{\phi_{3\mathrm{D}}I_0(q_{3\mathrm{D}}\rho)+\phi_{4\mathrm{D}}K_0(q_{3\mathrm{D}}\rho)}{\phi_{\infty\mathrm{D}}}\right]+\frac{1}{p}\frac{4\gamma}{(B_2-B_1)(B_2'-B_1')}\times\sum_{n=1}^{\infty}\left[\frac{\phi_{1\mathrm{D}}}{\phi_{0\mathrm{D}}}I_0(q_{1\mathrm{D}}\rho)+\frac{\phi_{2\mathrm{D}}}{\phi_{0\mathrm{D}}}K_0(q_{1\mathrm{D}}\rho)\right]F(B_1,B_2)F(B_1',B_2') \tag{22}$$

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and

$$\overline{\sigma_2}(\rho, p) = \left[\frac{2\gamma}{p}\frac{\phi_{5\mathrm{D}}K_0(q_{4\mathrm{D}}\rho)}{\phi_{\infty\mathrm{D}}K_0(q_{4\mathrm{D}}\rho_1)}\right] + \frac{1}{p}\frac{4\gamma}{(B_2 - B_1)(B_2' - B_1')} \times \sum_{n=1}^{\infty}\left[\frac{\phi_{\mathrm{D}}K_0(q_{2\mathrm{D}}\rho)}{\phi_{0\mathrm{D}}K_0(q_{2\mathrm{D}}\rho_1)}\right]F(B_1, B_2)F(B_1', B_2') \quad (23)$$

3. NUMERICAL IMPLEMENTATION

The Laplace transforms are commonly used to solve the differential and integral equations. In many engineering problems, the Laplace-domain solutions for the mathematical models are tractable, yet the corresponding solutions in the time domain may not be easily solved. Under such circumstances, the methods of numerical Laplace inversion such as the Stehfest method [22], Crump method [23], or Talbot method [24] may be used. The Laplace inversion transform of (22) and (23) are performed to three decimal places using the routine INLAP of IMSL [19], developed according to the work of de Hoog *et al.* [18].

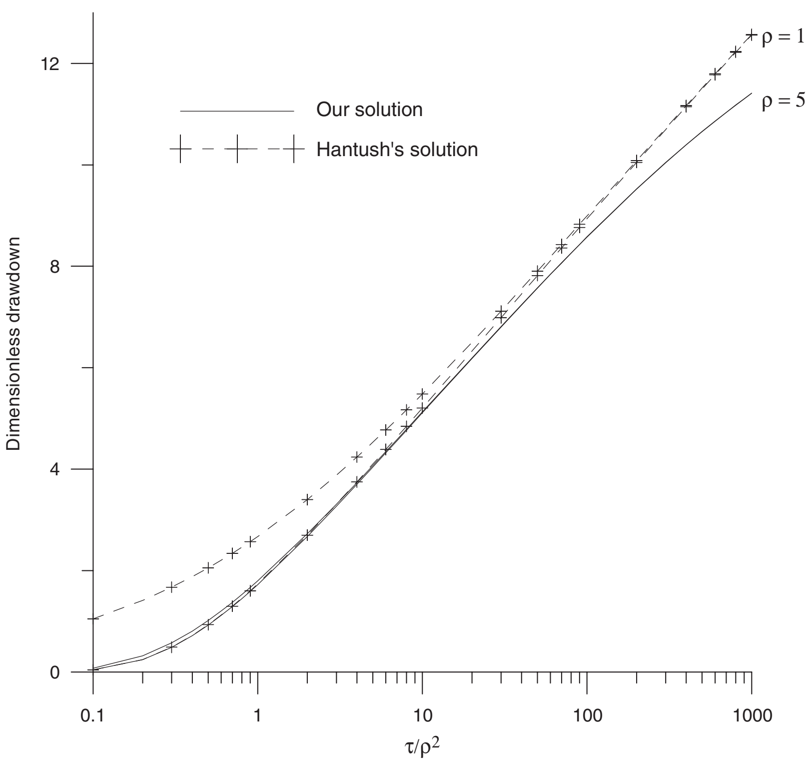

Figure 2. Comparison of the results between our solution and Hantush's solution for $\alpha = 0.1$ when $\rho = 1$ or 5.

4. DISCUSSION OF RESULTS

4.1. *Effect of well radius*

The effect of well radius on the drawdown due to a constant-flux pumping can be clearly explored by comparing the present solution with the Hantush solution [3]. The dimensionless drawdown curves according to our solution and the Hantush solution [3] is plotted in Figure 2 for $\alpha = 0.1$ when $\rho = 1$ or 5. For comparison purpose, the axis of dimensionless time is chosen as $\tau/\rho^2(Tt/Sr^2)$ which was also used in Hantush [3]. Figure 2 indicates that neglecting the effect of a well radius may make significant error in estimated dimensionless drawdowns, especially ρ is small. For $\rho = 1$ (at a well), the differences of dimensionless drawdowns between two solutions are large when $\tau/\rho^2 < 1$ and very small when $\tau/\rho^2 \geqslant 100$. For $\rho = 5$, the dimensionless drawdowns for the solutions with and without considering the well radius are almost identical. In fact, both the dimensionless drawdown curves approach Hantush's equation at very large dimensionless time.

4.2. *Effect of anisotropy*

This section investigates the effect of the anisotropy for $\phi = 0.1$, $\rho = 1$, $\rho_1 = 4$ and $L_D = 200$ when $\alpha = 0.01$, 0.1 or 1. The vertical flow occurs near a pumping well when the well is partially

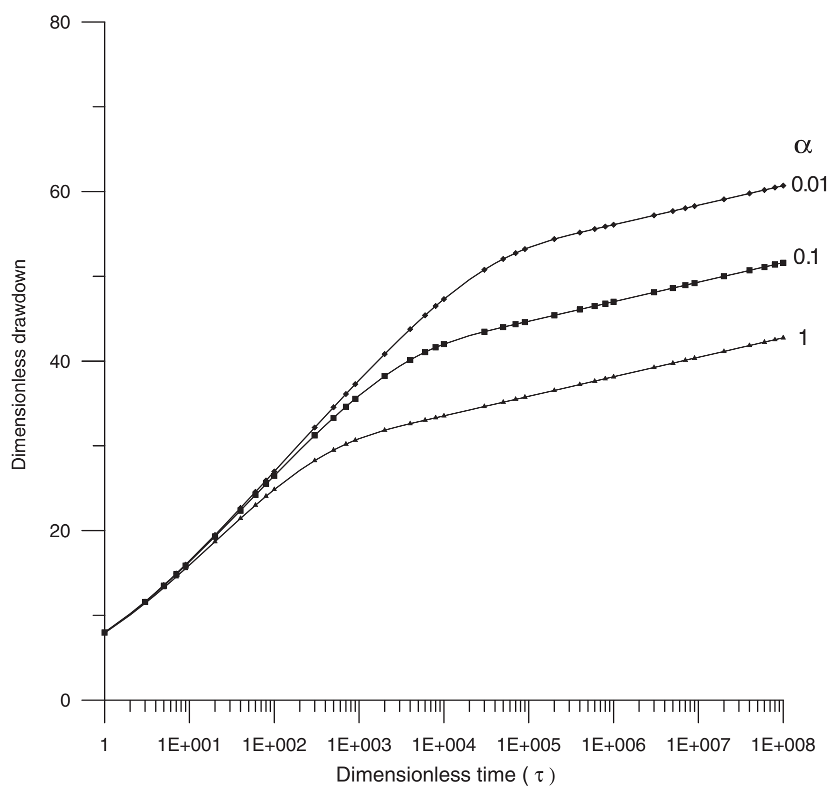

Figure 3. Dimensionless drawdown *versus* dimensionless time (τ) for $\phi = 0.1$, $\rho = 1$, $\rho_1 = 4$, and $L_D = 200$ when $\alpha = 0.01$, 0.1 or 1.

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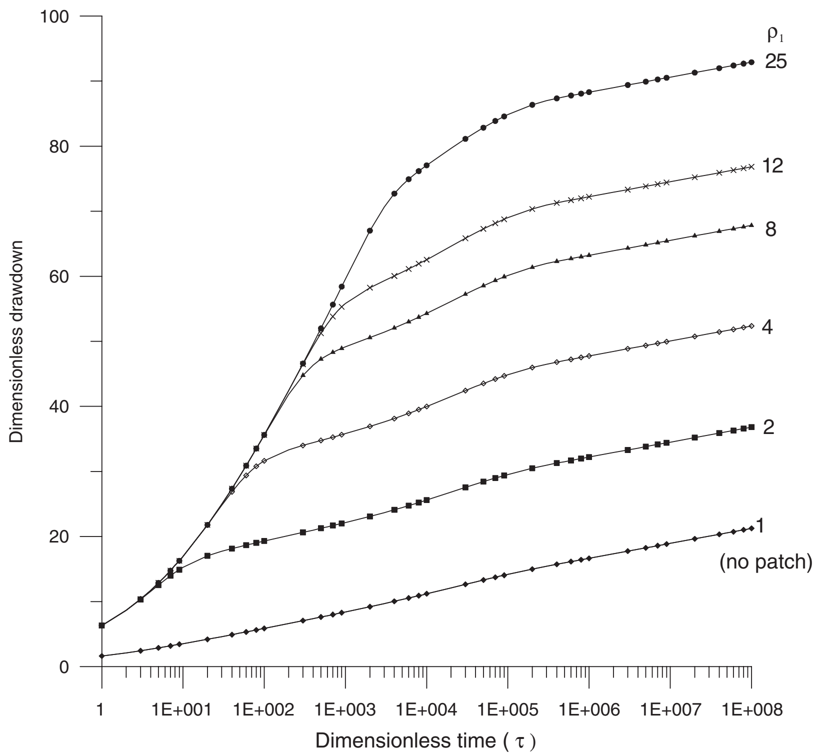

Figure 4. The effect of dimensionless patch thickness on dimensionless drawdown distribution for $\gamma=10$, $\rho=1$, $\alpha=0.1$, $B_1=20$, $B_2=180$ and $L_D=200$. The dimensionless patch thickness (ρ_1) ranges from 1 (no patch) to 25.

penetrated. Therefore, the vertical hydraulic conductivity is an important parameter to be considered. For field problems, the ratio of vertical hydraulic conductivity to radial hydraulic conductivity (α) ranges from 0.01 to 1. Figure 3 shows that the difference of dimensionless drawdowns is apparent when τ is large; in contrast, the difference of dimensionless drawdowns is small when τ is small, i.e. $\tau<100$. On the other hand, the effect of α on dimensionless drawdown is significant at large dimensionless time. In addition, the slope of three curves is approximately identical when $\tau \geqslant 10^5$. This result indicates that the vertical flow effect is noticeable when τ is small. Obviously, the erroneous results for a pumping-test data analysis performed in an anisotropic aquifer will be made if α is assumed to be one.

4.3. Effect of patch thickness

The effect of dimensionless patch thickness on dimensionless drawdown distribution is displayed in Figure 4. The curves are plotted for $\gamma=10$, $\rho=1$, $\alpha=0.1$, $B_1=20$, $B_2=180$, and $L_D=200$ when ρ_1 ranges from 1 to 25. Note that the dimensionless patch thickness is equal to ρ_1-1 and $\rho_1=1$ represents no patch case. The permeability of the patch for this case is one order of magnitude lower than that of a formation ($\gamma=10$). With a thicker patch zone, the presence of a patch zone is identifiable for a lower permeability. The dimensionless drawdowns for $\rho_1 = 2$–25 are greater than that of an uniform medium ($\rho_1 = 1$) at early dimensionless time, reflecting the effect of a patch zone. Obviously, without considering the presence of a patch for a two-zone aquifer system, the predicted drawdown will be under-estimated, for a negative patch and over-estimated for a positive patch. Notice that the slope of these dimensionless drawdown curves tends to equal that of an uniform medium when $\tau \geqslant 10^4$, implying that the patch effect diminishes at large dimensionless time.

4.4. *Effect of well partial penetration*

When $\gamma = 0.1$ or 10, Figure 5 depicts the relationship of dimensionless drawdown *versus* dimensionless time for $\alpha = 1$, $\rho = 1$, $\rho_1 = 4$, $L_D = 200$, $\phi = 0.1, 0.4, 0.8$ and 1.0. Note that $\phi = 1$ represents a fully penetrating well case. The dimensionless drawdown tends to increase rapidly with

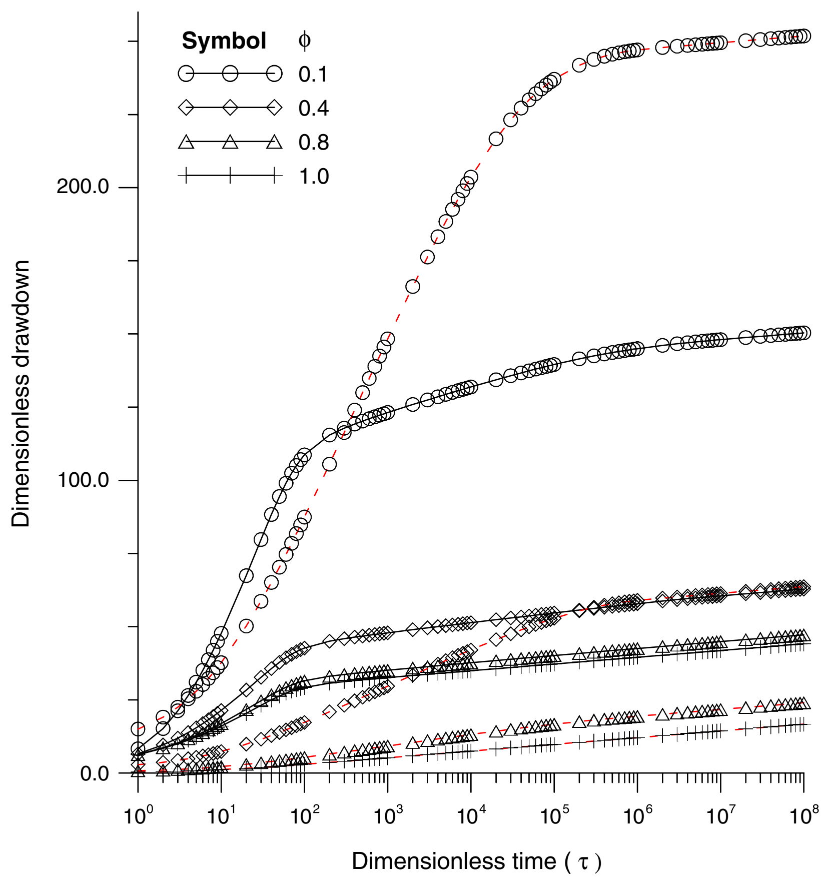

Figure 5. The plot of dimensionless drawdown *versus* dimensionless time for $\alpha = 1$, $\rho = 1$, $\rho_1 = 4$, $L_D = 200$, $\phi = 0.1, 0.4, 0.8$ and 1 when $\gamma = 0.1$ or 10. The dotted line represents the dimensionless time-drawdown curve for $\gamma = 0.1$ and the solid line represents the dimensionless time-drawdown curve for $\gamma = 10$.

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dimensionless time and stabilize when dimensionless time is very large ($\tau \geqslant 10^4$ for $\gamma = 0.1$ and $\tau \geqslant 10^2$ for $\gamma = 10$). The dimensionless drawdown with a partially penetrating well significantly differs from that with a fully penetrating well. Obviously, the dimensionless drawdown increases with decreasing ϕ at the same dimensionless time. The effect of a well partial penetration increases with dimensionless time. In addition, the effect of a well partial penetration on the drawdown with a negative patch ($\gamma = 0.1$) is larger than that with a positive patch ($\gamma = 10$).

5. CONCLUSIONS

New Laplace-domain solutions had been developed for a constant pumping at a partially penetrating well in a heterogeneous aquifer. The solutions were derived to account for the effects of the patch thickness and well partial penetration on the drawdown distributions. The derived solution considers the effects of well radius and provides appropriate mathematical models for the analyses of pumping test data. An efficient numerical inversion approach is used for evaluating this Laplace-domain solution. The results show that this solution can be used to investigate the effects of the patch thickness, well radius and well partial penetration on the drawdown distributions. The solution for the case with a well radius has shown to reduce to that presented by Hantush [3] if the well radius is neglected. This study demonstrates that Hantush's solution gives significant errors in the drawdown when the observation well is close to a pumping well and/or the pumping time is very small.

APPENDIX A: DERIVATIONS OF (10) AND (11)

The solutions of drawdown within the patch and formation zones are derived *via* Laplace transform with respect to time variable t and the finite Fourier transform with respect to spatial variable z. The appropriate finite Fourier transform is given by [25]

$$F[s(z)] = \tilde{s}(w_n) = \int_0^L s(z)\cos(w_n z)\,\mathrm{d}z, \quad 0 \leqslant z \leqslant L \tag{A1}$$

where $w_n = n\pi/L$, $n = 0, 1, 2, \ldots$. The transform has following operational property:

$$F\left\{\frac{\mathrm{d}^2 s(z)}{\mathrm{d}z^2}\right\} = (-1)^n \left.\frac{\mathrm{d}s(z)}{\mathrm{d}z}\right|_{z=L} - \left.\frac{\mathrm{d}s(z)}{\mathrm{d}z}\right|_{z=0} - w_n^2 \tilde{s}(w_n) \tag{A2}$$

Applying Laplace transform and the finite Fourier cosine transform, (1) and (2) give the following subsidiary equations:

$$\frac{\mathrm{d}^2\tilde{s}_1(r, w_n, p)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\tilde{s}_1(r, w_n, p)}{\mathrm{d}r} = q_1^2 \tilde{s}_1(r, w_n, p), \quad r_\mathrm{w} \leqslant r \leqslant r_1 \tag{A3}$$

and

$$\frac{\mathrm{d}^2\tilde{s}_2(r, w_n, p)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\tilde{s}_2(r, w_n, p)}{\mathrm{d}r} = q_2^2 \tilde{s}_2(r, w_n, p), \quad r_1 \leqslant r < \infty \tag{A4}$$

where p is the Laplace transform variable of time variable t, $\tilde{s}$ is the transformed drawdown, $q_1 = \sqrt{\alpha_1 w_n^2 + \beta_1 p}$, and $q_2 = \sqrt{\alpha_2 w_n^2 + \beta_2 p}$.

The transformed boundary conditions are

$$\widetilde{s}_2(\infty, w_n, p) = 0 \tag{A5}$$

and

$$\frac{\mathrm{d}\widetilde{s}_1(r_\mathrm{w}, w_n, p)}{\mathrm{d}r} = -\frac{1}{p}\frac{Q}{2\pi r_\mathrm{w}(b_2 - b_1)K_{\mathrm{r}1}} F(b_1, b_2) \tag{A6}$$

The continuity conditions required at the interface between the patch and formation zones are

$$\widetilde{s}_1(r_1, w_n, p) = \widetilde{s}_2(r_1, w_n, p) \tag{A7}$$

and

$$\frac{\mathrm{d}\widetilde{s}_1(r_1, w_n, p)}{\mathrm{d}r} = \gamma \frac{\mathrm{d}\widetilde{s}_2(r_1, w_n, p)}{\mathrm{d}r} \tag{A8}$$

The general solutions of (A3) and (A4) are

$$\widetilde{s}_1(r, w_n, p) = C_1 I_0(q_1 r) + C_2 K_0(q_1 r) \tag{A9}$$

and

$$\widetilde{s}_2(r, w_n, p) = C_3 I_0(q_2 r) + C_4 K_0(q_2 r) \tag{A10}$$

where C_1, C_2, C_3 and C_4 are the undetermined constants.

Substituting (A9) and (A10) into (A5)–(A8), one obtains

$$C_1 = -\frac{1}{p}\frac{Q}{2\pi r_\mathrm{w}(b_2 - b_1)K_{\mathrm{r}1}}\frac{\phi_1}{\phi_0} \tag{A11}$$

$$C_2 = -\frac{1}{p}\frac{Q}{2\pi r_\mathrm{w}(b_2 - b_1)K_{\mathrm{r}1}}\frac{\phi_2}{\phi_0} \tag{A12}$$

$$C_3 = 0 \tag{A13}$$

and

$$C_4 = -\frac{1}{p}\frac{Q}{2\pi r_\mathrm{w}(b_2 - b_1)K_{\mathrm{r}1}}\frac{\phi}{\phi_0 K_0(q_2 r_1)} \tag{A14}$$

Consequently, the solutions of the drawdowns within the patch and formation zones can be obtained by substituting the constants of (A11)–(A14) into (A9) and (A10) as

$$\widetilde{s}_1(r, w_n, p) = -\frac{1}{p}\frac{Q}{2\pi r_\mathrm{w}(b_2 - b_1)K_{\mathrm{r}1}} F(b_1, b_2)\left[\frac{\phi_1}{\phi_0} I_0(q_1 r) + \frac{\phi_2}{\phi_0} K_0(q_1 r)\right] \tag{A15}$$

and

$$\widetilde{s}_2(r, w_n, p) = -\frac{1}{p}\frac{Q}{2\pi r_\mathrm{w}(b_2 - b_1)K_{\mathrm{r}1}} F(b_1, b_2)\frac{\phi K_0(q_2 r)}{\phi_0 K_0(q_2 r_1)} \tag{A16}$$

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Applying the inverse finite Fourier transform gives

$$\overline{s_1}(r,z,p)=\frac{Q}{4\pi T_2}\left[\frac{1}{p}\,\frac{2T_2}{r_{\mathrm{w}}T_1}\,\frac{\phi_3 I_0(q_3 r)+\phi_4 K_0(q_3 r)}{\phi_\infty}\right]+\frac{Q}{4\pi T_2}\,\frac{1}{p}\,\frac{4T_2}{(b_2-b_1)r_{\mathrm{w}}T_1}$$

$$\times\sum_{n=1}^{\infty}\left[\frac{\phi_1}{\phi_0}I_0(q_1 r)+\frac{\phi_2}{\phi_0}K_0(q_1 r)\right]F(b_1,b_2)\cos(w_n z) \qquad \text{(A17)}$$

and

$$\overline{s_2}(r,z,p)=\frac{Q}{4\pi T_2}\left[\frac{1}{p}\,\frac{2T_2}{r_{\mathrm{w}}T_1}\,\frac{\phi_5 K_0(q_4 r)}{\phi_\infty K_0(q_1 r_1)}\right]+\frac{Q}{4\pi T_2}\,\frac{1}{p}\,\frac{4T_2}{(b_2-b_1)r_{\mathrm{w}}T_1}$$

$$\times\sum_{n=1}^{\infty}\left[\frac{\phi K_0(q_2 r)}{\phi_0 K_0(q_2 r_1)}\right]F(b_1,b_2)\cos(w_n z) \qquad \text{(A18)}$$

Equations (A17) and (A18) are, respectively, the Laplace-domain solutions for the drawdowns within the patch and formation zones.

NOMENCLATURE

b_1	lower z co-ordinate of well screen
b_2	upper z co-ordinate of well screen
B	b/r_{w}
B'	b'/r_{w}
K_{r}	hydraulic conductivity in a radial direction
K_z	hydraulic conductivity in a vertical direction
L	thickness of confined aquifer
L_{D}	L/r_{w}
S	storage coefficient
S_{s}	specific storage
s	drawdown distribution
T	transmissivity
t	time from the star of pumping
p	Laplace variable
Q	constant flow rate into or out the wellbore
q_1	$\sqrt{\alpha_1 w_n^2+\beta_1 p}$
q_2	$\sqrt{\alpha_2 w_n^2+\beta_2 p}$
q_3	$\sqrt{\beta_1 p}$
q_4	$\sqrt{\beta_2 p}$
q_5	$\sqrt{\alpha w_n^2+\beta p}$

q_6	$\sqrt{\beta p}$
q_{nD}	$q_n \times r_w, n = 1, 2, \ldots, 6$
w_{nD}	$n\pi/L_D, n = 1, 2, \ldots$
r	radial distance from the centreline of well
r_1	outer radius of patch region
r_w	radius of the pumping well
w_n	$n\pi/L, n = 0, 1, 2, \ldots$
z	vertical distance from a lower impermeable layer
$I_0(u)$, $K_0(u)$	modified Bessel functions of the first and second kinds of order zero
$I_1(u)$, $K_1(u)$	modified Bessel functions of the first and second kinds of order one
ϕ	$\phi_1 I_0(q_1 r_1) + \phi_2 K_0(q_1 r_1)$
ϕ_0	$q_1[\phi_2 K_1(q_1 r_w) - \phi_1 I_1(q_1 r_w)]$
ϕ_1	$q_1 K_0(q_2 r_1) K_1(q_1 r_1) - \gamma q_2 K_0(q_1 r_1) K_1(q_2 r_1)$
ϕ_2	$q_1 I_1(q_1 r_1) K_0(q_2 r_1) + \gamma q_2 I_0(q_1 r_1) K_1(q_2 r_1)$
ϕ_3	$q_3 K_0(q_4 r_1) K_1(q_3 r_1) - \gamma q_4 K_0(q_3 r_1) K_1(q_4 r_1)$
ϕ_4	$q_3 I_1(q_3 r_1) K_0(q_4 r_1) + \gamma q_4 I_0(q_3 r_1) K_1(q_4 r_1)$
ϕ_5	$\phi_3 I_0(q_3 r_1) + \phi_4 K_0(q_3 r_1)$
ϕ_∞	$q_3[\phi_4 K_1(q_3 r_w) - \phi_3 I_1(q_3 r_w)]$
ϕ_D	$\phi_{1D}(q_{1D}\rho_1) + \phi_{2D} K_0(q_{1D}\rho_1)$
ϕ_{0D}	$q_{1D}[\phi_{2D} K_1(q_{1D}) - \phi_{1D} I_1(q_{1D})]$
ϕ_{1D}	$q_{1D} K_0(q_{2D}\rho_1) K_1(q_{1D}\rho_1) - \gamma q_{2D} K_0(q_{1D}\rho_1) K_1(q_{2D}\rho_1)$
ϕ_{2D}	$q_{1D} I_1(q_{1D}\rho_1) K_0(q_{2D}\rho_1) + \gamma q_{2D} I_0(q_{1D}\rho_1) K_1(q_{2D}\rho_1)$
ϕ_{3D}	$q_{3D} K_0(q_{4D}\rho_1) K_1(q_{3D}\rho_1) - \gamma q_{4D} K_0(q_{3D}\rho_1) K_1(q_{4D}\rho_1)$
ϕ_{4D}	$q_{3D} I_1(q_{3D}\rho_1) K_0(q_{4D}\rho_1) + \gamma q_{4D} I_0(q_{3D}\rho_1) K_1(q_{4D}\rho_1)$
ϕ_{5D}	$\phi_{3D} I_0(q_{3D}\rho_1) + \phi_{4D} K_0(q_{3D}\rho_1)$
$\phi_{\infty D}$	$q_{3D}[\phi_{4D} K_1(q_{3D}) - \phi_{3D} I_1(q_{3D})]$
α_n	$K_{zn}/K_{rn}, n = 1, 2$
β_n	$S_{Sn}/K_{rn}, n = 1, 2$
γ	K_{r2}/K_{r1}
ρ	r/r_w
σ	$s(4\pi T)/Q$
τ	$K_r t/S_s r_w^2$
ϕ	$(B_2 - B_1)/L_D$
$F(b_1, b_2)$	$[\sin(w_n b_2) - \sin(w_n b_1)]/w_n$
$F(b'_1, b'_2)$	$[\sin(w_n b'_2) - \sin(w_n b'_1)]/w_n$

Subscripts

D	dimensionless
1	patch zone
2	formation zone
w	pumping well

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