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Topological horseshoes for perturbations of singular difference equations

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Abstract

In this paper, we study solutions of difference equations $\Phi_\lambda(y_n, y_{n+1}, \dots, y_{n+m}) = 0$, $n \in \mathbb{Z}$, of order m with parameter λ , and consider the case when Φ_λ has a singular limit depending on a single variable as $\lambda \rightarrow \lambda_0$, i.e. $\Phi_{\lambda_0}(y_0, \dots, y_m) = \varphi(y_N)$, where N is an integer with $0 \leq N \leq m$ and φ is a function. We prove that if φ has k simple zeros then for λ close enough to λ_0 , the difference equation has a k -horseshoe among its solutions, that is, the dynamics is conjugate to the full shift with k symbols. Moreover, we show that these horseshoes change continuously in the uniform topology as λ varies. As applications of these results, we establish the horseshoe structure in families of generalized Hénon-like maps and of Arneodo–Coullet–Tresser maps near their anti-integrable limits as well as in steady states for certain lattice models.

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1. Introduction

In the theory of dynamical systems, there are several approaches which aim to establish the presence of chaotic behaviour for orbits under appropriate hypotheses. Among others, a typical method is to find homoclinic points and then discover a horseshoe structure (for hyperbolic horseshoes see e.g. [3, 25, 26]; for merely topological horseshoes see [14, 15]). Recently, there appeared the so-called anti-integrable limit approach for systems with generating functions inspired by the concept of Aubry and Abramovici in [6]; see also [9, 20, 23]. For example, in [23], Qin obtained a two-horseshoe orbit structure in a family of generalized Hénon-like maps

$$H_\lambda(x_1, x_2, \dots, x_{m-1}, x_m) = (x_2, x_3, \dots, x_m, g(x_1, \dots, x_m) + \lambda\varphi(x_m)) \quad (1)$$

with sufficiently large λ (i.e. ‘near’ the anti-integrable limit for $\lambda_0 = \infty$), where $\varphi(x) = x(x - 1)$ and $g(x)$ is a C^1 function on $\mathbb{R}^m$ with bounded partial derivatives on $\mathbb{R}^m$ and $(\partial/\partial x_1)g(x_1, \dots, x_m)$ not vanishing, in which case H_λ is one-to-one. (Let us note in this connection that results on the stability of hyperbolic sets are known mainly for diffeomorphisms on manifolds, and only few are concerned with non-invertible smooth maps, e.g. [16, 24] for the latter).

The above family (1) can be regarded as a singular system at the anti-integrable limit $\lambda_0 = \infty$. Although there is no corresponding map H_{λ_0} at this limit value of parameter, one may use the usual elimination process for H_λ with finite λ to get a difference equation $\Phi_\lambda(y_n, y_{n+1}, \dots, y_{n+m}) = 0$, $n \in \mathbb{Z}$ (see section 4) and so, one obtains a well-defined function Φ_{λ_0} at this limit $\lambda_0 = \infty$. It is of the form $\varphi(y_n) = y_n(y_n - 1) = 0$, and thus its solutions are all bi-sequences of zeros and ones.

In this paper, we extend Qin’s example and Aubry and Abramovich’s anti-integrable limit approach to m -th order difference equations $\Phi_\lambda(y_n, y_{n+1}, \dots, y_{n+m}) = 0$, $n \in \mathbb{Z}$, that have a singular limit of the form $\Phi_{\lambda_0}(y_0, \dots, y_m) = \varphi(y_N)$ as $\lambda \rightarrow \lambda_0$, for some integer N with $0 \leq N \leq m$ and some function φ . We prove that if φ has k simple zeros, then for λ close enough to λ_0 , the difference equation has a topological k -horseshoe, i.e. a closed (in the product topology) shift-invariant set Γ_λ for which the restriction of the shift map σ is conjugate to $\sigma|_{\Sigma_k}$, the two-sided full shift on k symbols; see the precise statement of theorem 3.3. Moreover, we prove that the conjugacy map $\psi_\lambda : \Sigma_k \rightarrow \Gamma_\lambda$ depends continuously on λ not only in the product topology, but also in the uniform topology; more precisely, the family of maps $\lambda \mapsto \psi_\lambda(x)$ from the parameter space (close to λ_0) to ℓ_∞ forms an equicontinuous family in $x \in \Sigma_k$. The latter implies that the full orbit structure varies continuously starting from the ‘rigid horseshoe’ (i.e. from Σ_k) as λ varies.

Let us remark that our assumptions require neither invertibility of maps associated with the difference equation nor even continuity everywhere. For example, our arguments being applied to the generalized Hénon-like maps, allow us to extend the result of [23] so as to consider φ and g to be C^1 functions on $Q = [s_1, s_2] \setminus V$ and Q^m , respectively, for some real numbers $s_1 < s_2$ and open set $V \subset [s_1, s_2]$. This setting could be of use, for example, when φ is not differentiable or even not continuous at some points (such as for tent maps or Lorenz-like maps, in which case we may include these points in V). In such a situation we are interested only in those orbits under the map H_λ which always stay in $([s_1, s_2] \setminus V)^m$, and we show below that if φ has k simple zeros in $\text{int}([s_1, s_2] \setminus V)$ then for $|\lambda|$ sufficiently large, there still remain enough orbits to create a full shift on k symbols. Also note that for wide classes of polynomial maps, the non-wandering set is bounded (see [12, 17]), and thus, for appropriate s_1, s_2 , the set $[s_1, s_2]^m$ contains all non-wandering orbits.

The paper is organized as follows. In section 2, we prove a special version of the implicit function theorem in Banach spaces (theorem 2.1) in order to control simultaneously infinitely many branches of the implicit function while constructing later a conjugacy map from the full shift at the exceptional value of the parameter to solutions of difference equations at close values. Then we get some consequences from theorem 2.1 for the maps on l_∞ .

In section 3, we apply the results of section 2 to families of difference equations in the case when at the exceptional value of parameter they are reduced to functions of a single variable with $k \geq 2$ simple zeros. Here we especially pay attention to the usual situation when the difference equations for non-exceptional values of parameter correspond to well-defined maps f_λ (possibly non-invertible), in which case we prove that for f_λ with parameter λ close to the exceptional one, there is a closed invariant set Λ_λ such that the inverse limit of $f_\lambda|_{\Lambda_\lambda}$ is conjugate to the full shift on k symbols; see more properties in theorem 3.3. In section 4, with the help of this result we obtain a topological horseshoe structure near anti-integrable limits in

several families of generalized Hénon-like maps (see subsection 4.1) and in the families of the Arneodo–Coullet–Tresser maps (see subsection 4.2) and of quadratic volume preserving maps (see subsection 4.3). As another application, in subsection 4.4 we obtain chaotic behaviour for steady-state solutions in lattice models of discrete versions for some PDE of evolution type (see also [2] for Afraimovich and Chow’s approach to chaotic behaviour of steady states in lattice systems using homoclinic structure).

2. A version of the implicit function theorem and its consequences for shift commuting functions on ℓ_∞

The results of the paper will be based on simultaneous control of infinitely many branches of appropriate implicit functions. To this end we provide the following ‘uniform version’ of the implicit function theorem. In the statement and proof of this version, we will use notations $U(x, r)$ and $U[x, r]$ for the open and closed ball, respectively, of radius r centred at the point $x \in X$, where X is a metric space.

Theorem 2.1. *Let E be a metric space, and G and H be Banach spaces. Let a be a point in E and B be a subset of G . Denote $V_0 = U(a, \delta_0)$ and $W_0 = \bigcup_{b \in B} U(b, \eta_0)$ for some $\delta_0 > 0$ and $\eta_0 > 0$, and assume that $F : V_0 \times W_0 \rightarrow H$ is a function such that*

- (i) $F(a, b) = 0$ for all $b \in B$;
- (ii) F is continuous on $V_0 \times W_0$ and moreover, the family of functions $F(\cdot, y)$, $y \in W_0$, is equicontinuous on the domain V_0 , i.e. for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $y \in W_0$ one has $\|F(x_1, y) - F(x_2, y)\| < \epsilon$ provided $x_1, x_2 \in V_0$ with $\rho(x_1, x_2) < \delta$, where ρ is the metric on E ;
- (iii) the partial derivative operator $D_2 F(x, y)$ exists for any point $(x, y) \in V_0 \times W_0$ and moreover, it is continuous at $\{a\} \times B$ uniformly in $b \in B$ in the following sense: for any $\epsilon > 0$ there exists $\delta > 0$ and $\eta > 0$ such that for any $b \in B$, $\|D_2 F(x, y) - D_2 F(a, b)\| < \epsilon$ provided $x \in U(a, \delta)$ and $y \in U(b, \eta)$; and
- (iv) the operator $D_2 F$ at any point $(a, b) \in \{a\} \times B$ is invertible, and the inverse, $(D_2 F)^{-1}$, is uniformly bounded, i.e. there is a constant $M > 0$ such that for any $b \in B$, $\|(D_2 F(a, b))^{-1}\| \leq M$.

Then there exist $0 < \bar{\delta} < \delta_0$, $0 < \bar{\eta} < \eta_0$ and a family of continuous functions $\psi_b : U(a, \bar{\delta}) \rightarrow U(b, \bar{\eta})$, $b \in B$, such that for any $b \in B$,

$$\psi_b(a) = b \quad \text{and} \quad F(x, \psi_b(x)) = 0 \quad \text{for all } x \in U(a, \bar{\delta}).$$

Furthermore, the functions ψ_b are uniquely determined (that is, if $F(x, y) = 0$ and $(x, y) \in U(a, \bar{\delta}) \times U(b, \bar{\eta})$ for some $b \in B$ then necessarily $y = \psi_b(x)$) and form an equicontinuous family. Moreover, for any $x \in U(a, \bar{\delta})$, the map $b \mapsto \psi_b(x)$ is injective on B .

We postpone the proof of theorem 2.1 to the appendix.

Remark 2.2. It is easy to see that assumption (ii) of the theorem is surely satisfied provided E is a normed space, $D_1 F(x, y)$ exists, and $\|D_1 F(x, y)\|$ is bounded above on $V_0 \times W_0$. Also, both assumptions (ii) and (iii) are satisfied if $F(x, y)$ is C^2 on $V_0 \times W_0$ with bounded derivatives $D_2 D_2 F(x, y)$ and $D_1 D_2 F(x, y)$ and moreover, in this case, the resultant functions $\psi_b(x)$ are C^1 .

Let ℓ_∞ be the space of bounded real sequences endowed with the norm $\|\underline{y}\| = \sup\{|y_n| : n \in \mathbb{Z}\}$ for $\underline{y} = (y_n)$, $y_n \in \mathbb{R}$, i.e. with the topology of uniform convergence and let $\sigma : \ell_\infty \rightarrow \ell_\infty$ be the shift map, i.e. $\sigma(\underline{y}) = \underline{y}'$ with $y'_n = y_{n+1}$, $n \in \mathbb{Z}$.

Corollary 2.3. *Let the assumptions of theorem 2.1 be satisfied for the case when $G = H = \ell_\infty$ and B is σ -invariant: $\sigma(B) = B$. If for any $x \in U(a, \delta)$, the function $F(x, \cdot)$ commutes with σ , i.e. $F(x, \sigma y) = \sigma \circ F(x, y)$, then so is the map $b \mapsto \psi_b(x)$, i.e. $\psi_{\sigma(b)}(x) = \sigma \circ \psi_b(x)$.*

Proof. Denote the origin in ℓ_∞ by $\underline{0} = (\cdots 0.00 \cdots)$. Along with the conclusions of theorem 2.1, the commutative assumption implies that

$$F(x, \sigma \circ \psi_b(x)) = \sigma \circ F(x, \psi_b(x)) = \sigma(\underline{0}) = \underline{0},$$

while σ -invariance implies that

$$F(x, \psi_{\sigma(b)}(x)) = \underline{0}.$$

Moreover, we have $\|\sigma \circ \psi_b(x) - \sigma(b)\| = \|\psi_b(x) - b\| < \bar{\eta}$. Hence, by the uniqueness of ψ_b in theorem 2.1, we get $\sigma \circ \psi_b(x) = \psi_{\sigma(b)}(x)$. $\square$

In the following theorem (as well as in the next section) we will consider mainly subsets of ℓ_∞ endowed with the product (or Tichonov) topology on $\mathbb{R}^{\mathbb{Z}}$, i.e. with the topology of pointwise convergence. In such a case we will supply the notation of the appropriate sets with subscript prod, for example: $\ell_{\infty, \text{prod}}$, B_{prod} , etc. Since in the following theorem, the coordinate $x \in E$ will be fixed, it is convenient to re-denote $\psi_b(x)$ by $\bar{\psi}_x(b)$.

Theorem 2.4. *Let the assumptions of theorem 2.1 be satisfied for the case when $G = H = \ell_\infty$ and let B be a subset of $[s_1, s_2]^{\mathbb{Z}}$ for some real numbers $s_1 < s_2$. If for any $x \in U(a, \delta_0)$, the function $F(x, \cdot)$ is continuous with respect to the product topologies on the domain V_0 and codomain ℓ_∞ , then there exists $0 < \hat{\delta} < \bar{\delta}$ ($\bar{\delta}$ being from the statement of theorem 2.1) such that for any (fixed) $x \in U(a, \hat{\delta})$, the map $b \mapsto \bar{\psi}_x(b) := \psi_b(x)$ from B_{prod} to $\ell_{\infty, \text{prod}}$ is continuous.*

Proof. The equicontinuity property from theorem 2.1 implies that there is $0 < \hat{\delta} < \bar{\delta}$ such that for any $x \in U(a, \hat{\delta})$ and any $b \in B$, one has

$$\|\psi_b(x) - b\| < \frac{\bar{\eta}}{3},$$

where $\bar{\eta}$ is from the statement of theorem 2.1.

Fix $x \in U(a, \hat{\delta})$. Since $\ell_{\infty, \text{prod}}$ is separable, it is sufficient to show that given a sequence $b^{(m)} = (b_n^{(m)})_{n=-\infty}^{\infty} \in B$, $m \geq 0$, which tends to $b^* = (b_n^*)_{n=-\infty}^{\infty} \in B$ in the product topology, one has $\bar{\psi}_x(b^{(m)}) \xrightarrow{\text{prod}} \bar{\psi}_x(b^*)$ as $m \rightarrow \infty$.

By theorem 2.1, for any $b \in B$, one has $\|\bar{\psi}_x(b) - b\| < \bar{\eta}$ and thus $\bar{\psi}_x(b) \in [s_1 - \bar{\eta}, s_2 + \bar{\eta}]^{\mathbb{Z}}$. Since $[s_1 - \bar{\eta}, s_2 + \bar{\eta}]^{\mathbb{Z}}$ is compact in the product topology, it follows that the sequence $\bar{\psi}_x(b^{(m)})$ has an accumulation point, say y^* , in $[s_1 - \bar{\eta}, s_2 + \bar{\eta}]_{\text{prod}}^{\mathbb{Z}}$. We must show that $y^* = \bar{\psi}_x(b^*)$. Without loss of generality, we may suppose (by switching to an appropriate subsequence) that $\bar{\psi}_x(b^{(m)}) \xrightarrow{\text{prod}} y^*$ as $m \rightarrow \infty$. Since $b^{(m)} \xrightarrow{\text{prod}} b^*$, it follows that for any $N \in \mathbb{N}$ there is $M \in \mathbb{N}$ such that for $m > M$ one has

$$\|b_n^{(m)} - b_n^*\| < \frac{\bar{\eta}}{3} \quad \text{for all } -N \leq n \leq N,$$

and

$$\|\bar{\psi}_x(b^{(m)})_n - b_n^*\| \leq \|\bar{\psi}_x(b^{(m)})_n - b_n^{(m)}\| + \|b_n^{(m)} - b_n^*\| < \frac{2\bar{\eta}}{3},$$

where the last inequality follows from theorem 2.1. Letting $N \rightarrow \infty$, we get

$$\|y^* - b^*\| \leq \frac{2\bar{\eta}}{3} < \bar{\eta}. \quad (2)$$

On the other hand, because of the continuity of $F(x, \cdot)$ with respect to the product topology, one has

$$\underline{0} = F(x, \bar{\psi}_x(b^{(m)})) \xrightarrow{\text{prod}} F(x, y^*).$$

Therefore, $F(x, y^*) = \underline{0}$ and by using (2) one gets from theorem 2.1 that $y^* = \bar{\psi}_x(b^*)$. $\square$

We summarize the above results as follows.

Theorem 2.5. *Let the assumptions of theorem 2.1 be satisfied for the case when $G = H = \ell_\infty$ and let B be a σ -invariant and compact subset of $[s_1, s_2]_{\text{prod}}^{\mathbb{Z}}$ for some real numbers $s_1 < s_2$. If for any $x \in U(a, \delta_0)$, the function $F(x, \cdot)$ commutes with the shift map σ and is continuous with respect to the product topologies on the domain V_0 and codomain ℓ_∞ , then there exists $0 < \hat{\delta} < \bar{\delta}$ ($\bar{\delta}$ being from the statement of theorem 2.1) such that for any (fixed) $x \in U(a, \hat{\delta})$, the map $b \mapsto \bar{\psi}_x(b) := \psi_b(x)$ is a homeomorphism which conjugates $\sigma|_{B_{\text{prod}}}$ to $\sigma|_{\bar{\psi}_x(B)_{\text{prod}}}$. Moreover, the conjugacy map depends continuously on x , not only in the product topology but also in the ℓ_∞ topology; more precisely, the family of maps $x \mapsto \psi_b(x)$ from $U(a, \hat{\delta})$ to ℓ_∞ forms an equicontinuous family in $b \in B$.*

Proof. Since the domain is compact and the codomain is Hausdorff, the result follows immediately from the conclusions of theorems 2.1 and 2.4 and corollary 2.3. $\square$

Remark 2.6. Note that in theorem 2.4, the assumption on continuity of $F(x, \cdot)$ is with respect to the product topology and hence it is surely satisfied when each component of F is a continuous function depending on only finitely many coordinates of the domain (i.e. a finitary code function by terminology of the information theory):

$$(F(x, y))_n = f_n(x, y_{n+k(n)}, y_{n+k(n)+1}, \dots, y_{n+K(n)})$$

for each $x \in V_0$, $n \in \mathbb{Z}$, where $k(n)$ and $K(n)$ are some functions from $\mathbb{Z}$ to itself, and f_n is a continuous function from $\mathbb{R}^{K(n)+k(n)+2}$ to $\mathbb{R}$. If, in addition, $k(n)$ and $K(n)$ are constant (i.e. independent of n) then $F(x, \cdot)$ commutes with the shift map (in which case F is called a stationary code function).

Remark 2.7. One can check that theorems 2.4 and 2.5 still hold for a more general case when the spaces G and H are $\ell_\infty^d := (\ell_\infty)^d$, i.e. the direct product of d copies of ℓ_∞ , the set B is a subset of $([s_1, s_2]^{\mathbb{Z}})^d$ and $\sigma_d : \ell_\infty^d \rightarrow \ell_\infty^d$ is the direct product of d copies of the shift map σ .

3. Perturbation of zero-dimensional chaotic systems for difference equations

Let us consider a difference equation of the form

$$\Phi_\lambda(y_n, y_{n+1}, \dots, y_{n+m}) = 0, \quad (3)$$

where λ is a parameter from a metric space E and the function Φ_λ is defined on a closed subset $Q^{m+1} \subset \mathbb{R}^{m+1}$, where $Q = [s_1, s_2] \setminus V$ for some real numbers $s_1 < s_2$ and some open (possibly empty) set $V \subset \mathbb{R}$. We assume that for each $\lambda \in E$ the function $\Phi_\lambda : Q^{m+1} \rightarrow \mathbb{R}$ is C^1 and is continuous in λ on E and also that the partial derivatives $\partial_i \Phi_\lambda(x_1, \dots, x_{m+1})$, $i = 1, \dots, m+1$, $(x_1, \dots, x_{m+1}) \in Q^{m+1}$ are continuous in λ on E , where $\partial_i \Phi_\lambda$ is the partial derivative of Φ_λ with respect to the i th variable.

Given a $\lambda \in E$, let Y_λ be the set of solutions of the difference equation (3), i.e. the set of bi-sequences $\underline{y} = (y_n) = (\dots, y_{-1}, y_0, y_1, \dots)$ such that for any $n \in \mathbb{Z}$

1. $y_n \in Q$; and
2. $(m+1)$ consecutive components $y_n, y_{n+1}, \dots, y_{n+m}$ of $\underline{y}$ satisfy (3).

It is easy to see that Y_λ is a closed subset of the space ℓ_∞ . Note that $Y_{\lambda, \text{prod}}$ is also a closed subset of $[s_1, s_2]_{\text{prod}}^{\mathbb{Z}}$, and since the latter space is compact (by the Tichonov theorem), $Y_{\lambda, \text{prod}}$ is compact.

Note that the shift map σ , being considered as a map from ℓ_∞ to ℓ_∞ , is an isometric linear operator, while $\sigma : \ell_{\infty, \text{prod}} \rightarrow \ell_{\infty, \text{prod}}$ is a homeomorphism. It is evident that for any $\lambda \in E$, the set $Y_{\lambda, \text{prod}}$ is σ -invariant and the restriction $\sigma|_{Y_{\lambda, \text{prod}}}$ is a homeomorphism on a compact space. Thus, one can define the *topological entropy for solutions of the difference equation (3)* as $h_{\text{top}}(\sigma|_{Y_{\lambda, \text{prod}}})$.

In our applications, the difference equations for non-exceptional values of λ actually will be associated with well-defined maps. So we will need the following definition. We denote by σ_m the shift map acting on ℓ_∞^m .

Definition 3.1. We say that for a given λ , the difference equation (3) corresponds to a map $f_\lambda : P_\lambda \rightarrow \mathbb{R}^m$, where P_λ is a compact subset of $\mathbb{R}^m$ and f_λ is continuous, if $\sigma|_{Y_{\lambda, \text{prod}}}$ is conjugate to $\sigma_m|_{\tilde{P}_\lambda}$, where $\tilde{P}_\lambda = \{p = (p_n)_{n=-\infty}^\infty \in P_\lambda^{\mathbb{Z}} : p_{n+1} = f_\lambda(p_n)\}$, i.e. the space of full orbits for f_λ in P_λ endowed with the product topology.

We are concerned now with the case when the difference equation (3) corresponds to a map $f_\lambda : P_\lambda \rightarrow \mathbb{R}^m$. Let $T_\lambda : Y_\lambda \rightarrow \tilde{P}_\lambda$ denote the corresponding conjugacy map (we have omitted here the subscript prod, and we will do so again later if no doubts appear). So $\tilde{P}_\lambda$ is a compact and σ_m -invariant set. Let us denote $K_\lambda^+ = \bigcap_{n=0}^\infty f_\lambda^{-n}(P_\lambda)$. Then K_λ^+ is compact and f_λ -invariant, and $\sigma_m|_{\tilde{P}_\lambda}$ can be regarded as the inverse limit $\varprojlim (K_\lambda^+, f_\lambda)$ by identifying a sequence $(\dots, p_{-2}, p_{-1}, p_0) \in \varprojlim (K_\lambda^+, f_\lambda)$ with $(\dots, p_{-2}, p_{-1}, p_0, f_\lambda(p_0), f_\lambda^2(p_0), \dots) \in \tilde{P}_\lambda$.

Note that $K_\lambda := \pi_0(\tilde{P}_\lambda) = \bigcap_{n=0}^\infty f_\lambda^n(K_\lambda^+)$ (because of the compactness of K_λ^+), where by π_0 we denote the projection to the 0th coordinate. Obviously, π_0 is a semi-conjugacy map from $\sigma|_{\tilde{P}_\lambda}$ to $f_\lambda|_{K_\lambda}$, i.e. $f_\lambda \circ \pi_0 = \pi_0 \circ \sigma$. It is easy to see that if, in addition, $f_\lambda : P_\lambda \rightarrow \mathbb{R}^m$ is one-to-one, then π_0 is in fact a conjugacy. In any case, for a (possibly non-invertible) map f_λ , we may claim that π_0 is a semi-conjugacy which is an ‘almost conjugacy’; more precisely, for $\pi_0 : \tilde{P}_\lambda \rightarrow K_\lambda$ the following holds.

- (i) π_0 is surjective and so K_λ is compact and f_λ -invariant;
- (ii) π_0 yields a one-to-one correspondence between the sets of periodic points for $\sigma_m|_{\tilde{P}_\lambda}$ and $f_\lambda|_{K_\lambda}$; and
- (iii) π_0 preserves the topological entropy, i.e. for any compact f_λ -invariant set $L \subset K_\lambda$, $h_{\text{top}}(f_\lambda|_L) = h_{\text{top}}(\sigma_m|_{\pi_0^{-1}(L)})$.

Indeed, items (i) and (ii) are obvious while (iii) follows from the Bowen inequality in [7]

$$h_{\text{top}}(f_\lambda|_L) \leq h_{\text{top}}(\sigma_m|_{\pi_0^{-1}(L)}) \leq h_{\text{top}}(f_\lambda|_L) + \sup_{p \in L} h_{\text{top}}(\sigma_m, \pi_0^{-1}(p)),$$

since for any $p \in L$ we have that $h_{\text{top}}(\sigma_m, \pi_0^{-1}(p)) = 0$ because the diameter of the sets $(\sigma_m)^n(\pi_0^{-1}(p))$ tends to zero as $n \rightarrow +\infty$ (here we have chosen a metric compatible with the product topology).

We will call a map satisfying property (iii) above an entropy-preserving map. Notice that there are other properties of π_0 which recall the properties of a real conjugacy, because $\tilde{P}_\lambda$ is the inverse limit of the self-map f_λ on the compact space K_λ^+ (or on K_λ since

$\lim(K_\lambda^+, f_\lambda) = \lim(K_\lambda, f_\lambda)$. Roughly speaking, one may say that both systems have the same chaotic properties (with respect to several definitions of chaos); see the precise statements in [18]. So we have the following commutative diagram:

$$\begin{array}{ccccc} Y_\lambda & \xrightarrow{T_\lambda} & \tilde{P}_\lambda & \xrightarrow{\pi_0} & K_\lambda \\ \sigma \downarrow & \sigma_m \downarrow & & \downarrow f_\lambda & \\ Y_\lambda & \xrightarrow{T_\lambda} & \tilde{P}_\lambda & \xrightarrow{\pi_0} & K_\lambda. \end{array}$$

In this situation, the topological entropy $h_{\text{top}}(\sigma|Y_{\lambda, \text{prod}})$ is equal to $h_{\text{top}}(f_\lambda)$, the topological entropy of the map f_λ , which is meant as $h_{\text{top}}(f_\lambda|K_\lambda^+)$, the topological entropy of restriction of f_λ to the maximal f_λ -(forward) invariant set K_λ^+ . Indeed, we have $h_{\text{top}}(\sigma|Y_{\lambda, \text{prod}}) = h_{\text{top}}(\sigma|\tilde{P}_\lambda) = h_{\text{top}}(f_\lambda|K_\lambda)$ because T_λ is a conjugacy and π_0 is entropy-preserving by property (iii); further, the topological entropy $h_{\text{top}}(f_\lambda|K_\lambda)$ is the same as $h_{\text{top}}(f_\lambda|K_\lambda^+)$, because $K_\lambda = \pi_0(\tilde{P}_\lambda) = \bigcap_{n=0}^{\infty} f_\lambda^n(K_\lambda^+)$; see corollary 8.6.1 in [27].

Remark 3.2. Usually (see examples below and also [15, 17, 23]), the conjugacy T_λ^{-1} from $\tilde{P}_\lambda$ to Y_λ is of the form $(p_n)_{n=-\infty}^{\infty} \mapsto (y_n)_{n=-\infty}^{\infty} = (\pi(p_n))_{n=-\infty}^{\infty}$, where $\pi : \mathbb{R}^m \rightarrow \mathbb{R}$ is the projection to a chosen coordinate, while T_λ can be expressed by $p_n = G(y_{n-N}, y_{n+1-N}, \dots, y_{n+m-1-N})$ for some integer N and diffeomorphism $G : \mathbb{R}^m \rightarrow \mathbb{R}^m$.

Now according to notations from the previous section, we define $F : E \times Q^{\mathbb{Z}} \rightarrow \ell_\infty$ by

$$F(\lambda, \underline{y}) = (\Phi_\lambda(y_n, y_{n+1}, \dots, y_{n+m}))_{n=-\infty}^{\infty}. \quad (4)$$

Then $F(\lambda, \cdot)$ is σ -invariant; indeed,

$$\begin{aligned} F(\lambda, \sigma(\underline{y})) &= F(\lambda, (y_{n+1})_{n=-\infty}^{\infty}) = (\Phi_\lambda(y_{n+1}, \dots, y_{n+m+1}))_{n=-\infty}^{\infty} \\ &= \sigma(\Phi_\lambda(y_n, \dots, y_{n+m})_{n=-\infty}^{\infty}) = \sigma(F(\lambda, \underline{y})). \end{aligned}$$

Note that Y_λ is precisely the kernel of $F(\lambda, \cdot)$, i.e. the set of bi-sequences $\underline{y}$ satisfying $F(\lambda, \underline{y}) = \underline{0}$.

Let $\lambda_0 \in E$ be the initial value of parameter λ , i.e. λ_0 corresponds to the unperturbed difference equation (3) (in the previous section, such a value is denoted by a). Let B be a subset of Y_{λ_0} . We now check the assumptions of theorem 2.1 for our settings. By the definition of Y_λ the assumption (i) is readily satisfied. Given $\underline{y} = (y_n) \in Q^{\mathbb{Z}}$, $\lambda \in E$, and integers $n \in \mathbb{Z}$, $1 \leq i \leq m+1$, we denote for brevity $\partial_i \Phi_\lambda(\underline{y}_n) = \partial_i \Phi_\lambda(y_n, y_{n+1}, \dots, y_{n+m})$. Then the partial derivative operator $D_2 F(\lambda, \underline{y})$ exists at any point $(\lambda, \underline{y}) \in E \times Q^{\mathbb{Z}}$ and corresponds to the following bi-infinite band matrix

$$\begin{aligned} & D_2 F(\lambda, \underline{y}) \\ &= \begin{bmatrix} \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & \partial_1 \Phi_\lambda(\underline{y}_n) & \partial_2 \Phi_\lambda(\underline{y}_n) & \cdot & \cdots & \partial_{m+1} \Phi_\lambda(\underline{y}_n) & 0 & \cdots \\ \cdots & 0 & \partial_1 \Phi_\lambda(\underline{y}_{n+1}) & \partial_2 \Phi_\lambda(\underline{y}_{n+1}) & \cdots & \cdot & \partial_{m+1} \Phi_\lambda(\underline{y}_{n+1}) & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix} \leftarrow \begin{array}{l} \text{the} \\ n\text{th} \\ \text{row.} \end{array} \\ & \quad \uparrow \\ & \quad \text{the } n\text{th column} \end{aligned}$$

Since $\Phi_\lambda(x_1, \dots, x_{m+1})$ as well as the the partial derivatives $\partial_i \Phi_\lambda(x_1, \dots, x_{m+1})$, $i = 1, \dots, m+1$, are continuous both in $\lambda \in E$ and $(x_1, \dots, x_{m+1}) \in Q^{m+1}$, and since $E \times Q^{m+1}$

is compact, it follows that both $F(\lambda, y)$ and $D_2F(\lambda, y)$ are uniformly continuous on $E \times Q^{\mathbb{Z}}$. Therefore the assumptions (ii) and (iii) of theorem 2.1 are also satisfied.

As for (a more delicate) assumption (iv), we need to consider some special properties of the set B in order to guarantee (iv), and we are now going to do this. We consider the case when the unperturbed difference equation involves only one variable. More precisely, let

$$\Phi_{\lambda_0}(y_0, \dots, y_m) = \varphi(y_N),$$

where $0 \leq N \leq m$ and $\varphi : Q \rightarrow \mathbb{R}$ is a C^1 function. In addition, we suppose that φ has simple zeros at $d_1, d_2, \dots, d_k$ in $\text{int}(Q)$, i.e. $\varphi'(d_i) \neq 0$ for all $1 \leq i \leq k$. Then for any $\underline{d} = (d_i)_{i=-\infty}^{\infty} \in \{d_1, \dots, d_k\}^{\mathbb{Z}}$, the matrix $D_2F(\lambda_0, \underline{d})$ reduces to the diagonal matrix of the form

$$D_2F(\lambda_0, \underline{d}) = \sigma^N \cdot \text{diag}(\dots, \varphi'(d_{i_n}), \varphi'(d_{i_{n+1}}), \varphi'(d_{i_{n+2}}), \dots),$$

whose entries are

$$(D_2F(\lambda_0, \underline{d}))_{n,j} = \begin{cases} \varphi'(d_{i_n}), & \text{if } j = n + N, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$(D_2F(\lambda_0, \underline{d}))^{-1} = \sigma^{-N} \cdot \text{diag}\left(\dots, \frac{1}{\varphi'(d_{i_n})}, \frac{1}{\varphi'(d_{i_{n+1}})}, \frac{1}{\varphi'(d_{i_{n+2}})}, \dots\right)$$

and consequently

$$\|(D_2F(\lambda_0, \underline{d}))^{-1}\| \leq \left(\min_{1 \leq i \leq k} |\varphi'(d_i)| \right)^{-1}.$$

Hence the assumption (iv) of theorem 2.1 is satisfied for $B = \{d_1, \dots, d_k\}^{\mathbb{Z}}$, and we are able to use the results of section 2 (note that the additional assumptions contained in theorems 2.4 and 2.5 are surely satisfied). So we may use the conjugacy $\tilde{\psi}_\lambda|B$ from theorem 2.5 in order to get a closed σ -invariant subset $\Gamma_\lambda := \tilde{\psi}_\lambda(B)$ of $Y_{\lambda, \text{prod}}$ such that $\sigma|_{\Gamma_\lambda}$ is conjugate to $\sigma|_{\Sigma_k}$, the full shift on k symbols. As an immediate consequence we have the following theorem. For simplicity, we consider here one-parameter families with parameter $\lambda \in E := [\lambda_0, \lambda_1]$, where λ_0 might be either a real number or infinity, in which case E is the metric space of one point compactification with the metric $\rho(\lambda, \lambda') = |\frac{1}{\lambda} - \frac{1}{\lambda'}|$; $\frac{1}{\infty} = 0$. We summarize our arguments as follows.

Theorem 3.3. *Let*

$$\Phi_\lambda(y_n, y_{n+1}, \dots, y_{n+m}) = 0 \tag{5}$$

be a difference equation with parameter $\lambda \in [\lambda_0, \lambda_1]$ and let the function $\Phi_\lambda : Q^{m+1} \rightarrow \mathbb{R}$, where $Q = [s_1, s_2] \setminus V$ for some numbers $s_1 < s_2$ and some open set $V \subset [s_1, s_2]$, be such that it is C^1 for each λ and is continuous in λ and so are the partial derivatives $\partial_i \Phi_\lambda$, $i = 1, \dots, m+1$. Suppose that for $\lambda = \lambda_0$, the function Φ_{λ_0} depends on only one variable: $\Phi_{\lambda_0}(x_1, x_2, \dots, x_{m+1}) = \varphi(x_N)$, where N is an integer with $1 \leq N \leq m+1$ and $\varphi : Q \rightarrow \mathbb{R}$ is a C^1 function with k simple zeros in the interior of Q .

Then there exists $\bar{\delta} > 0$ such that for any $\lambda \in [\lambda_0, \lambda_0 + \bar{\delta})$ there is a closed σ -invariant subset Γ_λ of Y_λ , the set of solutions for (5) in the product topology, such that

- (i) $\sigma|_{\Gamma_\lambda}$ is topologically conjugate to $\sigma|_{\Sigma_k}$, the full shift on k symbols; in particular, $h_{\text{top}}(\sigma|_{Y_\lambda}) \geq \log k$;
- (ii) the conjugacy map $\tilde{\psi}_\lambda : \Sigma_\lambda \rightarrow \Gamma_\lambda$ is the identity map for $\lambda = \lambda_0$ and is continuous in λ ; moreover, the map $\lambda \mapsto \tilde{\psi}_\lambda(\underline{x})$ from $[\lambda_0, \lambda_0 + \bar{\delta})$ to ℓ_∞ (in the uniform topology) forms an equicontinuous family in $\underline{x} \in \Sigma_k$; and

(iii) if, in addition, the difference equation (5) for given λ corresponds to a map $f_\lambda : P_\lambda \rightarrow \mathbb{R}^m$, then $h_{\text{top}}(f_\lambda) \geq \log k$ and one has the following commutative diagram:

$$\begin{array}{ccccccc} \Sigma_k & \xrightarrow{\bar{\psi}_\lambda} & Y_\lambda & \xrightarrow{T_\lambda} & \tilde{P}_\lambda & \xrightarrow{\pi_0} & K_\lambda \\ \sigma \downarrow & & \sigma \downarrow & & \sigma_m \downarrow & & \downarrow f_\lambda \\ \Sigma_k & \xrightarrow{\bar{\psi}_\lambda} & Y_\lambda & \xrightarrow{T_\lambda} & \tilde{P}_\lambda & \xrightarrow{\pi_0} & K_\lambda, \end{array}$$

where $\bar{\psi}_\lambda$ is injective, T_λ is bijective and π_0 is a (surjective) projection which is entropy preserving and is bijective when restricted to the set of periodic points; here $\tilde{P}_\lambda = \{\underline{p} = (p_n)_{n=-\infty}^\infty \in P_\lambda^{\mathbb{Z}} : p_{n+1} = f_\lambda(p_n)\}$ and $K_\lambda = \bigcap_{i=0}^\infty f_\lambda^i(\bigcap_{n=0}^\infty f_\lambda^{-n}(P_\lambda))$. In particular, the map $\theta_\lambda := \pi_0 \circ T_\lambda \circ \bar{\psi}_\lambda$ semiconjugates the full shift on k symbols to the restriction of f_λ to the closed f_λ -invariant subset $\Lambda_\lambda := \theta_\lambda(\Sigma_k)$ and thus the map $f_\lambda|_{\Lambda_\lambda}$ is transitive and its periodic points are dense in Λ_λ . Moreover, the inverse limit of $f_\lambda|_{\Lambda_\lambda}$ is conjugate to $\sigma|_{\Sigma_k}$. If $f_\lambda : P_\lambda \rightarrow \mathbb{R}^m$ is injective, then the above semiconjugacy $\theta_\lambda|_{\Sigma_k}$ is in fact a conjugacy.

4. Examples

4.1. The generalized Hénon-like maps

We now apply the above results to generalized Hénon-like maps (for relevant properties of Hénon-like maps see [10, 12, 15, 21, 23, 26]). Let H_λ be the following m -dimensional ($m \geq 2$) one-parameter family of Hénon-like maps.

$$H_\lambda(x_1, x_2, \dots, x_{m-1}, x_m) = (x_2, x_3, \dots, x_m, g(x_1, \dots, x_m) + \lambda\varphi(x_m)), \quad (6)$$

where $\lambda \in \mathbb{R}$ is a parameter, $g : \mathbb{R}^m \rightarrow \mathbb{R}$ is a C^1 function, and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 function with k simple zeros. We do not require that $\partial_1 g(x_1, \dots, x_m) \neq 0$ and so H_λ need not to be a diffeomorphism on $\mathbb{R}^m$ (even locally). Furthermore, we allow the domains of C^1 functions φ and g to be $Q = [s_1, s_2] \setminus V$ and Q^m , respectively, for some real numbers $s_1 < s_2$ and open set $V \subset [s_1, s_2]$. In such a situation we are interested only in those orbits under H_λ which stay always in $([s_1, s_2] \setminus V)^m$, and we show below that if φ has k simple zeros in $\text{int}([s_1, s_2] \setminus V)$ then for $|\lambda|$ sufficiently large, there still remain enough orbits to create a full shift on k symbols. Note that if $V = \emptyset$ and $\partial_1 g(x_1, \dots, x_m) \neq 0$ on Q^m then H_λ is injective for any λ ; this can be verified directly by the contrary.

For an initial point $p = (x_{0,1}, x_{0,2}, \dots, x_{0,m})$, the n th iteration $H_\lambda^n(p)$ of p under H_λ will be also denoted by $(x_{n,1}, x_{n,2}, \dots, x_{n,m})$. Then for any $n \in \mathbb{Z}$, we have the following system consisting of $m(m-1) + 1$ equations (the index i for labelling the equations below takes the values $1, 2, \dots, m-1$):

$$\begin{cases} x_{n+i,1} = x_{n+i-1,2}, & (1, i) \\ x_{n+i,2} = x_{n+i-1,3}, & (2, i) \\ \vdots & \vdots \\ x_{n+i,m-1} = x_{n+i-1,m}, & (m-1, i) \\ x_{n+i,m} = g(x_{n+i-1,1}, \dots, x_{n+i-1,m}) + \lambda\varphi(x_{n+i-1,m}), & (m, i) \\ x_{n+m,1} = x_{n+m-1,2}. & (1, m+1) \end{cases}$$

By the elimination process with respect to the first coordinate in $\mathbb{R}^m$, we get

$$\begin{aligned} x_{n+m,1} &= x_{n+m-1,2} = \dots = x_{n+1,m} \\ &= g(x_{n,1}, x_{n,2}, \dots, x_{n,m}) + \lambda\varphi(x_{n,m}) \\ &= g(x_{n,1}, x_{n+1,1}, \dots, x_{n+m-1,1}) + \lambda\varphi(x_{n+m-1,1}). \end{aligned}$$

Let $\pi_1 : \mathbb{R}^m \rightarrow \mathbb{R}$ denote the projection to the first coordinate. Then for any $p = (x_{0,1}, x_{0,2}, \dots, x_{0,m}) \in \mathbb{R}^m$, the projection $\pi_1(H_\lambda^n(p)) = x_{n,1} := y_n$ of the full orbit of p under H_λ satisfies the following difference equation.

$$\Phi_\lambda(y_n, y_{n+1}, \dots, y_{n+m}) := \frac{1}{\lambda} y_{n+m} - \frac{1}{\lambda} g(y_n, y_{n+1}, \dots, y_{n+m-1}) - \varphi(y_{n+m-1}) = 0. \quad (7)$$

Conversely, solutions $(y_n)_{n=-\infty}^\infty$ of (7) uniquely determine the full orbits of H_λ ; indeed, it is easily seen from the above system of equations that the correspondence $\underline{y} = (y_n)_{n=-\infty}^\infty \mapsto \underline{p} = (p_n)_{n=-\infty}^\infty \in (\mathbb{R}^m)^\mathbb{Z}$ with $p_n = (y_n, y_{n+1}, \dots, y_{n+m-1})$ provides the full orbits of H_λ . So we have that the correspondence map T_λ provides a conjugacy between the shift map acting on Y_λ , the set of solutions of (7) with $y_n \in Q$ for all $n \in \mathbb{Z}$, and $\tilde{P}_\lambda$, the set of full orbits of H_λ in $P_\lambda := Q^m$, both sets being considered with respect to the product topologies.

$$\begin{array}{ccc} Y_\lambda & \xrightarrow{\sigma} & Y_\lambda \\ T_\lambda \downarrow & & \downarrow T_\lambda \\ \tilde{P}_\lambda & \xrightarrow[\sigma_m]{} & \tilde{P}_\lambda \end{array} \quad \begin{array}{l} (T_\lambda(\underline{y}))_n = (y_n, \dots, y_{n+m-1}) \\ (T_\lambda^{-1}(\underline{p}))_n = \pi_1(p_n). \end{array}$$

Note that if H_λ is injective then $\sigma_m : \tilde{P}_\lambda \rightarrow \tilde{P}_\lambda$ is conjugate to the Hénon map itself.

Now let $\lambda \rightarrow \infty$. Then for $\lambda_0 := \infty$, we have the limit difference equation

$$\Phi_{\lambda_0}(y_n, y_{n+1}, \dots, y_{n+m}) = -\varphi(y_{n+m-1}) = 0,$$

which does not correspond to any map but satisfies our assumptions above. Thus theorem 3.3 implies the following.

Proposition 4.1. *Let H_λ be a one-parameter family of Hénon-like maps of the form (6), where φ and g are C^1 functions on $Q = [s_1, s_2] \setminus V$ and Q^m , respectively, for some real numbers $s_1 < s_2$ and open set $V \subset [s_1, s_2]$. If φ has k simple zeros in $\text{int}(Q)$, then for $|\lambda|$ sufficiently large the statements of theorem 3.3 hold (with $f_\lambda = H_\lambda$); in particular, by item (iii), there is a closed H_λ -invariant subset Λ_λ of Q^m such that the inverse limit of $H_\lambda|_{\Lambda_\lambda}$ is conjugate to the full shift on k symbols. If $V = \emptyset$ and $\partial_1 g(x_1, \dots, x_m) \neq 0$ on Q^m then in fact, $H_\lambda|_{\Lambda_\lambda}$ itself is conjugate to the full shift on k symbols.*

Remark 4.2. It is allowed that the function (6) might have discontinuities. For example, one may include Lorenz-type maps instead of the usual quadratic maps there, more precisely, if one considers Hénon-like maps of the form

$$H_\lambda(x_1, x_2, \dots, x_{m-1}, x_m) = (x_2, x_3, \dots, x_m, g(x_1, \dots, x_m) + \lambda x - \frac{\lambda - 1}{2}(1 + \text{sgn}(x - 1/2))). \quad (8)$$

Then by repeating the above arguments one gets for $\lambda_0 = \infty$ the equations

$$\Phi_{\lambda_0}(y_n, y_{n+1}, \dots, y_{n+m}) = y_{n+m-1} - \frac{1}{2}(1 + \text{sgn}(y_{n+m-1} - \frac{1}{2})) = 0,$$

which have two simple zeros $\{0, 1\}$.

There are families of Hénon-like maps for which the above propositions do not apply directly; nevertheless, they might be of use when adapted by appropriate scaling. For example, consider the (original) Hénon maps in the form $H_\lambda(x, y) = (y, bx + \lambda - ay^2)$ with $a > 0$ and parameter $\lambda \rightarrow +\infty$. Then after rescaling $\hat{x} = x/\sqrt{\lambda}$ and $\hat{y} = y/\sqrt{\lambda}$, we have for the new coordinates,

$$\hat{H}_\lambda(\hat{x}, \hat{y}) = (\hat{y}, b\hat{x} + \sqrt{\lambda}(1 - a\hat{y}^2))$$

and the corresponding difference equation reads

$$\hat{\Phi}_\lambda(y_n, y_{n+1}, y_{n+2}) = \frac{1}{\sqrt{\lambda}} y_{n+2} - \frac{b}{\sqrt{\lambda}} y_n - (1 - ay_{n+1}^2) = 0.$$

So we are able to use theorem 3.3 (with $\lambda_0 = +\infty$, $k = 2$, and s_1, s_2 any two numbers such that $s_1 < -\sqrt{a}$ and $s_2 > \sqrt{a}$) to provide a conjugacy between orbits of the full shift on two symbols and orbits of H_λ with λ sufficiently large.

The same arguments work for a family of m -dimensional Hénon-like maps of the form

$$H_\lambda(x_1, x_2, \dots, x_m) = (x_2, x_3, \dots, x_m, g(x_1, \dots, x_m) + \lambda - ax_m^{2j}), \quad (9)$$

where $a > 0$, $j \in \mathbb{N}$, and $g(x_1, \dots, x_m)$ is a real polynomial with $\deg(g) < 2j$. After rescaling $\hat{x}_i = x_i / \sqrt[2j]{\lambda}$ for $i = 1, \dots, m$, we have

$$\hat{H}_\lambda(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_m) = \left(\hat{x}_2, \hat{x}_3, \dots, \hat{x}_m, \frac{1}{\sqrt[2j]{\lambda}} g(\sqrt[2j]{\lambda} \hat{x}_1, \dots, \sqrt[2j]{\lambda} \hat{x}_m) + \frac{\lambda}{\sqrt[2j]{\lambda}} (1 - a\hat{x}_m^{2j}) \right)$$

and the corresponding difference equation reads

$$\hat{\Phi}_\lambda(y_n, y_{n+1}, \dots, y_{n+m}) := \frac{\sqrt[2j]{\lambda}}{\lambda} y_{n+m} - \frac{1}{\lambda} g(\sqrt[2j]{\lambda} y_n, \dots, \sqrt[2j]{\lambda} y_{n+m-1}) + (1 - ay_{n+m-1}^{2j}) = 0.$$

Since $\deg(g) < 2j$, we have that $(1/\lambda)g(\sqrt[2j]{\lambda} y_n, \dots, \sqrt[2j]{\lambda} y_{n+m-1}) \rightarrow 0$ as $\lambda \rightarrow +\infty$ and since the function $\varphi(x) := 1 - ax^{2j}$ has two simple zeros $\{-\sqrt[2j]{a^{-1}}, \sqrt[2j]{a^{-1}}\}$, we may apply theorem 3.3 with constants $s_1 < -\sqrt[2j]{a^{-1}}$ and $s_2 > \sqrt[2j]{a^{-1}}$ and get the following result.

Proposition 4.3. *Let H_λ be a one-parameter family of Hénon-like maps of the form (9) where $a > 0$, $j \in \mathbb{N}$ and $g(x_1, \dots, x_m)$ is a real polynomial with $\deg(g) < 2j$. Then for $\lambda > 0$ sufficiently large the statements of theorem 3.3 hold (with $f_\lambda = H_\lambda$, $k = 2$); in particular, by item (iii) there is a closed H_λ -invariant subset $\Lambda_\lambda \subset \mathbb{R}^m$ for which the inverse limit of $H_\lambda|_{\Lambda_\lambda}$ is conjugate to the full shift on two symbols.*

4.2. The Arneodo–Coullet–Tresser maps

Now we consider the family of the so-called ACT maps $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ (due to Arneodo, Coullet and Tresser; refer to [11]); they are of the form

$$f(x, y, z) = (ax - b(y - z), bx + a(y - z), cx - dx^k + ez), \quad (10)$$

where $a, b, c, d, e \in \mathbb{R}$ are parameters and $k \geq 2$. If $(a^2 + b^2)e \neq 0$, then f is a diffeomorphism with the inverse

$$f^{-1}(x, y, z) = \left(\hat{x}, \frac{-bx + ay}{a^2 + b^2} + \hat{z}, \hat{z} \right), \quad \text{where } \hat{x} = \frac{ax + by}{a^2 + b^2} \text{ and } \hat{z} = \frac{z - c\hat{x} + d\hat{x}^k}{e}.$$

If $bd \neq 0$, then there are interesting dynamical properties and bifurcations in several regions of the parameter space; see [11]. For an initial point $p = (x_0, y_0, z_0)$, denote the n th iteration of p under f by (x_n, y_n, z_n) . Then for any $n \in \mathbb{Z}$, we have the following system consisting of 7 equations (the index i for labelling the equations below takes the values 1 and 2):

$$\begin{cases} x_{n+i} = ax_{n+i-1} - b(y_{n+i-1} - z_{n+i-1}), & (1, i) \\ y_{n+i} = bx_{n+i-1} + a(y_{n+i-1} - z_{n+i-1}), & (2, i) \\ z_{n+i} = cx_{n+i-1} - dx_{n+i-1}^k + ez_{n+i-1}, & (3, i) \\ x_{n+3} = ax_{n+2} - b(y_{n+2} - z_{n+2}). & (1, 3) \end{cases}$$

Assuming $b \neq 0$, From equations (1, 1) and (2, 1), we can express y_{n+1} in terms of x_n and x_{n+1} . Then plugging y_{n+1} into equation (1, 2), we can express z_{n+1} in terms of x_n , x_{n+1} and x_{n+2} . The expressions are

$$\begin{aligned} y_{n+1} &= \frac{a^2 + b^2}{b} x_n - \frac{a}{b} x_{n+1}, \\ z_{n+1} &= \frac{a^2 + b^2}{b} x_n - \frac{2a}{b} x_{n+1} + \frac{1}{b} x_{n+2}. \end{aligned}$$

Similarly, from equations (1, 2), (2, 2) and (1, 3), we can express z_{n+2} in terms of x_{n+1} , x_{n+2} and x_{n+3} . Then, plugging the expressions of z_{n+1} and z_{n+2} into equation (3, 2), we get the difference equation

$$dx_{n+1}^k - \frac{a^2 e + b^2 e}{b} x_n + \frac{a^2 + b^2 - bc + 2ae}{b} x_{n+1} - \frac{2a + e}{b} x_{n+2} + \frac{1}{b} x_{n+3} = 0. \quad (11)$$

Suppose the parameters a , b and e to be fixed and let $c \rightarrow \infty$, $d \rightarrow \infty$ in such a way that $c/d = \text{constant} := A > 0$. Denote $\lambda = 1/d$, then $c = A/\lambda$ and for the x -coordinate of orbit under the ACT map $f = f_\lambda$, we have the difference equation

$$x_{n+1}(x_{n+1}^{k-1} - A) + \lambda \left(-\frac{a^2 e + b^2 e}{b} x_n + \frac{a^2 + b^2 + 2ae}{b} x_{n+1} - \frac{2a + e}{b} x_{n+2} + \frac{1}{b} x_{n+3} \right) = 0. \quad (12)$$

The difference equation (12) corresponds to the map f_λ for $\lambda \neq 0$ because between solutions $\underline{x} = (x_n)_{n=-\infty}^\infty$ of (12) and full orbits $\underline{p} = (p_n)_{n=-\infty}^\infty$ of f_λ we have a conjugacy $\underline{x} \mapsto \underline{p}$, given by

$$p_n = \left(x_n, \frac{a^2 + b^2}{b} x_{n-1} - \frac{a}{b} x_n, \frac{a^2 + b^2}{b} x_{n-1} - \frac{2a}{b} x_n + \frac{1}{b} x_{n+2} \right),$$

while the inverse $\underline{p} \mapsto \underline{x}$ is given by $x_n = \pi_1(p_n)$. For the limit value $\lambda_0 = 0$ of parameter, we have

$$\varphi(x_{n+1}) := x_{n+1}(x_{n+1}^{k-1} - A) = 0.$$

Thus, the function φ has at least two simple zeros; more precisely φ has two simple zeros $\{0, \sqrt[k-1]{A}\}$ when k is even, and three simple zeros $\{0, \pm \sqrt[k-1]{A}\}$ when k is odd. Thus we are able to use theorem 3.3 for the above family of the ACT maps (with $s_1 < -\sqrt[k-1]{A}$ and $s_2 > \sqrt[k-1]{A}$) and have the following result.

Proposition 4.4. *Suppose that for the family of the ACT maps (10), the parameters a , $b \neq 0$ and e are fixed while $c \rightarrow \infty$ and $d \rightarrow \infty$ in such a way that $c/d = \text{constant} > 0$. Let $\lambda = 1/d$. Then for all $|\lambda|$ sufficiently small, the ACT map $f = f_\lambda$ has a closed invariant set Λ_λ such that $f_\lambda|_{\Lambda_\lambda}$ is conjugate to the full shift on either two or three symbols depending on whether k is even or odd, respectively.*

There are also other parameter routes in families of ACT maps, for which the above arguments apply. Namely, if we consider a and c to be fixed while $e \rightarrow 0$, and $b \rightarrow \infty$ and $d \rightarrow \infty$ in such a way that $b/d = \text{constant} > 0$, then similarly along the lines of the previous proposition, we will have the following.

Proposition 4.5. *Suppose that for the family of the ACT maps (10), the parameters a and c are fixed while $e \rightarrow 0$, $b \rightarrow \infty$ and $d \rightarrow \infty$ in such a way that $b/d = \text{constant} > 0$. Let $\lambda = 1/d$. Then for all sufficiently small $|e|$ and $|\lambda|$, the ACT map $f = f_{e,\lambda}$ has a closed invariant set $\Lambda_{e,\lambda}$ such that $f_{e,\lambda}|_{\Lambda_{e,\lambda}}$ is conjugate to the full shift on either two or three symbols depending on whether k is even or odd, respectively.*

One can also replace the variable x_i by $\sqrt[k-1]{|c|}\hat{x}_i$ so that (11) becomes

$$d \sqrt[k-1]{|c|} \hat{x}_{n+1}^k - \sqrt[k-1]{|c|} \left(\frac{a^2 e + b^2 e}{b} \lambda \hat{x}_n + \frac{a^2 + b^2 - bc + 2ae}{b} \hat{x}_{n+1} - \frac{2a + e}{b} \hat{x}_{n+2} + \frac{1}{b} \hat{x}_{n+3} \right) = 0.$$

By using the above method, we have the following result.

Proposition 4.6. *Suppose that for the family of the ACT maps (10), the parameters $a, b \neq 0, d \neq 0, e$ are fixed. Then for all sufficiently small $|c|$ with $c \neq 0$, the ACT map $f = f_c$ has a closed invariant set Λ_c such that $f_c|_{\Lambda_c}$ is conjugate to the full shift on either two or three symbols provided that k is even or k is odd and $d < 0$, respectively.*

4.3. Quadratic volume preserving maps

In this subsection, we consider the family of volume preserving maps $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$f(x, y, z) = (\eta + \alpha x + \beta y + z + Q(x, y), x, y), \quad (13)$$

where η, α, β are real parameters and $Q(x, y) = ax^2 + bxy + cy^2$ is a quadratic form. As shown by Lomeli and Meiss in [19], generically every quadratic volume preserving map is conjugate to a map of the above form, while for volume preserving quadratic automorphisms (i.e. for diffeomorphisms with quadratic inverse) in generic form, one may have $a + b + c = 1$ and $\beta = 0$.

It is easy to see that the corresponding difference equation for (13) is of the form

$$\eta + \alpha x_n + \beta x_{n-1} + x_{n-1} + ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2 - x_{n+1} = 0.$$

By using the above method, we have the following result.

Proposition 4.7. *Suppose that for the family of quadratic volume preserving maps (13), the parameters η, β, b, c are fixed while $\alpha \rightarrow \infty$ and $a \rightarrow \infty$ in such a way that $a/\alpha = \text{constant} < 0$. Let $\lambda = 1/a$. Then for all sufficiently small $|\lambda|$, the map $f = f_\lambda$ has a closed invariant set Λ_λ such that $f_\lambda|_{\Lambda_\lambda}$ is conjugate to the full shift on two symbols.*

Notice that for quadratic automorphisms, the presented method does not apply because the equality $a + b + c = 1$ implies that in the anti-integrable limit both variables x_n and x_{n-1} should be involved, and so the limit function contains two variables. In a forthcoming paper [13], we consider such a case when at the singular value of parameter, the limit function depends on two (not necessarily consecutive) variables.

4.4. Steady states in lattice models

Finally, we consider steady states in lattice models which are discrete versions of partial differential equations of evolution type (for lattice models and their chaotic and stability properties see section 4.3 of [3] and [1, 2, 4, 8, 22]). The discrete versions are obtained by replacing derivatives with respect to time and space by appropriate differences (see [22]). In general, the time-space discrete version of these PDEs are of the form

$$u_n^{t+1} = f(u_n^t) + \epsilon g(u_{n-s}^t, u_{n-s+1}^t, \dots, u_{n+s}^t), \quad (14)$$

where $t \in \mathbb{Z}$ is the time variable and $n \in \mathbb{Z}$ is the space one. The function f is called the local map and g is called the interaction of finite size s .

If we look for the steady-state (or stationary) solutions u_n^t of (14), then u_n^t must be independent of the time coordinate t , i.e., $u_n^t := x_n$ for all $t \in \mathbb{Z}$. In this case, equation (14) can be reduced to the difference equation

$$x_n = f(x_n) + \epsilon g(x_{n-s}, x_{n-s+1}, \dots, x_{n+s}), \quad n \in \mathbb{Z}. \quad (15)$$

Thus, by using theorem 3.3, we have the following result on the chaotic structure of steady-state solutions.

Proposition 4.8. *Let the function $x \mapsto x - f(x)$ have k simple zeros and be of class C^1 in a neighbourhood U of these zeros, and let g be of class C^1 in U^{2s+1} . Then for sufficiently small ϵ , there exists a closed (in the product topology) σ -invariant subset Λ_ϵ of the set of steady-state solutions for (14) such that $\sigma|_{\Lambda_\epsilon}$ is topologically conjugate to $\sigma|_{\Sigma_k}$, the full shift on k symbols, and the conjugacy map can be chosen to be continuous in ϵ with respect not only to the product topology but also to the uniform topology.*

Let us remark that unlike in [2, 4, 22] we do not require that the functions f and g are defined and smooth on the whole $\mathbb{R}$ and $\mathbb{R}^{2s+1}$, respectively, with bounded partial derivatives.

There are several partial differential equations of evolution type whose discrete versions can be considered as lattice dynamical systems of the form (14). For instance, the one-dimensional nonlinear reaction diffusion equation

$$u_t = h(u) + \epsilon u_{xx}, \quad (16)$$

where ϵ is the diffusion coefficient, has its discrete version in the following form:

$$u_n^{t+1} = u_n^t + \delta h(u_n^t) + \delta \eta^{-2} \epsilon (u_{n+1}^t - 2u_n^t + u_{n-1}^t),$$

where δ and η are the discretization steps of time and space, respectively, h is a nonlinear function and t and n are the time and space variables, respectively. Hence the steady-states are solutions of the difference equation

$$h(u_n) + \eta^{-2} \epsilon (u_{n+1} - 2u_n + u_{n-1}) = 0. \quad (17)$$

The well-known particular cases of reaction diffusion equation are:

1. Kolmogorov–Petrovsky–Piskunov equation with $h(u) = \alpha u(1 - u)$, where $\alpha > 0$;
2. Huxley equation with $h(u) = \alpha u(1 - u)(u - a)$, where $0 < a < 1$ and $\alpha > 0$;
3. Frenkel–Kontorowa model with $h(u) = \sin(u)$, refer to [5, 6].

As another example, we can consider the Ginzburg–Landau equation,

$$u_t = h(u) + (\epsilon + i\delta)u_{xx},$$

whose real version has the form (16) with $h(u) = u(\alpha - \beta u^2)$ and real parameters ϵ, α, β . Since the function h for the Kolmogorov–Petrovsky–Piskunov equation has two simple zeros while for the Huxley equation and Ginzburg–Landau equation it has three simple zeros, it follows that proposition 4.8 applies for these systems whenever the parameter ϵ is sufficiently small. Therefore, the steady-state solutions of these systems contain chaotic dynamical behaviour conjugate to full shift on two or three symbols, respectively.

As one can notice, the results of section 3 can be presented in the setting of high dimensional difference equations. To this end we can consider the difference equation (5) with coordinates $y_n \in \mathbb{R}^d$, the operator Φ_λ with domain $(Q^d)^{m+1} \subset (\mathbb{R}^d)^{m+1}$ and co-domain $\mathbb{R}^d$, and the shift map σ_d acting on ℓ_∞^d . So, the high dimensional reaction diffusion equation

$$\frac{\partial \vec{u}}{\partial t} = h(\vec{u}) + \epsilon A \Delta \vec{u},$$

where A is the coupling matrix, can be also considered in our context. For example, for the two dimensional FitzHugh–Nagumo equation, the function h is of the form

$$h(u_1, u_2) = (du_1(u_1 - \theta)(1 - u_1) - au_2, bu_1 - cu_2),$$

where $0 < \theta < 1$ and $a, b, c, d > 0$. Thus, proposition 4.8 is applicable here because in certain regions of parameters, the function h has three zeros, namely,

$$(0, 0) \text{ and } \left(u_{\pm}, \frac{b}{c}u_{\pm}\right), \text{ where } u_{\pm} = \frac{1}{2} \left(1 + \theta \pm \sqrt{(1 - \theta)^2 - \frac{4ab}{cd}}\right),$$

with the non-zero Jacobian determinant at these zeros.

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Appendix

Here we give the proof of our uniform version of the implicit function theorem.

Proof of Theorem 2.1. For any $b \in B$, we define a function $g_b : V_0 \times U[b, \eta_0] \rightarrow G$ by

$$g_b(x, y) = y - (D_2F(a, b))^{-1}F(x, y).$$

Then for any $b \in B$,

$$g_b(a, b) = b \quad \text{and} \quad D_2g_b(a, b) = 0.$$

From now on, we denote $T_b = D_2F(a, b)$. By assumptions (iii) and (iv), there are $0 < \delta_1 < \delta_0$ and $0 < \eta_1 < \eta_0$ such that for any $b \in B$ and any $(x, y) \in U[a, \delta_1] \times U[b, \eta_1]$ one has

$$\|D_2F(x, y) - T_b\| \leq \frac{1}{2}M^{-1} \leq \frac{1}{2}\|T_b^{-1}\|^{-1},$$

and therefore

$$\begin{aligned} \|D_2g_b(x, y)\| &= \|I - T_b^{-1} \cdot D_2F(x, y)\| = \|T_b^{-1} \cdot T_b - T_b^{-1} \cdot D_2F(x, y)\| \\ &\leq \|T_b^{-1}\| \cdot \|T_b - D_2F(x, y)\| \leq \frac{1}{2}. \end{aligned}$$

This implies with that the help of the mean value theorem applied to $g_b(x, \cdot)$, that for any $b \in B$, $x \in U[a, \delta_1]$, and any two points $y_1, y_2 \in U[b, \eta_1]$,

$$\|g_b(x, y_1) - g_b(x, y_2)\| \leq \frac{1}{2}\|y_1 - y_2\|. \quad (18)$$

We now fix η_1 and by using assumptions (ii) and (iv), we choose $0 < \delta_2 < \delta_1$ such that for any $b \in B$ and any $x \in U[a, \delta_2]$,

$$\|F(x, b)\| = \|F(x, b) - F(a, b)\| < \frac{1}{2}\|T_b^{-1}\|^{-1}\eta_1,$$

and therefore

$$\|g_b(x, b) - b\| = \|T_b^{-1} \cdot F(x, b)\| \leq \|T_b^{-1}\|\|F(x, b)\| < \frac{1}{2}\eta_1. \quad (19)$$

Thus for any $b \in B$, $x \in U[a, \delta_1]$ and $y \in U[b, \eta_1]$, one has by using (18) and (19), that

$$\|g_b(x, y) - b\| \leq \|g_b(x, y) - g_b(x, b)\| + \|g_b(x, b) - b\| < \frac{1}{2}\|y - b\| + \frac{1}{2}\eta_1 \leq \eta_1. \quad (20)$$

The inequality (20) combining with (18) implies that for any $b \in B$ and any (fixed) $x \in U[a, \delta_2]$, the map $y \mapsto g_b(x, y)$ is a contraction of the complete metric space $U[b, \eta_1]$ into itself. Hence by the contraction mapping principle, there exists a unique fixed point,

say $\psi_b(x)$, and so $g_b(x, \psi_b(x)) = \psi_b(x)$ or, equivalently, $F(x, \psi_b(x)) = 0$. Also, we have obtained that for any $x \in U[a, \delta_2]$, the element $y = \psi_b(x)$ is the only solution of the equation $F(x, y) = 0$ in the region $y \in U[b, \eta_1]$. Note that in fact $\psi_b(x) \in U(b, \eta_1)$ due to the strict inequality in (20).

Next, we prove the equicontinuity property. Let $b \in B$ and $x, x' \in U[a, \delta_2]$. Then

$$\begin{aligned} \|\psi_b(x) - \psi_b(x')\| &= \|g_b(x, \psi_b(x)) - g_b(x', \psi_b(x'))\| \\ &\leq \|g_b(x, \psi_b(x)) - g_b(x, \psi_b(x'))\| + \|g_b(x, \psi_b(x')) - g_b(x', \psi_b(x'))\| \\ &\leq \frac{1}{2} \|\psi_b(x) - \psi_b(x')\| + \|g_b(x, \psi_b(x')) - g_b(x', \psi_b(x'))\|. \end{aligned}$$

Thus

$$\begin{aligned} \|\psi_b(x) - \psi_b(x')\| &\leq 2\|g_b(x, \psi_b(x')) - g_b(x', \psi_b(x'))\| \\ &= 2\|T_b^{-1} \cdot (F(x, \psi_b(x')) - F(x', \psi_b(x')))\| \\ &\leq 2M\|F(x, \psi_b(x')) - F(x', \psi_b(x'))\| \end{aligned}$$

and the result follows from the continuity assumption (ii).

Finally, take $\bar{\eta} = \eta_1/2$. By using the equicontinuity property for the family of functions $\psi_b(x)$, we can choose $0 < \bar{\delta} < \delta_2$ such that for any $b \in B$ one has

$$\|b - \psi_b(x)\| < \bar{\eta} \quad \text{whenever } x \in U(a, \bar{\delta}). \quad (21)$$

Then the sets $U(b, \bar{\eta})$, $b \in B$, are disjoint; indeed, otherwise one would have $y^* \in U(b, \bar{\eta}) \cap U(b', \bar{\eta})$ for some $y^* \in G$ and $b, b' \in B$ with $b \neq b'$, and hence $\|b - b'\| \leq \|b - y^*\| + \|y^* - b'\| < 2\bar{\eta} = \eta_1$ which implies that b and b' would be different solutions of the equation $F(a, y) = 0$ in the region $y \in U(b, \eta_1)$, a contradiction to the uniqueness of solutions. Since, by (21), $\psi_b \in U(b, \bar{\eta})$ for any $x \in U(a, \bar{\delta})$, it follows that the map $b \mapsto \psi_b(x)$ is injective.

The proof of theorem 2.1 is completed. $\square$

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