

Analysis and synthesis of geometric models using tree-structured relations

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In this study, we propose a methodology capable of analysing and synthesizing relational geometric design models, based on a mapping between geometric nodes (points) on the model and the physical dimensions of the solid model. The relations that include translation and rotation between the geometric nodes are stored in a tree structure. Based on this methodology, an implementation is developed that allows a designer to freely synthesize and analyse geometric relations among admissible physical dimensions of the object. Meanwhile the management of the underlying parameterized relational geometric data automatically occurs. © 1997 Elsevier Science Ltd.

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INTRODUCTION

Relational geometric design, based on parameterized solid models, not only has the capability to reduce the time required to investigate geometric design modifications, but can also establish a consistent geometric record of the original design relations. Such a record allows designers to investigate the effects of various geometric modifications.

Geometric modelling systems can assemble simple geometric primitives into more complex geometric models (i.e. CSG modellers). These modellers use matching conditions between points on adjoining a model's subcomponents to specify parameters for the geometric constraints. In those systems, the multiple constraints must be satisfied simultaneously. Lin *et al.*⁴ and Light and Gossard³ presented an example of such a geometric modeller. In their system, geometric constraints are defined by dimensions and characteristic points on a model that describe its geometry. Those constraints are converted into a set of linear equations and are subsequently solved using iterative numerical methods. Rocheleau and Lee⁵ applied geometric constraints

defined by so-called 'against' and 'fit' conditions to solid models for the purpose of connecting one body to another based on the concept of a 'virtual link'. Krishnan and Patnaik² proposed an entities-relationship model. Their model was employed to describe the concepts that underlie a geometric database.

Geometric modellers rely on matching conditions for constraint satisfaction. However, the consistency of the constraints imposed on model dimensions is difficult to trace, particularly when a large number of constraints exists. To overcome this problem, Van Emmerik⁷ designed a new scheme for satisfying geometric constraints by using a geometric tree constructed from points on the object, and relative relations among the points. Within his system, the definition of a geometric tree and the CSG hierarchy between subcomponents of a model are separated. The relative positions and orientations of geometric nodes are directly evaluated via the geometric tree's structure.

Van Emmerik's modeller possesses the advantage of allowing the user to attain a more thorough understanding of the possible geometric variations that can be made into an object. The modeller also improves control over those variations. However, the drawback of his modeller is that obtaining an appropriate geometric tree comprised of a large number of dimensional relations is difficult and non-intuitive.

To overcome the problems associated with Van Emmerik's geometric modeller, this study outlines the procedures for both synthesizing the required relations among the geometric nodes in a model and analysing the associated variations in the model's physical dimension. More specifically, the synthesis procedures are used to develop a mapping between the geometric tree and the required variations of the geometric model's physical dimensions in a form that ensures that certain prescribed dimensions of the model remain the same. This is important since actual 3D objects typically contain many geometric nodes. The analysis procedures, in accordance with the geometric tree, automatically analyse the variations in the physical dimensions that result from changes made to each geometric node.

This new methodology, based on graph theory and matrix operations, can automatically construct the geometry tree subject to the satisfaction of user-specified dimensional constraints. In this study, we develop the user interface that converts more general

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geometric constraints into lower level geometric relations. The geometric modeller described herein has been used in applications such as the design of mechanical assembly and structural design.

THE RELATION BETWEEN THE GEOMETRIC TREE AND SOLID MODEL

The geometric node, say P_i , as shown in Figure 1, is defined by a set of six parameters, three translation parameters x_i, y_i, z_i along the X_j, Y_j , and Z_j axes, and three rotation parameters in terms of Euler angles $\alpha_i, \beta_i, \gamma_i$, along the X_j, Y_j , and Z_j axis of a reference node, say P_j . Every geometric node has a reference node which can either be another geometric node or the origin of the global coordinate system.

A geometric tree is defined as a set of geometric nodes connected by different relations (edges). The top level node is called the root node. The reference node of the root node is always the origin of the global coordinate system. In the example in Figure 2a, P_1 is the root node whose reference node is the origin of the global coordinate system. Following the conventional terminology for tree-structure¹ the starting node is called the parent node and the other nodes are called the children nodes of this starting node. Thus, P_3, P_4 and P_5 are the children nodes of P_2 or P_1 , and P_3 is the parent node of P_4 .

From the geometric tree in Figure 2a, primitives can be created as shown in Figure 2b. The block is defined by the bottom-centre node at P_1 , and the corner node at P_2 . The cylinder is defined by the origin of P_3 , at the bottom-centre point and P_2 on the top-circle. The sphere is defined by P_4 at the centre point and P_5 on the surface of the sphere. Additional primitives which are not included in this example include the cone, torus, and pyramid, and can be defined in a similar manner to the primitives described above using a set of geometric nodes.

The overall part as shown in Figure 2c can then be constructed by integrating of primitives using a Constructive Solid Geometry (CSG) tree. Shown here is the union operator applied to the cylinder and block, followed by a difference with the sphere.

Following the creation of geometry, kinematic connectivities can be established to integrate different parts

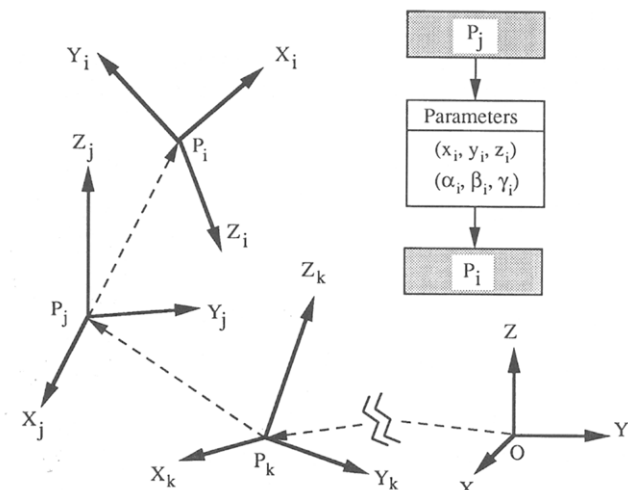


Figure 1 A geometric node P_i is defined from the other geometric node P_j

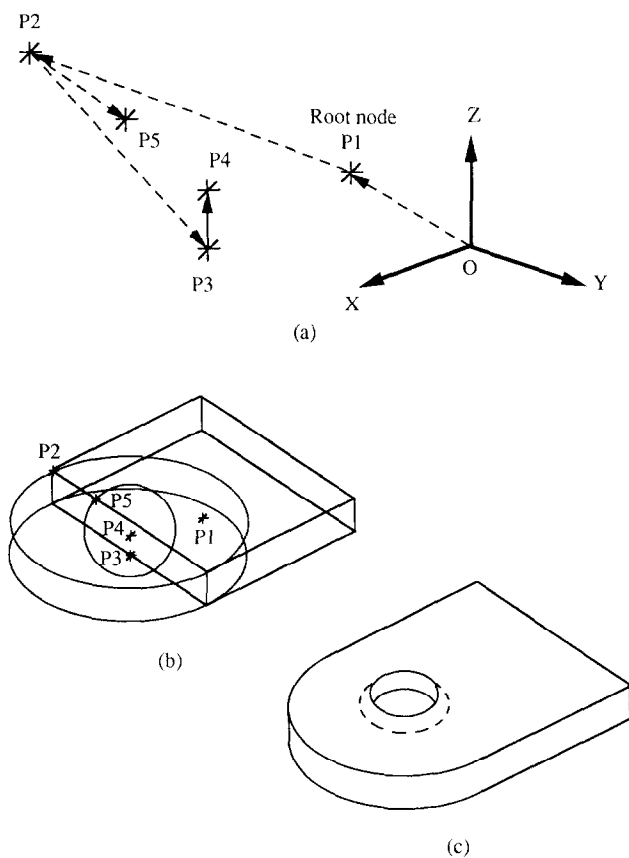


Figure 2 (a) A geometric tree. (b) Creation of primitives based on the geometric nodes. (c) Creation of a part based on the CSG tree

into an overall assembly. Tilove⁶ addresses this issue in detail. By using this approach, a systematic relational geometric design process has been developed as shown in Figure 3.

The fundamental rule for geometric tree operations is the propagation rule. If any geometric node P_i is moved, all of its children nodes undergo a rigid body motion by

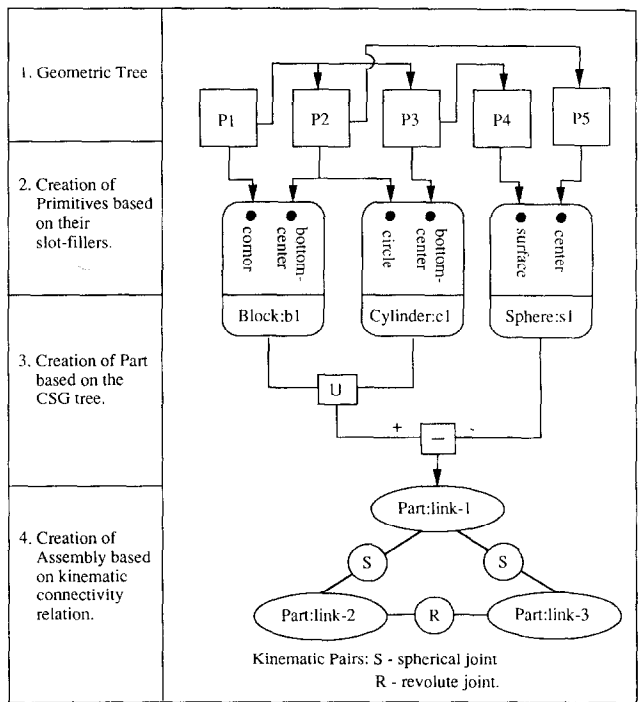


Figure 3 A four-stage relational geometric design

that displacement through node P_i ; however, its parent nodes remain still.

Physical dimensions can be interpreted as relative translations or rotations between the geometric nodes. Figure 4a presents an example showing a circular plate with an off-centre hole. For clarity and ease of explanation, all the geometric nodes have been selected to lie on the YZ plane of the root node P_1 . Obviously, the type of variation depends on the geometric tree's structure. For instance, Figure 4b displays the top-side view of the circular plate in Figure 4a. Figure 4c denotes the dimensional variation while P_1 is the reference node for all other geometric nodes (P_2 , P_3 , and P_4). By moving P_2 along the Y_1 axis, the radius of the off-centre hole ($R1$), $L1$, and $L3$ vary; meanwhile, the other dimensions remain unchanged. Figure 4d shows the same geometric tree but now the moving node is P_3 , which moves along the Y_1 axis. In this case, $R1$, $L2$, and $L4$ vary, while the other dimensions remain unchanged. Figure 4e shows the dimensional variations that result while P_3 is the reference node for P_4 , P_1 is the reference node of P_3 and P_2 . Moving P_3 affects P_4 , and the physical dimensions represented by $L2$, $L3$, $R1$ and $R2$ vary; meanwhile, physical dimensions $L1$ and $L4$ remain unchanged since the relation between P_3 and P_4 is unchanged.

For the case described above with four geometric nodes, there are 52 non-isomorphic geometric trees, and only two are in Figure 4. This figure also confirms that the number of possible tree structures grows

exponentially with an increasing number of geometric nodes. Hence, it is not feasible to enumerate all the possible structures to determine the tree structure that can satisfy the specified relations. In the following sections, we propose a systematic method for analysing and synthesizing the geometric tree.

THE ANALYSIS OF GEOMETRICAL TREE

From the geometric tree, we can define the transition matrix $A_{n \times n}$ as follows. For each element a_{ij} in A ,

$$a_{ij} = \begin{cases} -1 & \text{if geometric node } j \text{ is the reference node of} \\ & \text{geometric node } i, \text{ and geometric node } i \text{ is the} \\ & \text{moving node;} \\ 1 & \text{if geometric node } j \text{ is the reference node of} \\ & \text{node } i, \text{ which is not the moving node, or} \\ & i = j = 1; \\ 0 & \text{if geometric node } j \text{ is not the reference node of} \\ & \text{geometric node } i \end{cases}$$

Notably, a situation in which node $i = 1$ always denotes the root node. Based on the definition of the transition matrix A , the first row of A is always $[1, 0, \dots, 0]_{1 \times n}$, since node 1 is the root node, which has no parents. There is exactly one non-zero element in each row of A , since each node other than the root node has exactly one parent. Consequently, there are exactly n non-zero elements in A since there are $n - 1$ edges in the geometry tree plus $a_{11} = 1$. If the i th node is a leaf of the geometric tree, then the i th column of A is a zero column.

From the above statement, the characteristic equation of the transition matrix A is

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \Rightarrow (-\lambda)^{n-1} (1 - \lambda) &= 0 \end{aligned}$$

then the eigenvalue of A is $\lambda = 1$.

There exists a vector V such that $AV = V$. This vector is simply the eigenvector of A corresponding to the eigenvalue $\lambda = 1$. The eigenvector V of the form $V = [1, v_2, \dots, v_n]^T$, has the following property:

$$v_i = \begin{cases} -1 & \text{if the absolute coordinate of the } i\text{th node is} \\ & \text{changed due to a self-variation or a propagated} \\ & \text{variation;} \\ 1 & \text{otherwise} \end{cases}$$

Hence, the eigenvector V of A is called the nodal variation vector. This property can be proven by the following.

Let

$$C_1 = [c_{11}, c_{21}, \dots, c_{i1}, \dots, c_{n1}]^T = [1, 0, 0, \dots, 0]_{n \times 1}^T, \quad 1 \leq i \leq n$$

and

$$\begin{aligned} C_2 &= [c_{12}, c_{22}, \dots, c_{i2}, \dots, c_{n2}]^T = AC_1 \\ &= [\text{column 1 of } A] = [1, a_{21}, \dots, a_{i1}, \dots, a_{n1}]^T, \\ &1 \leq i \leq n, \end{aligned}$$

where a_{i1} is the element of A defined previously

$$C_3 = [c_{13}, c_{23}, \dots, c_{i3}, \dots, c_{n3}]^T = AC_2$$

where $c_{i3} = (\text{row } i \text{ of } A) * [1, a_{21}, \dots, a_{i1}, \dots, a_{n1}]^T$. As

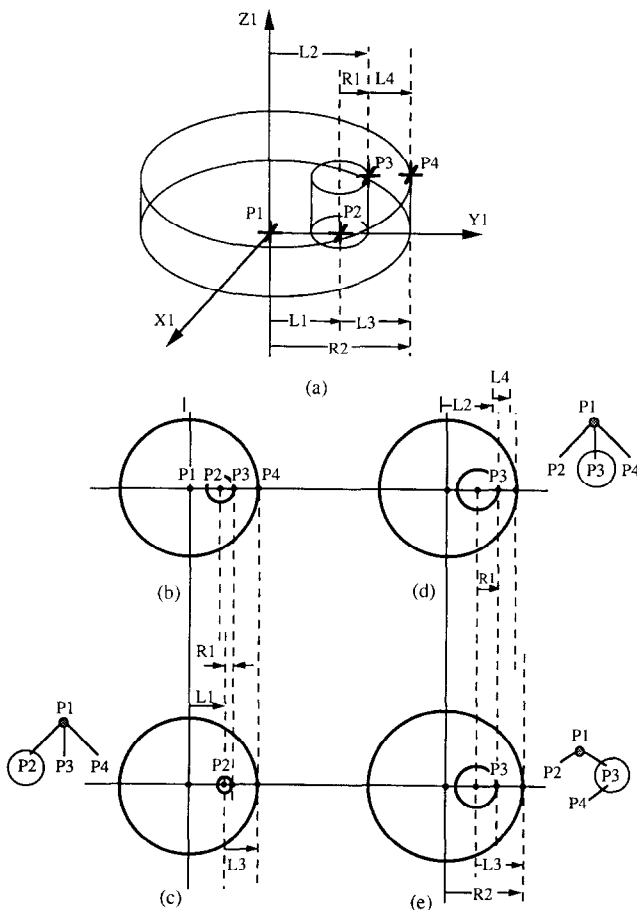


Figure 4 Variation of physical dimensions due to different moving geometric nodes and trees; (a) original configuration of the circular plate, (b) top view of the circular plate, (c) moving node is P_2 , (d) moving node is P_3 , (e) moving node is P_3 but with a different geometric tree

stated previously (that each row of $\mathbf{A}$ contains exactly one non-zero element), let the non-zero element in row i of $\mathbf{A}$ be a_{ij} which indicates that node j is the reference node of node i , where $1 \leq j \leq n$, the conditions for $c_{i3} = a_{ij} * a_{j1} = \pm 1$ is that node 1 (root node) is the reference node of node j . Otherwise, we would have $a_{j1} = 0$ which yields $c_{i3} = 0$. The sign ' $\pm$ ' is determined by the condition of whether either node i and node j is the moving node. If either node i or node j is the moving node, then $c_{i3} = a_{ij} * a_{j1} = -1$; otherwise, $c_{i3} = a_{ij} * a_{j1} = 1$. As shown in Figure 5, yielding a non-zero element c_{i3} requires that there must be a two-edge, directed path on the geometric tree from node 1 to node i , and node j is on the midway of the directed path from node 1 (root node) to node i .

Since $\mathbf{C}_2$ is the first column of $\mathbf{A}$, i.e. $c_{i2} = a_{i1}$, $c_{i2} = \pm 1$ indicates the only non-zero element in row i of $\mathbf{A}$ is a_{i1} and we obtain $c_{i3} = a_{i1} * a_{11} = c_{i2} * 1 = c_{i2}$. Consequently, a general statement that if $c_{ik} = 1$ (or -1), then $c_{ik+1} = 1$ (or -1) can be induced from as the following:

- (1) Let $m = 3$, it is true that if $c_{jm-1} = 1$ (or -1) then $c_{jm} = 1$ (or -1), $1 \leq j \leq n$.
- (2) $c_{im} = a_{ij} * c_{jm-1}$, where a_{ij} is the non-zero element in row i of $\mathbf{A}$.
- (3) $c_{im+1} = a_{ij} * c_{jm} = a_{ij} * c_{jm-1} = c_{im}$.
- (4) From (3), it was derived that (1) is valid for $m = 4$.
- (5) The induction continues to $m = 5, 6, \dots$ to k .

Above equation indicates that the non-zero element of $\mathbf{C}_i$ will remain stationary by further multiplications of $\mathbf{A}$. We can conclude that

$$\mathbf{C}_{k+1} = [c_{1k}, c_{2k}, \dots, c_{ik}, \dots, c_{nk}]^T = \mathbf{A}\mathbf{C}_k$$

where $c_{ik} = 0$ when the length of the directed path from node 1 to node i is greater than k ;

$$c_{ik} = \begin{cases} 1 & \text{when the length of the directed path from node 1 to node } i \text{ is not greater than } k, \text{ and the moving node is on this directed path;} \\ 1 & \text{otherwise} \end{cases}$$

Thus, $\mathbf{C}_n = \mathbf{A}\mathbf{C}_{n-1}$. The above equation indicates that $\mathbf{C}_{n+m} = \mathbf{C}_{n+m-1}$ for $m \geq 1$, when the lengths of all the directed paths are less than n . If $\mathbf{V} = \mathbf{C}_n$, and based on the propagation rule, v_i will have the properties above. QED

A matrix $\mathbf{D} = [d_{ij}]_{n \times n} = \mathbf{V}\mathbf{V}^T$, referred to as a dimensional variation matrix where $\mathbf{V}$ is the nodal variation vector, has the following properties:

$$d_{ij} = \begin{cases} -1 & \text{if the dimension, i.e. relative translation and/or rotation, between the } i\text{th and } j\text{th nodes is changed;} \\ 1 & \text{if the dimension between the } i\text{th and } j\text{th nodes remains unchanged or } i = j \end{cases}$$

It can be proven that $d_{ij} = v_i v_j = -1$ when the absolute coordinates of either node i or node j are changed, therefore the relative dimension between i and j has changed, and $d_{ij} = v_i v_j = 1$ when the absolute coordinates of both node i and node j remain the same or both node i and node j are changed. Therefore, the relative dimension remains unchanged. QED

The example in Figure 5 demonstrates the corresponding transition matrix $\mathbf{A}$ subjected to the moving node P_2 . The nodal variation vector $\mathbf{V} = [1 \ -1 \ -1 \ 1]^T$ indicates

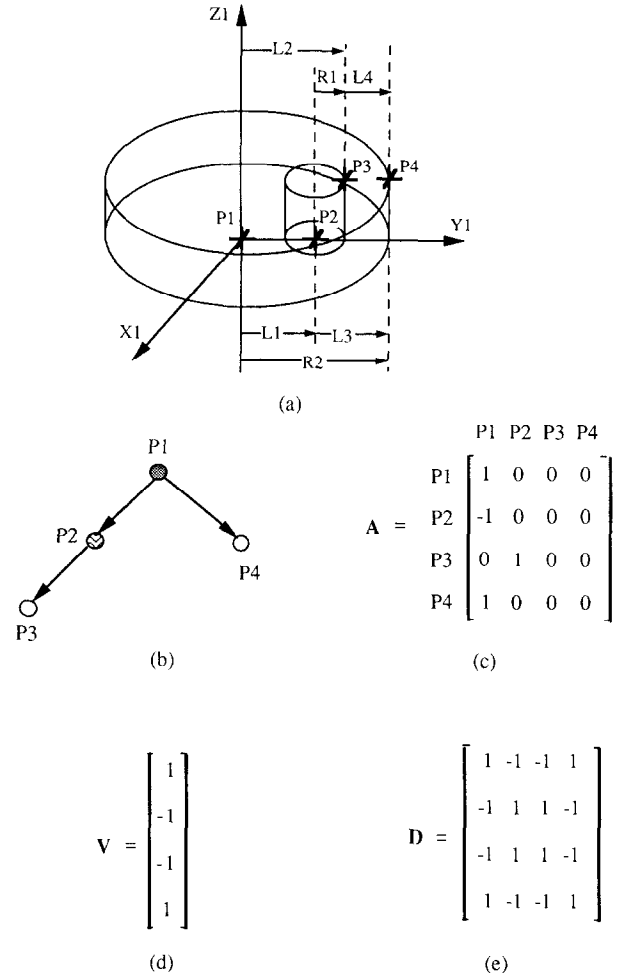


Figure 5 (a) The original configuration of the circular plate. (b) The geometric tree with the moving node at P_2 . (c) The corresponding $\mathbf{A}$ matrix. (d) The eigenvector of $\mathbf{A}$, the nodal variation vector. (e) The corresponding dimensional matrix

that the absolute coordinates of P_2 and P_3 are moved and those of P_1 and P_4 remain unchanged. By the dimensional variation matrix $\mathbf{D}$, the variational conditions of all physical dimensions in the geometric tree can be obtained. We obtain $\mathbf{D} = \mathbf{V}\mathbf{V}^T$, which indicates that $L1, L2, L3, L4$ are altered while $R1$ and $R2$ are not.

THE SYNTHESIS OF GEOMETRIC TREE

In the synthesis problem, in contrast with the analysis one, the user will specify a set of preferred variations of the physical dimensions. The corresponding geometric tree will then be constructed. The specified variation will be referred to as the dimensional variation set.

According to the dimensional variation sets, the dimensional variation entries d_{ij} between pairs of nodes i and j can be determined. For instance, if the dimension between node i and j is specified to be changed, then we set $d_{ij} = -1$ where d_{ij} is the element of i th row and j th column of matrix $\mathbf{D}$; if they are specified not to be changed, then we should set $d_{ij} = 1$. After the $n - 1$ independent entries in the dimension variation matrix $\mathbf{D}$ have been specified, the rest of the entries can be derived from a chain-rule like relation as follows:

$$\text{For } v_p^2 = 1, \text{ then } d_{ij} = v_p^2 v_i v_j = d_{ip} d_{pj}. \quad \text{QED}$$

Consequently, the nodal variation vector $\mathbf{V}$ can then be determined from the definition of the dimensional variation matrix $\mathbf{D}$. According to the definition of $\mathbf{D}$, the nodal variation vector $\mathbf{V}$ is equal to the first column of $\mathbf{D}$. This can be shown as follows:

$$\text{Let } \mathbf{V} = \begin{bmatrix} 1 \\ \mathbf{U} \end{bmatrix}, \quad \text{then } \mathbf{D} = \mathbf{V}\mathbf{V}^T = \begin{bmatrix} 1 & \mathbf{U}^T \\ \mathbf{U} & \mathbf{U}\mathbf{U}^T \end{bmatrix}$$

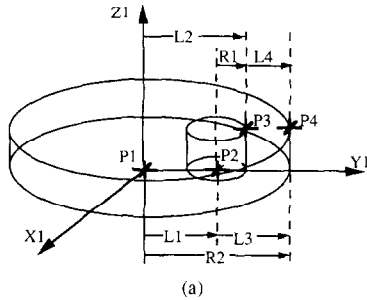
For instance, Figure 6b shows three-dimensional variation sets, which can be described as follows.

- A: 'L1 changes but R1 and R2 should remain unchanged'
 B: 'L2 changes but L1 and R2 should remain unchanged'
 C: 'R2 changes but L1 and R1 should remain unchanged'

Figures 6c and 6d show the corresponding dimensional variation matrices $\mathbf{D}_a$, $\mathbf{D}_b$ and $\mathbf{D}_c$ and the nodal variation vectors $\mathbf{V}_a$, $\mathbf{V}_b$ and $\mathbf{V}_c$.

In a synthesis problem, the satisfaction of multiple-dimensional variation sets should be allowed. Since each dimensional variation set maps to a single dimensional variation matrix and a corresponding nodal variation vector, the ability to satisfy multiple-dimensional variation sets can be verified by the compatibility of each pair of nodal variation vectors corresponding to the dimensional variation sets.

For a nodal variation vector with a specified moving node, say node i , the i th element of this vector must be



Dimensional matrix entry Dimensional variation set	d12 (L1)	d13 (L2)	d24 (L3)	d34 (L4)	d23 (R1)	d14 (R2)
A	-1	unspecified			1	1
B	1	-1	unspecified			1
C	1	unspecified			1	-1

$$\mathbf{D}_a = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad \mathbf{D}_b = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \quad \mathbf{D}_c = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

$$\mathbf{V}_a = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{V}_b = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{V}_c = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Figure 6 (a) The original configuration of the circular plate. (b) The dimensional variation sets. (c) The corresponding dimensional variation matrices. (d) The corresponding nodal variation vectors

-1 to indicate that the absolute coordinate of node i (the moving node) has been changed. On the contrary, for a nodal variation vector with an unspecified moving node, the moving node can be any one of the nodes with element value -1. To synthesize the tree that would subsequently satisfy all the nodal variation vectors, the compatibility among them must be examined.

The compatibilities among nodal variation vectors can be classified into two types: inclusive and exclusive relations. The nodal variation vector $\mathbf{V}_a$ is 'inclusive' of $\mathbf{V}_b$ if and only if the position of -1 in $\mathbf{V}_a$ covers all the positions of -1 in $\mathbf{V}_b$; equivalently, $\mathbf{V}_a$ must satisfy the bitwise 'AND' Boolean operation when -1 is logic-low and 1 is logic-high that

$$\text{AND}(\mathbf{V}_a, \mathbf{V}_b) = \mathbf{V}_a$$

The nodal variation vector $\mathbf{V}_a$ is 'exclusive' of $\mathbf{V}_b$ if and only if the positions of -1 in $\mathbf{V}_a$ and $\mathbf{V}_b$ are mutually exclusive, thereby implying that $\mathbf{V}_b$ is 'exclusive' of $\mathbf{V}_a$. Equivalently, it must satisfy the bitwise 'OR' Boolean operation that

$$\text{OR}(\mathbf{V}_a, \mathbf{V}_b) = \mathbf{1}$$

Figure 7 presents an example that demonstrates the relations. The -1 positions of $\mathbf{V}_a$ cover all the -1 positions of $\mathbf{V}_b$ as shown in Figure 7a. The -1 position of $\mathbf{V}_c$ is excluded from the -1 position of $\mathbf{V}_d$, and vice versa, as shown in Figure 7b.

For two different nodal variation vectors, say $\mathbf{V}_a$ is inclusive of $\mathbf{V}_b$, if the moving node of $\mathbf{V}_a$ is node i then the i th element of $\mathbf{V}_a$ must be -1, and the i th element of $\mathbf{V}_b$ must be 1. The above result is derived from that the moving node of $\mathbf{V}_a$ must lie on the directed path from node 1 (root node) to the moving node of $\mathbf{V}_b$; otherwise, $\mathbf{V}_a$ cannot be inclusive of $\mathbf{V}_b$. Hence, the moving node of $\mathbf{V}_a$ must be chosen from the -1 element of a resultant vector of

$$\text{NOT}(\text{XOR}(\mathbf{V}_a, \mathbf{V}_b))$$

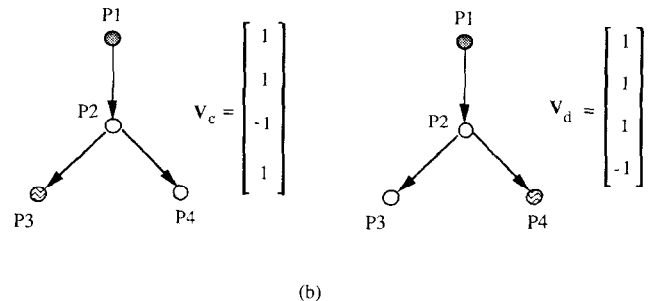
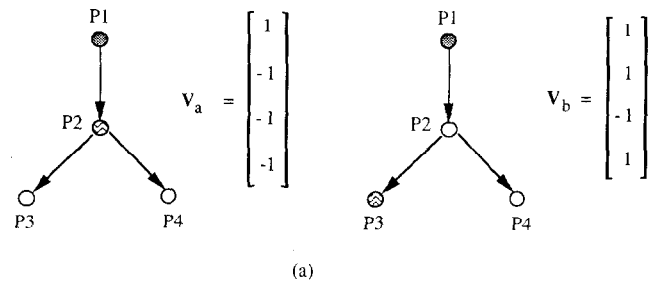


Figure 7 (a) Inclusive relation between nodal variation vectors. (b) Exclusive relation between nodal variation vectors

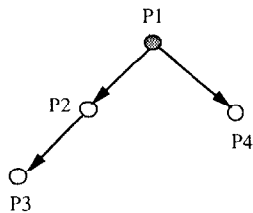


Figure 8 The geometric tree that satisfies all the dimensional variation sets specified in Figure 6

If V_a and V_b are mutually exclusive, then the moving node of V_a cannot lie on the directed path from node 1 (root node) to the moving node of V_b and vice versa. For the mutually exclusive ones, each nodal variation vector can be selected with a different moving node whose corresponding element value is -1 .

If any two nodal variation vectors, say V_p and V_q , of the dimensional variation sets have failed to satisfy either compatibility conditions, i.e. the inclusive and exclusive relations, the synthesis process must then be terminated since no valid geometric tree can be found to satisfy all the dimensional variations specified. The fact that the directed path from the root node of the moving node of V_p partially covers the directed path from the root node to the moving node of V_q causes the dimensional variation to be violated when either of the two nodal variation vectors is put into action.

If all two nodal variation vectors of the dimensional variation sets satisfy one of the compatibility conditions,

the possibility still arises that no valid geometric tree exists to satisfy all the dimensional variations specified. For instance, $V_1 = [1 \ -1 \ -1 \ -1]^T$, $V_2 = [1 \ -1 \ 1 \ -1]^T$, and $V_3 = [1 \ 1 \ -1 \ 1]^T$, where V_1 is inclusive of both V_2 and V_3 , and V_2 is exclusive of V_3 . From the inclusive relation between V_1 and V_2 , the moving node of V_1 is obtained to be node 3. From the inclusive relation between V_1 and V_3 , the moving node can be either node 2 or 4. However, there is no intersection between node 3 and node 2 or 4; in addition, no moving node can be specified for V_1 . Thus it must be also verified that

$$\text{AND}[\text{AND}(\text{NOT}(V_1), V_2), \text{AND}(\text{NOT}(V_1), V_3)] \neq 0$$

to guarantee the existence of a moving node of V_1 .

After the nodal variation vectors and the corresponding moving nodes are obtained, the geometric tree can be determined using the procedures described in following. We denote the transition matrix and its corresponding nodal variation vector, with the i th node being the moving node, as A_i and $V_i = [v_1, v_2, \dots, v_i, \dots, v_n]^T$, respectively. The following equations are valid for V_i

$$(-1)^{\delta(i,k)} v_{p(k)} = v_k \quad \text{for } 2 \leq k \leq n$$

where $p(k)$ denotes the reference node number of node k , and $\delta(i,k)$ is Kronecker delta function. This can be proven from the property of eigenvector, that $V = AV$, thus $v_k = (\text{row } k \text{ of } A) * V$. Since there is only one non-zero element in each individual row of A , when node j is

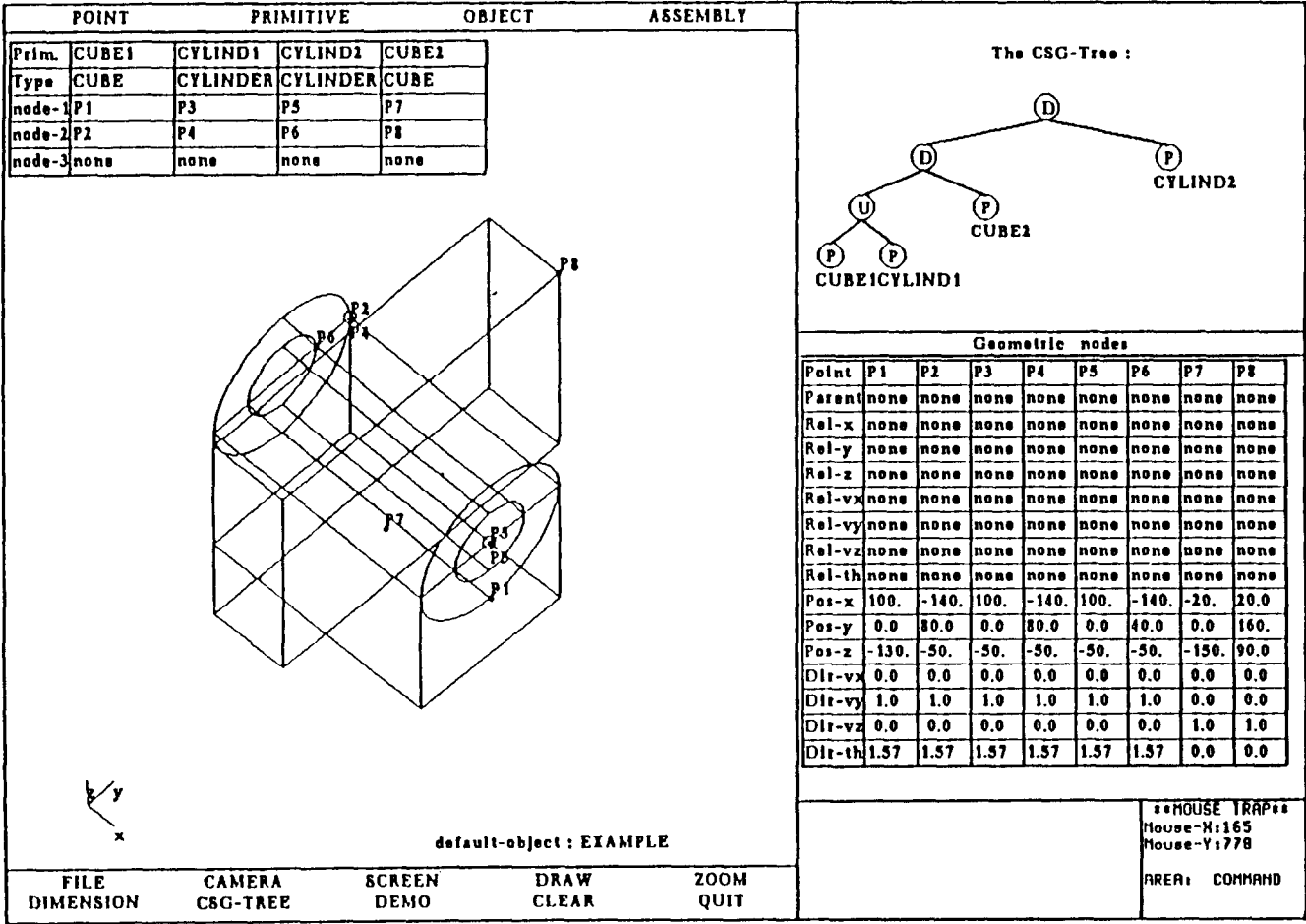


Figure 9 Example of geometric model defined by the geometric nodes and a CSG tree

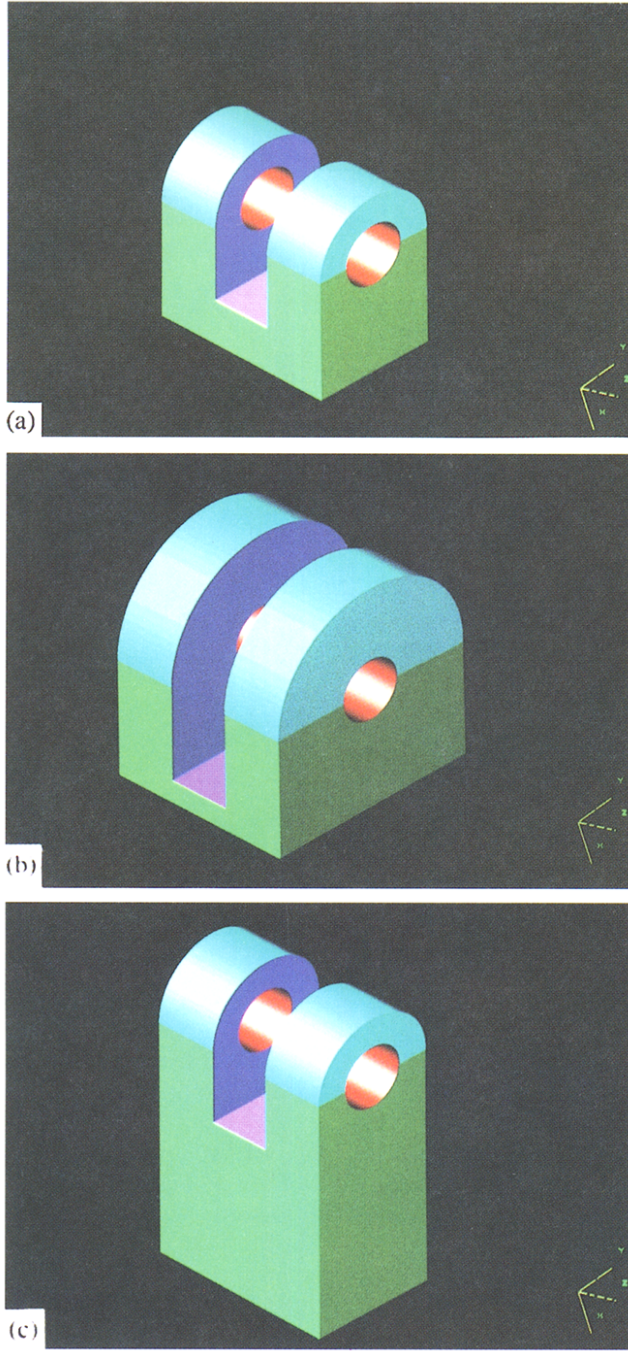


Figure 10 (a) Rendered image of the geometric model according to Figure 9. (b) Rendered image of the geometric model while the geometric node P_2 is moved. (c) Rendered image of the geometric model while the geometric node P_3 is moved

the reference node of node k , i.e. $p(k) = j$, we have $v_k = a_{kj} * v_j = a_{kj} * v_{p(k)}$. If node k is the moving node, i.e. $k = i$, then we have $a_{kj} = -1$ by the element definition of $\mathbf{A}$ and $v_k = -v_{p(k)}$, else we have $a_{kj} = 1$ and $v_k = v_{p(k)}$. Therefore, the relation between moving node i , arbitrary node k , v_k and $v_{p(k)}$ is represented using the Kronecker delta function. **QED**

Assume node i is the moving node, we have $v_i = -v_{p(i)}$ and $v_k = v_{p(k)}$ for node $k \neq i$, to yield

$$\mathbf{V}_j = [v_1, v_2, \dots, v_j, \dots, v_n]^T,$$

where $1 \leq i \leq n$ and $2 \leq j \leq n$,

$$= [1, v_{p(2)}, v_{p(3)}, \dots, v_{p(i-1)}, -v_{p(i)}, v_{p(i+1)}, \dots, v_{p(n)}]^T_j$$

Define the conjugate vector $\mathbf{V}'_j$ of $\mathbf{V}_j$, which negates the i th element of $\mathbf{V}_j$ as follows:

$$\mathbf{V}'_j = [v_1, v_2, \dots, -v_i, \dots, v_n]^T_j,$$

where $1 \leq i \leq n$ and $2 \leq j \leq n$,

$$= [1, v_{p(2)}, v_{p(3)}, \dots, v_{p(i-1)}, v_{p(i)}, v_{p(i+1)}, \dots, v_{p(n)}]^T_j$$

Allow a matrix $\mathbf{Q}_{n \times (n-1)}$ to be composed of all the $\mathbf{V}_j$ as follows:

$$\mathbf{Q} = [\mathbf{V}_2, \mathbf{V}_3, \dots, \mathbf{V}_n]_{n \times (n-1)}$$

$$= \begin{bmatrix} 1 & 1 & \dots & 1 & \dots & 1 \\ (v_2)_2 & (v_2)_3 & \dots & (v_2)_j & \dots & (v_2)_n \\ \vdots & \vdots & & \vdots & & \vdots \\ (v_k)_2 & (v_k)_3 & \dots & (v_k)_j & \dots & (v_k)_n \\ \vdots & \vdots & & \vdots & & \vdots \\ (v_n)_2 & (v_n)_3 & \dots & (v_n)_j & \dots & (v_n)_n \end{bmatrix}_{n \times (n-1)}$$

Allow a matrix $\mathbf{S}_{n \times (n-1)}$ to be composed of all $\mathbf{V}'_j$ as follows:

$$\mathbf{S} = [\mathbf{V}'_2, \mathbf{V}'_3, \dots, \mathbf{V}'_n]_{n \times (n-1)}$$

$$= \begin{bmatrix} 1 & 1 & \dots & 1 & \dots & 1 \\ -(v_2)_2 & (v_2)_3 & \dots & (v_2)_j & \dots & (v_2)_n \\ (v_3)_2 & -(v_3)_3 & \dots & (v_3)_j & \dots & (v_3)_n \\ \vdots & \vdots & & \vdots & & \vdots \\ (v_n)_2 & (v_n)_3 & \dots & (v_n)_j & \dots & -(v_n)_n \end{bmatrix}_{n \times (n-1)}$$

$$= \begin{bmatrix} 1 & 1 & \dots & 1 & \dots & 1 \\ (v_{p(2)})_2 & (v_{p(2)})_3 & \dots & (v_{p(2)})_j & \dots & (v_{p(2)})_n \\ (v_{p(3)})_2 & (v_{p(3)})_3 & \dots & (v_{p(3)})_j & \dots & (v_{p(3)})_n \\ \vdots & \vdots & & \vdots & & \vdots \\ (v_{p(n)})_2 & (v_{p(n)})_3 & \dots & (v_{p(n)})_j & \dots & (v_{p(n)})_n \end{bmatrix}_{n \times (n-1)}$$

As generally known, $p(k)$, the reference node number of node k , should be consistent in all different nodal variation vectors $\mathbf{V}_j$. Therefore it is necessary that $s_k = q_{p(k)}$ for all $2 \leq k \leq n$, where s_k is the k th row vector of $\mathbf{S}$, and $q_{p(k)}$ is the $p(k)$ th row vector of $\mathbf{Q}$. Thus, if the i th row vector of $\mathbf{S}$ is identical to the j th row vector of $\mathbf{Q}$, i.e. $s_i = q_j$, then node j is the reference node of node i , i.e. $j = p(i)$. The reference-node relation is a directed edge from node i to node j in graph representation. When all the directed edges are obtained from matrices $\mathbf{S}$ and $\mathbf{Q}$, the corresponding geometric tree can be constructed.

For instance in Figure 7, because the corresponding moving node of $\mathbf{V}_b$ is P_3 ; therefore, $\mathbf{V}_3 = \mathbf{V}_b$. Similarly, $\mathbf{V}_2 = \mathbf{V}_a$ and $\mathbf{V}_4 = \mathbf{V}_c$. After the nodal variation vectors have been identified, matrices $\mathbf{S}$ and $\mathbf{Q}$ can be obtained based on their definitions as follows:

$$\mathbf{Q}_{4 \times 3} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

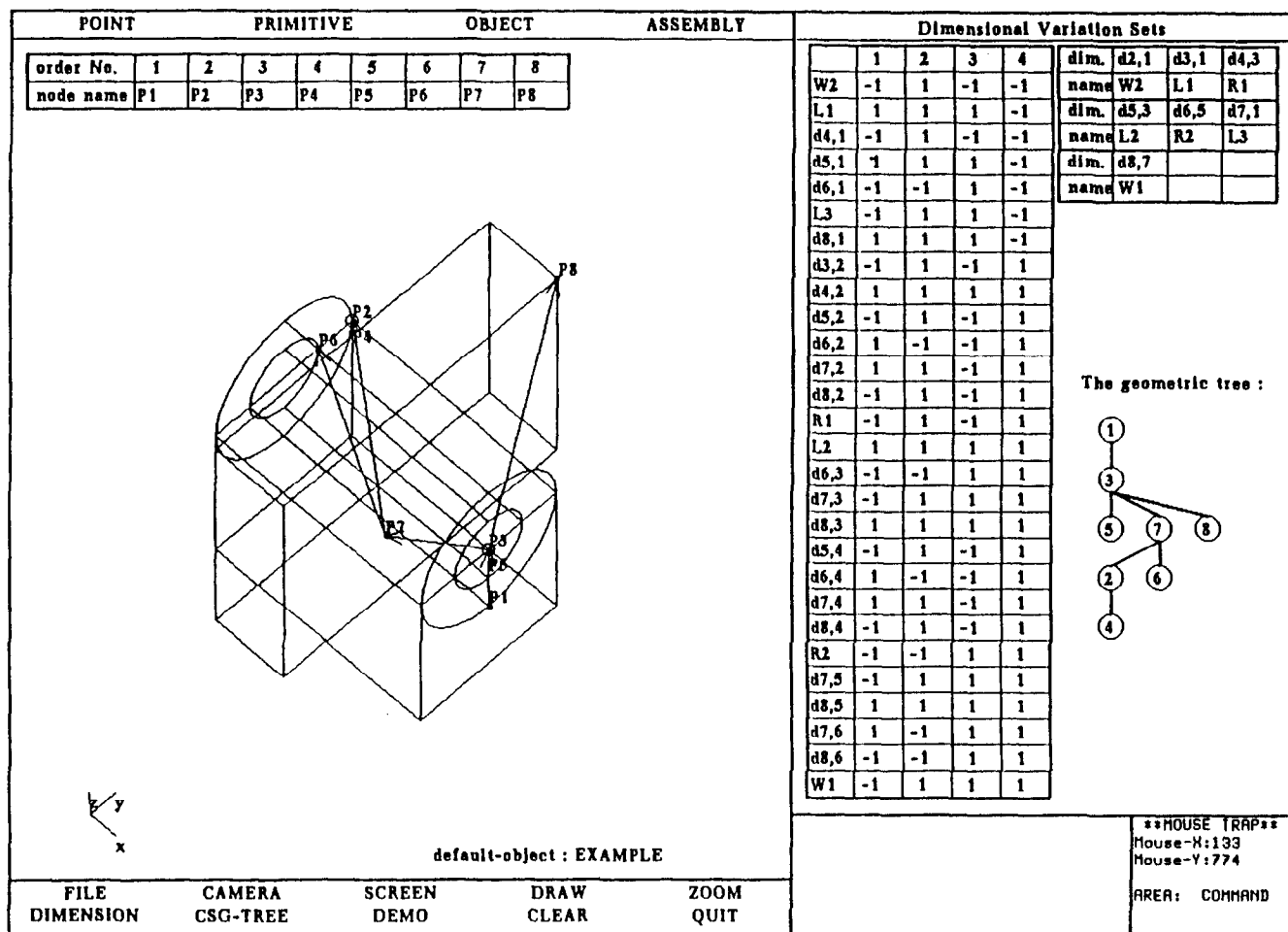


Figure 11 Example of specification of dimensional variation sets and the corresponding geometric tree

$$S_{4 \times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

In the above matrices, row 3 of S is identical to row 2 of Q , therefore the parent node is P_3 is P_2 . Similarly, we find that the parent node of P_2 and P_4 is P_1 . Figure 8 shows the subsequent geometric tree.

IMPLEMENTATION AND THEIR APPLICATION

A computer program has been implemented on PC using the geometric model. As shown in Figure 9, the example problem uses eight geometric nodes, $n = 8$, to define four primitives, i.e. cube-1 is defined by P_1 and P_2 , cylinder-1 is defined by P_3 and P_4 , cylinder-2 is defined by P_5 and P_6 , and cube-2 is defined by P_7 and P_8 . Those nodes are initially defined from the global coordinate system. Figure 10a displays the rendered image of the geometric model following the application of CSG operations. There are seven, i.e. $n - 1 = 7$, independent physical dimensions that have been selected, which are $W2$ (x -component of the relative translation between node 1 and node 2), $L1$ (z -component of the relative translation between node 3 and node 1), $R1$, $L2$, $R2$, $L3$, and $W1$.

During the synthesis phase, four user-specified dimensional variation sets are available.

- Set 1. When $L3$ changes, $W2$, $R1$, $R2$, $W1$ may change accordingly and rest of the independent dimensions must remain unchanged.
- Set 2. When $R2$ changes, all other independent dimensions must remain unchanged.
- Set 3. When $W2$ changes, $R1$ may be changed accordingly and the rest of the independent dimensions must remain unchanged.
- Set 4. When $L1$ changes, $W2$ and $L3$ may be changed accordingly while the rest of the independent dimensions must remain unchanged.

After the dimensional variation sets are specified, based on the previously described synthesis procedures, the program will automatically construct the geometric tree and determine the moving node for each of the variation set. Figure 11 presents an example which satisfies all the dimensional variation sets. The moving node for the first variation set can only be P_7 since the dimension $L3$ is defined through P_7 and P_1 , when P_1 is the root node and moving P_1 can only induce a rigid body motion, not the desired variation. The moving nodes for the other three variation sets are P_6 , P_2 , and P_3 , respectively.

The validity of the synthesis result can be verified during the analysis phase as demonstrated by the following example. In this example, moving the geometric node P_2 ,

the transition matrix of the geometric tree shown in Figure 11 is obtained on the basis of its definition as follows:

$$\mathbf{A}_{8 \times 8} = \begin{matrix} & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

The eigenvector for the transition matrix is given as follows:

$$\mathbf{V} = \begin{matrix} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 \\ \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix}^T \end{matrix}$$

and the dimensional variation matrix yields

$$\mathbf{D} = \mathbf{V}\mathbf{V}^T = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

This equation is identical to the user specified dimensional variation Set 3 given above. Hence, the above matrix indicates that the dimensional variation Set 3 can be satisfied by the subsequent geometric tree and the moving node for the dimensional variation Set 3 should be P_2 . Moving P_2 along the positive y-direction will increase the thickness the part as shown in Figure 10b. Similarly, Figure 10c shows a situation in which the geometric node P_3 , which is specified as the moving node of dimensional variation Set 4, is moved along positive z-direction in order to make the part taller.

CONCLUSION

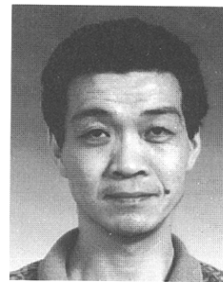
A systematic approach to tree-structured relational geometric design has been presented. Variations in physical dimensions can be detected on the basis of the analysis method. The relational tree structure that satisfies user-specified variations in physical dimensions can be obtained on the basis of the synthesis method. Owing to the simplicity of the synthesis and analysis methods, a quicker and easier approach for relational geometric design has been achieved. The 3D solid model engineering graphics can be designed and modified with less effort.

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