

# 行政院國家科學委員會專題研究計畫 期中進度報告

## 類神經網路的數學研究(2/3)

計畫類別：個別型計畫

計畫編號：NSC93-2115-M-009-005-

執行期間：93 年 08 月 01 日至 94 年 07 月 31 日

執行單位：國立交通大學應用數學系(所)

計畫主持人：石至文

報告類型：精簡報告

處理方式：本計畫可公開查詢

中 華 民 國 94 年 5 月 25 日

## 離散時間類神經網路的全局一致性

我們研究離散時間型的 Grossberg's Model 之全局一致性;亦即討論在什麼條件下,任一起始點皆會趨近於一平衡狀態。一般而言,此類問題的研究皆採用 LaSalle's invariance principles, 如果可以找到適合的 Lyapunov function, 而在這個模型中,並不知道存在有任何 Lyapunov function。我們運用數學分析,研究解的各 component 的 dominant behavior 而獲致全局一致性的結果。

# Global Consensus for Discrete-time Neural Networks

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May 25, 2005

## Abstract

A discrete-time version of Grossberg's competitive system is studied. We are interested in how the dynamics of the system reach global consensus. Analytical arguments are developed to conclude that every orbit converges to a fixed point of the system as time tends to infinity, without knowing a Lyapunov function.

## 1 Introduction

Global consensus (also named convergence of dynamics, or complete stability) has been an important dynamical behavior in the theory and applications of competitive systems and neural network systems. The commonest ways to guarantee convergence of dynamics is to find a Lyapunov function for the system, that is, a continuous real valued function  $V$  on state space, which is nonincreasing along trajectories of the system. One then applies the LaSalle's invariance principle to conclude the convergence. For example, Cohen and Grossberg (1983) [1] proved one convergence theorem for neural network systems of the form

$$\dot{x}_i = a_i(\mathbf{x})[b_i(x_i) - \sum_{j=1}^n \omega_{ij}g_j(x_j)], \quad i = 1, \dots, n, \quad (1.1)$$

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where  $a_i \geq 0$ , the matrix  $[\omega_{ij}]$  of coupling weights is symmetric, and  $g'_j \geq 0$  for all  $j$ . There exists a Lyapunov function

$$V(\mathbf{x}) = - \sum_{i=1}^n \int_0^{x_i} b_i(\xi) g'_i(\xi) g \xi + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \omega_{ij} g_i(x_i) g_j(x_j).$$

They showed that if  $a_i > 0$  and  $g'_i > 0$  for every  $i$ , then  $V$  is a strict Liapunov function and therefore the system is quasi-convergent, see also [9]. Forti et. al. (1995) [2] proved global stability of Hopfield-type neural network of the form

$$\dot{x}_i = -d_i x_i + \sum_{j=1}^n T_{ij} g_j(x_j) + I_i, \quad (1.2)$$

where  $d_i > 0$ ,  $g_j$  is nondecreasing function. Again, the results obtained therein employed a Lyapunov function of the so-called generalized Lur'e-Postnikov type. However, it is not always easy to find a suitable Lyapunov function when considering convergent dynamics. Grossberg (1978) [3] proved a convergence theorem for a class of “competitive systems” for which no Lyapunov functions are known. He considered systems of the form

$$\dot{x}_i = a_i(\mathbf{x})[b_i(x_i) - c(\mathbf{x})], \quad (1.3)$$

where  $a_i \geq 0$ ,  $\frac{\partial c}{\partial x_i} \geq 0$ , for  $i = 1, \dots, n$ . Herein, each  $b_i$  is a function of only one variable  $x_i$ , and the function  $c$  does not depend on  $i$ . In this kind of system, population  $x_i$  at neuron  $i$  competes indirectly with other  $x_j$  through a scalar  $c(\mathbf{x})$ , i.e., the interaction among neurons are through function  $c(\mathbf{x})$ . Worth noticed, it is difficult to find a suitable Lyapunov function for (1.3). In fact, systems (1.1), (1.2) both can be written in the form

$$\dot{x}_i = a_i(\mathbf{x})[b_i(x_i) - c_i(\mathbf{x})],$$

which has a crucial difference from (1.3).

The “competition” for (1.3) by Grossberg means  $a_i \geq 0, \frac{\partial c}{\partial x_j} \geq 0$ , for all  $i, j$  and therefore has a little different sense from the commonly used one. Usually, a system  $\dot{x}_i = G_i(x_1, x_2, \dots, x_n)$  is competitive if  $\frac{\partial G_i}{\partial x_j} \leq 0$ , for  $i \neq j$ . The sense of competition in Grossberg’s paper can be seen if we consider functions  $a_i$  as positive constants. The assumption on  $a_i$  for the studied dynamics is more general though.

Let us give more details to Grossberg’s model. In (1.3),  $n$  is any integer greater than 1,  $\mathbf{x}(t) = (x_1(t), x_2(t), \dots, \dots, x_n(t)) \in \mathbb{R}^n$ . Such a system can have

any number of competing populations, any interpopulation signal functions  $b_i(x_i)$ , any mean competition function, or adaptation level  $c(\mathbf{x})$ , and any state-dependent amplifications  $a_i(\mathbf{x})$  of the competitive balance. That work in [3] proved that any initial value  $\mathbf{x}(0) \geq 0$  (i.e.  $x_i(0) \geq 0$ , for any  $i$ ) generates a limiting pattern  $\mathbf{x}(\infty) = (x_1(\infty), x_2(\infty), \dots, x_n(\infty))$  with  $0 \leq x_i(\infty) := \lim_{t \rightarrow \infty} x_i(t) < \infty$ , under some conditions on  $a_i$ ,  $b_i$ ,  $c$ . We shall summarize the main ideas of Grossberg's work in Section 4.

Recently, discrete-time systems have attracted much scientific interests, cf. [5], [6], [8]. In this study, we consider the following discrete-time version of Grossberg's model

$$x_i(k+1) = x_i(k) + \beta a_i(\mathbf{x}(k)) [b_i(x_i(k)) - c(\mathbf{x}(k))], \quad (1.4)$$

where  $i = 1, 2, \dots, n$ ,  $k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$ . Viewing from the  $\delta$ -operator, (1.3) can be approximated by

$$x_i((k+1)\delta) = x_i(k\delta) + \delta a_i(\mathbf{x}(k\delta)) [b_i(x_i(k\delta)) - c(\mathbf{x}(k\delta))]. \quad (1.5)$$

One usually takes  $x_i[k]_\delta := x_i(k\delta)$  as the  $k$ -th iteration of  $x_i$  and  $\mathbf{x}[k]_\delta := (x_1[k]_\delta, x_2[k]_\delta, \dots, x_n[k]_\delta)$  the  $k$ -th iteration of  $\mathbf{x}$ .

In this presentation, we mainly consider (1.4) with  $\beta = 1$ , i.e.

$$x_i(k+1) = x_i(k) + a_i(\mathbf{x}(k)) [b_i(x_i(k)) - c(\mathbf{x}(k))]. \quad (1.6)$$

We define  $\Delta x_i(k) := x_i(k+1) - x_i(k)$ , hence system (1.6) can be rewritten in the form

$$\Delta x_i(k) = a_i(\mathbf{x}(k)) [b_i(x_i(k)) - c(\mathbf{x}(k))]. \quad (1.7)$$

The main purpose of this investigation is to find out under what conditions on functions  $a_i$ ,  $b_i$ , and  $c$ , systems (1.4) or (1.6) possesses a global limiting pattern  $\mathbf{x}(\infty) := (x_1(\infty), x_2(\infty), \dots, x_n(\infty))$  with  $-\infty < x_i(\infty) := \lim_{t \rightarrow \infty} x_i(t) < \infty$  for every  $i$ , given any initial value  $\mathbf{x}(0)$ .

Below, in Section 2, we state the main results of this presentation. In Section 3, we prove three key lemmas for our main result Theorem 2.2. In Section 4, we summarize the work of Grossberg [3] and make a generalization. A comparison of the analysis in deriving the global consensus for the continuous-time and the discrete-time competitive systems is also made.

## 2 Main Results

**Definition 2.1. (*Global Consensus*)** A competitive system is said to achieve global consensus (or global pattern information) if, given any initial value  $\mathbf{x}(0)$ , the limit  $x_i(\infty) := \lim_{k \rightarrow \infty} x_i(k)$  exist, for all  $i = 1, 2, \dots, n$ .

The main results require the following conditions :

Condition (I): Each  $a_i(\mathbf{x})$  is continuous, and

$$0 < a_i(\mathbf{x}) \leq 1, \text{ for all } \mathbf{x} \in \mathbb{R}^n, i = 1, \dots, n. \quad (2.1)$$

Condition (I)': Each  $a_i(\mathbf{x})$  is continuous, and

$$0 < a_i(\mathbf{x}) \leq A, \text{ for all } \mathbf{x} \in \mathbb{R}^n, i = 1, \dots, n. \quad (2.2)$$

Condition (II):  $c(\mathbf{x})$  is bounded and continuously differentiable with bounded derivatives; namely, there exist constants  $M_1, M_2, r_j$  such that

$$M_1 \leq c(\mathbf{x}) \leq M_2, \quad (2.3)$$

$$0 \leq \frac{\partial c}{\partial x_j}(\mathbf{x}) \leq r_j, \quad (2.4)$$

for all  $\mathbf{x} \in \mathbb{R}^n$ , and  $j = 1, 2, \dots, n$ .

Condition (III):  $b_i(\xi)$  is continuously differentiable, strictly decreasing and there exist  $d_i > 0, l_i \in \mathbb{R}, u_i \in \mathbb{R}$  such that

$$-d_i \leq b'_i(\xi) < 0, \text{ for all } \xi \in \mathbb{R}, \quad (2.5)$$

and

$$b_i(\xi) > M_2, \text{ for } \xi \leq l_i, \quad (2.6)$$

$$b_i(\xi) < M_1, \text{ for } \xi \geq u_i. \quad (2.7)$$

Condition (IV): For  $i = 1, \dots, n$ ,

$$0 < d_i \leq 1 - \sum_{j=1}^n r_j < 1. \quad (2.8)$$

Condition (IV)': For  $i = 1, \dots, n$ ,

$$0 < d_i \leq \frac{1}{\beta} - \sum_{j=1}^n r_j < \frac{1}{\beta}. \quad (2.9)$$

Condition (IV)'': For  $i = 1, \dots, n$ ,

$$0 < d_i \leq \frac{1}{A\beta} - \sum_{j=1}^n r_j < \frac{1}{A\beta}. \quad (2.10)$$

Set

$$d := \min\{d_i : i = 1, 2, \dots, n\}, \quad (2.11)$$

$$M := \max\{|M_1|, |M_2|\}. \quad (2.12)$$

**Theorem 1.** *System (1.6) with functions  $a_i, b_i$ , and  $c$  satisfying Conditions (I), (II), (III), and (IV) achieves global consensus.*

The proof of Theorem 2.2 consists of three lemmas stated below. For system (1.4), the following corollary can be derived.

**Corollary 2.** *System (1.4) with functions  $a_i, b_i$ , and  $c$  satisfying Conditions (I), (II), (III), and (IV)' achieves global consensus.*

In fact, we only need that function  $a_i$  is continuous, positive and bounded above by some real number, say  $A$ , for all  $i$ , instead of Condition (I). It is due to that (1.6) can be rewritten as

$$x_i(k+1) = x_i(k) + \frac{a_i(\mathbf{x}(k))}{A} [Ab_i(x_i(k)) - Ac(\mathbf{x}(k))].$$

We thus derive the following Corollary.

**Corollary 3.** *System (1.4) whose functions  $a_i, b_i$ , and  $c$  satisfy Condition (I)', (II), (III), and (IV)'' achieves global consensus.*

**Remark 2.1.** *From Corollary 2, we find that the smaller  $\beta$  in (1.4) ( $\delta$  in (1.5)) is, the weaker restrictions on functions  $a_i, b_i, c$  are. In other words, when we consider (1.4) instead of (1.3), and want to have the global consensus proposition, we must choose sufficiently small  $\beta$  in (1.4), basically.*

In order to state the key lemmas for our main result, Theorem 2.2, we introduce some notations and definition as follows:

**Notation 2.2.**

$$\begin{aligned}
g_i(k) &= b_i(x_i(k) - c(\mathbf{x}(k))), \\
\Delta g_i(k) &= g_i(k+1) - g_i(k), \\
g_{\max}(k) &= \max\{g_i(k) : i = 1, 2, \dots, n\}, \\
g_{\min}(k) &= \min\{g_i(k) : i = 1, 2, \dots, n\}, \\
I(k) &= \min\{i : g_i(k) = g_{\max}(k)\}, \\
J(k) &= \min\{i : g_i(k) = g_{\min}(k)\}, \\
\hat{x}(k) &= x_{I(k)}(k), \\
\check{x}(k) &= x_{J(k)}(k), \\
\hat{b}(k) &= b_{I(k)}(\hat{x}(k)), \\
\check{b}(k) &= b_{J(k)}(\check{x}(k)), \\
\Delta \hat{b}(k) &= \hat{b}(k+1) - \hat{b}(k), \\
\Delta \check{b}(k) &= \check{b}(k+1) - \check{b}(k), \\
\Delta b_i(x_i(k)) &= b_i(x_i(k+1)) - b_i(x_i(k)).
\end{aligned}$$

**Definition 2.3.** (i) A jump of type-1 is said to occur from  $i$  to  $j$  at  $k$ th iteration if  $I(k) = i, I(k+1) = j$ , (ii) A jump of type-2 is said to occur from  $i$  to  $j$  at  $k$ th iteration if  $J(k) = i, J(k+1) = j$ .

**Lemma 1.** Consider system (1.6) with  $a_i, b_i$ , and  $c$  satisfying (2.1), (2.3), (2.5), (2.6) and (2.7). Given any initial value  $\mathbf{x}(0)$ ,  $\{\mathbf{x}(k)\}$  will be attracted to some compact cube contained in  $\mathbb{R}^n$ . Hence sequence  $\{x_i(k) \mid k \in \mathbb{N}_0\}$  are bounded above and below for all  $i = 1, 2, \dots, n$ .

If Lemma 1 is valid, consider an arbitrary orbit  $\{\mathbf{x}(k)\}$ . Then  $\{|a_i(\mathbf{x}(k))| \mid k \in \mathbb{N}_0\}$  is bounded below by some positive number, say  $0 < \rho_i \leq |a_i(\mathbf{x}(k))|$  for all  $k \in \mathbb{N}_0$  and  $\{b'_i(x_i(k)) \mid k \in \mathbb{N}_0\}$  are bounded above by some negative number, say  $b'_i(x_i(k)) \leq -\epsilon_i < 0$  for all  $k \in \mathbb{N}_0$ . We define

$$\rho := \min\{\rho_i : i = 1, 2, \dots, n\}, \quad (2.13)$$

$$\epsilon := \min\{\epsilon_i : i = 1, 2, \dots, n\}. \quad (2.14)$$

**Lemma 2.** Consider system (1.6) with  $a_i, b_i$ , and  $c$  satisfying (2.1), (2.4), (2.5) and (2.8). Then

(I) for function  $g_{\max}$ , either case  $(g_{\max}-(i))$  or case  $(g_{\max}-(ii))$  holds, where

$(g_{\max}-(i))$ :  $g_{\max}(k) < 0$ , for all  $k \in \mathbb{N}_0$ ,

$(g_{\max}-(ii))$ :  $g_{\max}(k) \geq 0$ , for all  $k \geq k_1$ , for some  $k_1 \in \mathbb{N}_0$ ;

(II) for function  $g_{\min}$ , either case  $(g_{\min}-(i))$  or case  $(g_{\min}-(ii))$  holds, where

$(g_{\min}-(i))$ :  $g_{\min}(k) > 0$ , for all  $k \in \mathbb{N}_0$ ,

$(g_{\min}-(ii))$ :  $g_{\min}(k) \leq 0$ , for all  $k \geq k_2$ , for some  $k_2 \in \mathbb{N}_0$ .

If Lemma 2 holds, there are only four possibilities to consider.

**case (i):** Both  $(g_{\max}-(i))$  and  $(g_{\min}-(i))$  hold. This case is impossible from our definition of  $g_{\max}$  and  $g_{\min}$ .

**case (ii):** Both  $(g_{\max}-(i))$  and  $(g_{\min}-(ii))$  hold, then sequence  $\{x_i(k)\}$  will always decrease as  $k$  increases, for each  $i = 1, 2, \dots, n$ . By Lemma 1,  $\{x_i(k)\}$  are bounded below for every  $i$ , hence the limit  $x_i(\infty)$  exists, for  $i = 1, 2, \dots, n$ .

**case (iii):** Both  $(g_{\max}-(ii))$  and  $(g_{\min}-(i))$  hold, then sequence  $\{x_i(k)\}$  will always increase as  $k$  increases, for all  $i = 1, 2, \dots, n$ . By Lemma 1,  $\{x_i(k)\}$  are bounded above for every  $i$ . Hence the limit  $x_i(\infty)$  exists, for every  $i = 1, 2, \dots, n$ .

**case (iv):** Both  $(g_{\max}-(ii))$  and  $(g_{\min}-(ii))$  hold.

Accordingly, we are left with the case **case (iv)** only, for the conclusion of global consensus for (1.6). We thus assume that  $g_{\max}(0) \geq 0$ ,  $g_{\min}(0) \leq 0$ , without loss of generality.

**Lemma 3.** Consider system (1.6) with  $a_i$ ,  $b_i$ , and  $c$  satisfying Conditions (I), (II), (III), and (IV) then,

(i)  $\lim_{k \rightarrow \infty} \hat{b}(k)$  exists, denoted by  $\hat{B}$ , and  $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) = \hat{B}$ ,

(ii)  $\lim_{k \rightarrow \infty} \check{b}(k)$  exists, denoted by  $\check{B}$ , and  $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) = \check{B}$ .

If Lemma 3 holds, we find that

$$\lim_{k \rightarrow \infty} \hat{b}(k) = \lim_{k \rightarrow \infty} \check{b}(k) =: \bar{B}, \quad (2.15)$$

since  $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) = \hat{B} = \check{B}$ . For any  $i = 1, 2, \dots, n$ ,  $g_{\min}(k) \leq g_i(k) \leq g_{\max}(k)$ , for all  $k \in \mathbb{N}_0$ . Equivalently,

$$\check{b}(k) - c(\mathbf{x}(k)) \leq b_i(x_i(k)) - c(\mathbf{x}(k)) \leq \hat{b}(k) - c(\mathbf{x}(k)),$$

for all  $k \in \mathbb{N}_0$ . Thus,  $\check{b}(k) \leq b_i(x_i(k)) \leq \hat{b}(k)$ , for all  $k \in \mathbb{N}_0$ . Therefore

$$\lim_{k \rightarrow \infty} \check{b}(k) \leq \lim_{k \rightarrow \infty} b_i(x_i(k)) \leq \lim_{k \rightarrow \infty} \hat{b}(k).$$

We obtain

$$\lim_{k \rightarrow \infty} \hat{b}(k) = \lim_{k \rightarrow \infty} b_i(x_i(k)) = \lim_{k \rightarrow \infty} \check{b}(k) = \bar{B},$$

by (2.15). Therefore we conclude that

$$\lim_{k \rightarrow \infty} b_i(x_i(k)) = \bar{B}, \text{ for all } i = 1, 2, \dots, n. \quad (2.16)$$

Moreover,  $\lim_{k \rightarrow \infty} x_i(k)$  exists, for every  $i = 1, 2, \dots, n$ , by (2.5) and (2.16). Hence, global consensus of system (1.6) is achieved, if functions  $a_i$ ,  $b_i$ , and  $c$  satisfy Conditions (I), (II), (III), (IV).

### 3 Proofs of Lemmas

**Proof of Lemma 1 :** For any initial value  $\mathbf{x}(0)$ , we consider the iteration sequence  $\{x_i(k)\}$  and their components  $x_i(k)$ . We divide the proof into several steps.

(i) By (2.3) and (2.7),

$$b_i(x_i) - c(\mathbf{x}) < 0, \quad (3.1)$$

for all  $x_i \geq u_i$ . Therefore

$$\Delta x_i(k) = a_i(\mathbf{x}(k)) [b_i(x_i(k)) - c(\mathbf{x}(k))] < 0, \quad (3.2)$$

if  $x_i(k) \geq u_i$ . Similarly, By (2.3) and (2.6),

$$b_i(x_i) - c(\mathbf{x}) > 0, \quad (3.3)$$

for all  $x_i \leq l_i$ . Therefore

$$\Delta x_i(k) = a_i(\mathbf{x}(k)) [b_i(x_i(k)) - c(\mathbf{x}(k))] > 0, \quad (3.4)$$

if  $x_i(k) \leq l_i$ . We claim that for all  $k \in \mathbb{N}_0$ ,

$$|b_i(x_i(k))| \leq d_i |x_i(k)| + |b_i(0)|. \quad (3.5)$$

This follows from

$$b_i(x_i(k)) - b_i(0) = b'_i(\cdot) x_i(k),$$

where “ $\cdot$ ” means some real number between  $x_i(k)$  and 0. Thus, by (2.5),

$$\begin{aligned} |b_i(x_i(k))| &= |b_i(0) + b'_i(\cdot) x_i(k)| \\ &\leq |b_i(0)| + |b'_i(\cdot) x_i(k)| \\ &\leq |b_i(0)| + d_i |x_i(k)|. \end{aligned}$$

(ii) Next, we show that for fixed constant  $L_i$ , there exist some constants  $u'_i$  and  $d'_i$ , where  $u'_i > 0$ ,  $0 < d_i < d'_i < 1$  such that

$$d_i|x_i| + L_i < d'_i|x_i|, \text{ if } |x_i| \geq u'_i. \quad (3.6)$$

Let us verify this. Notably,

$$\frac{d_i|x_i| + L_i}{|x_i|} = d_i + \frac{L_i}{|x_i|} \rightarrow d_i < 1,$$

as  $|x_i| \rightarrow \infty$ . Therefore, there exist some  $u'_i$  and  $d'_i$ , where  $u'_i > 0$ ,  $0 < d_i < d'_i < 1$  such that  $(d_i|x_i| + L_i)/|x_i| < d'_i$ , if  $|x_i| \geq u'_i$ .

(iii)

$$\begin{aligned} |\Delta x_i(k)| &= |a_i(\mathbf{x}(k))[b_i(x_i(k) - c(\mathbf{x}(k)))]| \\ &\leq |b_i(x_i(k) - c(\mathbf{x}(k)))| \quad (\text{by (2.1)}) \\ &\leq |b_i(x_i(k))| + |c(\mathbf{x}(k))| \\ &\leq d_i|x_i(k)| + |b_i(0)| + |c(\mathbf{x}(k))| \quad (\text{by (3.5)}) \\ &\leq d_i|x_i(k)| + |b_i(0)| + M \quad (\text{by (2.3), (2.12)}). \end{aligned}$$

Hence, by (3.6), we choose  $|b_i(0)| + M = L_i$ , there exist some constants  $u'_i$  and  $d'_i$  where  $u'_i > 0$ ,  $0 < d_i < d'_i < 1$  such that

$$|\Delta x_i(k)| < d'_i|x_i(k)| < |x_i(k)|, \text{ if } |x_i(k)| \geq u'_i. \quad (3.7)$$

(iv) Set, for each  $i$ ,

$$q'_i := \max\{|u_i|, |l_i|, u'_i\}. \quad (3.8)$$

Let  $Q' := [-q'_1, q'_1] \times \cdots \times [-q'_n, q'_n]$ .  $Q'$  is a compact set, hence  $|a_i(\mathbf{x})[b_i(x_i) - c(\mathbf{x})]|$  is bounded on  $Q'$ , say

$$|a_i(\mathbf{x})[b_i(x_i) - c(\mathbf{x})]| \leq K, \quad (3.9)$$

for all  $\mathbf{x} \in Q'$ , for all  $i$ . Set

$$q_i := q'_i + K, \quad (3.10)$$

$$Q := [-q_1, q_1] \times \cdots \times [-q_n, q_n]. \quad (3.11)$$

We shall utilize (3.2), (3.4), (3.7), (3.8), (3.9), (3.10) in the following discussions.

(v) If  $-q_i \leq x_i(0) \leq q_i$ , then  $-q_i < x_i(k) < q_i$ , for all  $k \in \mathbb{N}_0$ .

case (a): If  $x_i(0) \in [-q_i, -q'_i]$ , then  $\Delta x_i(0) > 0$ , due to  $x_i(0) \leq -q'_i \leq l_i$ , and  $|\Delta x_i(0)| < |x_i(0)|$ , due to  $x_i(0) \leq -u'_i$ , hence  $x_i(1)$  still stays in  $(-q_i, -q'_i]$ , or moves into  $(-q'_i, q'_i)$ . If the former case occurs, we consider  $x_i(1)$  as case (a) again. If the latter case occurs, we consider  $x_i(1)$  as in the following case (b).

case (b): If  $x_i(0) \in (-q'_i, q'_i)$ , then  $|\Delta x_i(0)| < K$ , by (3.9), hence  $x_i(1)$  will stay in  $[-q_i, -q'_i]$  or  $(-q'_i, q'_i)$  or  $[q'_i, q_i]$ . Then we can still consider  $x_i(1)$  as in case (a), case (b), and case (c), respectively.

case (c): If  $x_i(0) \in [q'_i, q_i]$ , then  $\Delta x_i(0) < 0$ , by  $x_i(0) \geq q'_i \geq u_i$ , and  $|\Delta x_i(0)| < |x_i(0)|$ , by  $x_i(0) \geq u'_i$ , hence  $x_i(1)$  still stays in  $[q'_i, q_i]$ , or moves into  $(-q'_i, q'_i)$ . If the former case occurs, we consider  $x_i(1)$  as in case (c) again. If the latter case occurs, we consider  $x_i(1)$  as in case (b). From the above arguments, we find that if  $-q_i \leq x_i(0) \leq q_i$ , then  $-q_i < x_i(1) < q_i$ , and we can prove that  $-q_i < x_i(k) < q_i$ , for all  $k \geq 2$ , by induction.

(vi): If  $x_i(0) < -q_i$ , then

case (d):  $\{x_i(k)\}$  either increases as  $k$  increases and remains bounded above by  $-q_i$ , or

case (e):  $\{x_i(k)\}$  enter  $[-q_i, q_i]$  at some iteration, and never leaves  $[-q_i, q_i]$  again.

(vii) if  $x_i(0) > q_i$ , then

case (f):  $\{x_i(k)\}$  either decreases as  $k$  increases and remains bounded below by  $q_i$ , or

case (g):  $\{x_i(k)\}$  enters  $[-q_i, q_i]$  at some iteration, and never leaves  $[-q_i, q_i]$  again.

We find that no matter which case above occurs,  $\{x_i(k)\}$  are bounded above and below for all  $i$ . Therefore,  $\{|a_i(\mathbf{x}(k))|\}$  are bounded below by some positive number, say  $0 < \rho_i \leq |a_i(\mathbf{x}(k))|$ , and  $\{b'_i(x_i(k))\}$  are bounded above by some negative number, say  $b'_i(x_i(k)) \leq -\epsilon_i < 0$ . In fact, it is impossible for the above case (d) and case (f) to occur. This is due to that if case (d) occurs, then

$$\begin{aligned} b_i(x_i(k)) - c(\mathbf{x}(k)) &= b_i(x_i(k)) - b_i(l_i) + b_i(l_i) - c(\mathbf{x}(k)) \\ &> b_i(x_i(k)) - b_i(l_i) \\ &= b'_i(\cdot)[x_i(k) - l_i] \\ &\geq \epsilon_i K, \end{aligned}$$

for all  $x_i(k) \leq -q_i \leq l_i - K$ , by (2.5), (3.3), where “.” means some real number between  $x_i(k)$  and  $l_i$ . Therefore  $\Delta x_i(k) = a_i(\mathbf{x}(k))[b_i(x_i(k)) - c(\mathbf{x}(k))] > \epsilon_i K \rho_i$ .

Hence  $\{x_i(k)\}$  will increase unboundedly, and this yields a contradiction. Therefore case (d) never occurs. Similarly, case (f) never occurs too. By the arguments above, we can find that given any initial value  $\mathbf{x}(0)$ ,  $\{\mathbf{x}(k)\}$  will be attracted by  $Q$ .

**Proof of Lemma 2:**

For function  $g_{\max}$ , if  $g_{\max}(k) \geq 0$  for some  $k$ , say  $I(k) = i$ , then  $g_j(k) \leq g_i(k)$ , for all  $j \neq i$ . Consider two possibilities  $|\Delta g_i(k)| \leq g_i(k)$ , and  $|\Delta g_i(k)| > g_i(k)$ .

case (i)  $|\Delta g_i(k)| \leq g_i(k)$ : It follows that

$$g_{\max}(k+1) \geq g_i(k+1) = g_i(k) + \Delta g_i(k) \geq 0.$$

case (ii)  $|\Delta g_i(k)| > g_i(k)$ . Let us elaborate.

$$\begin{aligned} \Delta g_i(k) &= g_i(k+1) - g_i(k) \\ &= b_i(x_i(k+1)) - c(\mathbf{x}(k+1)) - [b_i(x_i(k)) - c(\mathbf{x}(k))] \\ &= b_i(x_i(k+1)) - b_i(x_i(k)) - [c(\mathbf{x}(k+1)) - c(\mathbf{x}(k))] \\ &= b'_i(\cdot)[x_i(k+1) - x_i(k)] - \sum_{j=1}^n \frac{\partial c}{\partial x_j}(\bullet)[x_j(k+1) - x_j(k)], \end{aligned}$$

where “ $\cdot$ ” means some real number between  $x_i(k+1)$  and  $x_i(k)$ , “ $\bullet$ ” means some vector between  $\mathbf{x}(k+1)$  and  $\mathbf{x}(k)$ . Thus,

$$\begin{aligned} \Delta g_i(k) &= b'_i(\cdot)a_i(\mathbf{x}(k))g_i(k) - \sum_{j=1}^n \frac{\partial c}{\partial x_j}(\bullet)a_j(\mathbf{x}(k))g_j(k) \\ &\geq -d_i a_i(\mathbf{x}(k))g_i(k) - \sum_{j=1}^n r_j a_j(\mathbf{x}(k))g_i(k) \\ &\quad (\text{by (2.3), (2.5) and } g_j(k) \leq g_i(k) \geq 0) \\ &\geq -d_i g_i(k) - \sum_{j=1}^n r_j g_i(k) \quad (\text{by (2.1)}) \\ &= (-d_i - \sum_{j=1}^n r_j)g_i(k) \\ &\geq -g_i(k) \quad (\text{by (2.8)}). \end{aligned}$$

Hence  $\Delta g_i(k) > 0$ , since  $|\Delta g_i(k)| > g_i(k)$  and  $\Delta g_i(k) \geq -g_i(k)$ . Therefore,  $g_{\max}(k+1) \geq g_i(k+1) = g_i(k) + \Delta g_i(k) > 0$ .

For function  $g_{\min}$ , if  $g_{\min}(k) \leq 0$  for some  $k$ , say  $J(k) = i$ . Then  $g_j(k) \geq g_i(k)$ , for all  $j \neq i$ . Then either  $|\Delta g_i(k)| \leq -g_i(k)$  or  $|\Delta g_i(k)| > -g_i(k)$  holds.

case (i)  $|\Delta g_i(k)| \leq -g_i(k)$ : It follow sthat  $g_{\min}(k+1) \leq g_i(k+1) = g_i(k) + \Delta g_i(k) \leq 0$ .

case(ii)  $|\Delta g_i(k)| > -g_i(k)$ :

$$\begin{aligned}
\Delta g_i(k) &= g_i(k+1) - g_i(k) \\
&= b_i(x_i(k+1)) - c(\mathbf{x}(k+1)) - [b_i(x_i(k)) - c(\mathbf{x}(k))] \\
&= b_i(x_i(k+1)) - b_i(x_i(k)) - [c(\mathbf{x}(k+1)) - c(\mathbf{x}(k))] \\
&= b'_i(\cdot)[x_i(k+1) - x_i(k)] - \sum_{j=1}^n \frac{\partial c}{\partial x_j}(\bullet)[x_j(k+1) - x_j(k)],
\end{aligned}$$

where “ $\cdot$ ” means some real number between  $x_i(k+1)$  and  $x_i(k)$ , “ $\bullet$ ” means some vector between  $\mathbf{x}(k+1)$  and  $\mathbf{x}(k)$ . Thus

$$\begin{aligned}
|\Delta g_i(k)| &= b'_i(\cdot)a_i(\mathbf{x}(k))g_i(k) - \sum_{j=1}^n \frac{\partial c}{\partial x_j}(\bullet)a_j(\mathbf{x}(k))g_j(k) \\
&\leq -d_i a_i(\mathbf{x}(k))g_i(k) - \sum_{j=1}^n r_j a_j(\mathbf{x}(k))g_j(k) \\
&\quad (\text{by (2.3), (2.5) and } g_j(k) \geq g_i(k) \leq 0) \\
&\leq -d_i g_i(k) - \sum_{j=1}^n r_j g_i(k) \quad (\text{by (2.1)}) \\
&= (d_i + \sum_{j=1}^n r_j)(-g_i(k)) \\
&\leq -g_i(k) \quad (\text{by (2.8)}).
\end{aligned}$$

Hence  $\Delta g_i(k) < 0$ , since  $|\Delta g_i(k)| > -g_i(k)$  and  $\Delta g_i(k) \leq -g_i(k)$ . So,  $g_{\min}(k+1) \leq g_i(k+1) = g_i(k) + \Delta g_i(k) < 0$ .

From the above arguments, we find that function  $g_{\max}$  may keep negative at all iterations. But once it becomes nonnegative at some iteration, it will always remain nonnegative after this iteration. Similarly,  $g_{\min}$  may keep positive at all iterations. But once it get nonpositive at some iteration, it will always be nonpositive after this iteration. This completes the proof of Lemma 2. With Lemma 2, we assume that  $g_{\max}(0) \geq 0$ ,  $g_{\min}(0) \leq 0$ , without loss of generality.

### Proof for Lemma 3:

We assert that  $\lim_{k \rightarrow \infty} \hat{b}(k)$  exists, and denote it by  $\hat{B}$ ; moreover,  $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) = \hat{B}$ .

Case (i): There exist finitely many jumps of type-1.

In this case, there exist some  $K_3 \in \mathbb{N}$ , some  $i$ , say 1, such that  $g_{\max}(k) = g_1(k) \geq 0$ , for all  $k \geq K_3$ . Hence  $\{x_1(k)\}$  will be non-decreasing as  $k$  increases. By Lemma 1,

$\{x_1(k)\}$  are bounded above. Therefore,  $\lim_{k \rightarrow \infty} x_1(k)$  exists, hence  $\lim_{k \rightarrow \infty} b_1(x_1(k))$  exists, denoted by  $\hat{B}$ . Restated,  $\lim_{k \rightarrow \infty} \hat{b}(k) = \hat{B}$ .

Next, we justify that  $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) = \hat{B}$ . Assume otherwise,  $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) \neq \hat{B}$ . It follows from  $g_{\max}(k) = g_1(k) \geq 0$ , for all  $k \geq K_3$ , that  $b_1(x_1(k)) \geq c(\mathbf{x}(k))$ , for all  $k \geq K_3$ . There exists some  $\varepsilon > 0$ , and subsequence  $\{k_l\}_{l=1}^{\infty}$  of positive integer number with  $k_1 > K_3$  such that  $|c(\mathbf{x}(k_l)) - \hat{B}| > \varepsilon$ , for all  $l \in \mathbb{N}$ . Because  $\lim_{k \rightarrow \infty} b_1(x_1(k)) = \hat{B}$ , for such  $\varepsilon$ , there exists  $K_4 \in \mathbb{N}$ , such that  $|b_1(x_1(k)) - \hat{B}| \leq \frac{\varepsilon}{2}$ , for all  $k \geq K_4$ . Therefore  $g_1(k_l) = b_1(x_1(k_l)) - c(\mathbf{x}(k_l)) > \frac{\varepsilon}{2}$ , for all  $k_l \geq K_4$ . We find that  $\{x_1(k)\}$  is always increasing after  $K_4$ -th iteration. In fact,

$$\Delta x_1(k_l) = a_1(\mathbf{x}(k_l))[b_1(x_1(k_l)) - c(\mathbf{x}(k_l))] > \rho \frac{\varepsilon}{2},$$

if  $k_l \geq K_4$ . Hence  $\{x_1(k)\}$  will increase unboundedly, and yields a contradiction to Lemma 1.

Case (ii): There exist infinitely many jumps of type-1.

We shall justify that  $\{\hat{b}(k)\}$  decreases as  $\{k\} \uparrow \infty$ . Consider a fixed  $k \in \mathbb{N}$ .

Subcase (ii-a): no jump of type-1 occurs at  $k$ -th iteration.

Suppose  $I(k) = I(k+1) = i$ , then  $g_i(k) \geq 0, g_i(k+1) \geq 0$ . In addition,

$$\begin{aligned} \hat{b}(k+1) &= b_i(x_i(k+1)) \\ &\leq b_i(x_i(k)) \\ &= \hat{b}(k), \end{aligned}$$

thank to (2.5), and  $\Delta x_i(k) = a_i(\mathbf{x}(k))g_i(k) \geq 0$ . Thus  $\{\hat{b}(k)\}$  decreases as  $k$  increases.

Subcase (ii-b): jump of type-1 occurs at  $k$ -th iteration and  $g_i(k) \geq 0, g_j(k) \geq 0$ , where  $I(k) = i \neq I(k+1) = j$ .

It follows that

$$\begin{aligned} \hat{b}(k+1) &= b_j(x_j(k+1)) \\ &\leq b_j(x_j(k)) \\ &\leq b_i(x_i(k)) \\ &= \hat{b}(k), \end{aligned}$$

due to (2.5),  $\Delta x_j(k) = a_j(\mathbf{x}(k))g_j(k) \geq 0$ , and by  $I(k) = i \neq j$ .

Subcase (ii-c): jump of type-1 occurs at  $k$ -th iteration and  $g_i(k) \geq 0, g_j(k) < 0$ , where  $I(k) = i \neq I(k+1) = j$ .

Notably, we still have  $g_j(k+1) \geq 0$ . We claim that

$$b_j(x_j(k+1)) - b_j(x_j(k)) \leq b_i(x_i(k)) - b_j(x_j(k)). \quad (3.12)$$

Indeed,

$$\begin{aligned} LHS &= b'_j(\cdot) \Delta x_j(k) \\ &= b'_j(\cdot) a_j(\mathbf{x}(k)) g_j(k) \\ &\leq b'_j(\cdot) g_j(k) \text{ (by (2.1))} \\ &\leq -d_i g_j(k) \text{ (by (2.5), and } g_j(k) < 0) \\ &\leq g_i(k) - g_j(k) \text{ (by } (1 - d_i)g_j(k) < 0 \leq g_i(k)) \\ &= b_i(x_i(k)) - b_j(x_j(k)) \\ &= RHS. \end{aligned}$$

Herein, “ $\cdot$ ” is defined as before. Hence,  $\hat{b}(k+1) = b_j(x_j(k+1)) \leq b_i(x_i(k)) = \hat{b}(k)$ . All these cases indicate that  $\hat{b}(k)$  decreases as  $k$  increases. By Lemma 1,  $\{\mathbf{x}(k)\}$  are attracted into some compact cube  $Q$  contained in  $\mathbb{R}^n$ . Therefore,  $\{b_i(x_i(k))\}$  are bounded below, and so are  $\{\hat{b}(k)\}$ . Hence  $\{\hat{b}(k)\}$  decreases and converges to some number  $\hat{B}$  as  $k$  tends to infinity (denoted by  $\{\hat{b}(k)\} \downarrow \hat{B}$ ).

Next, we verify that  $\lim_{n \rightarrow \infty} c(\mathbf{x}(k)) = \hat{B}$ . Assume otherwise:  $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) \neq \hat{B}$ . There exist some positive  $\mu$ , subsequence  $\{k_l\}_{l=1}^{\infty}$  of positive integers, such that

$$|c(\mathbf{x}(k_l)) - \hat{B}| > \frac{\mu}{\epsilon \rho}, \quad (3.13)$$

Where  $\epsilon, \rho$  are defined in (2.13) and (2.14). Because  $\{\hat{b}(k)\} \downarrow \hat{B}$ , for  $\mu' := \min\{\frac{\mu}{\epsilon \rho}, \mu\} > 0$ , there exists  $L \in \mathbb{N}$  such that

$$\hat{B} \leq b_{I(k)}(x_{I(k)}(k)) \leq \hat{B} + \mu', \quad (3.14)$$

for all  $k \geq L$ . Moreover

$$g_{\max}(\ell) = b_{I(\ell)}(x_{I(\ell)}(\ell) - c(\mathbf{x}(\ell))) \geq 0, \quad (3.15)$$

for all  $\ell \in \mathbb{N}$ . Let  $k_L > L$ . By (3.13), (3.14), and (3.15), we have

$$g_{\max}(k_L) = b_1(x_1(k_L)) - c(\mathbf{x}(k_L)) > \frac{\mu}{\epsilon \rho},$$

where, for convenience, we set  $I(k_L)=1$  without loss of generality. There are two possibilities at  $k_L$ -th iteration, either jump of type-1 occurs or not. If it does not occur, then

$$\begin{aligned}
|\Delta \hat{b}(k_L)| &= |\hat{b}(k_L + 1) - \hat{b}(k_L)| \\
&= |b_1(x_1(k_L + 1)) - b_1(x_1(k_L))| \\
&= |b'_1(\cdot)| |x_1(k_L + 1) - x_1(k_L)| \\
&= |b'_1(\cdot)| |a_1(\mathbf{x}(k_L))| |g_1(k_L)| \\
&= |b'_1(\cdot)| |a_1(\mathbf{x}(k_L))| |g_{\max}(k_L)| \\
&> \epsilon \rho \frac{\mu}{\epsilon \rho} \\
&= \mu.
\end{aligned}$$

But it is impossible, because of (3.14).

If jump of type-1 occurs at  $k_L$ -th iteration. Assume that  $I(k_L + 1)=2$ . Below we consider three different cases for  $b_2(x_2(k_L))$ :

Case (a):  $\hat{B} \leq b_2(x_2(k_L)) < b_1(x_1(k_L))$ . Then  $g_2(k_L) > \frac{\mu}{\epsilon \rho}$ , and  $|\Delta b_2(x_2(k_L))| = |b'_2(\cdot)| |a_2(\mathbf{x}(k_L))| |g_2(k_L)| > \epsilon \rho \frac{\mu}{\epsilon \rho} = \mu$ . It is impossible, due to (3.14).

Case (b):  $\hat{B} > b_2(x_2(k_L)) \geq c(\mathbf{x}(k_L))$ . Then  $g_2(k_L) \geq 0$ , and  $x_2(k_L + 1) \geq x_2(k_L)$ . Thus,

$$\begin{aligned}
\hat{b}(k_L + 1) &= b_2(x_2(k_L + 1)) \\
&\leq b_2(x_2(k_L)) \\
&< \hat{B}.
\end{aligned}$$

It is impossible, since  $\{\hat{b}(k)\} \downarrow \hat{B}$ .

Case (c):  $b_2(x_2(k_L)) < c(\mathbf{x}(k_L))$ . Then  $g_2(k_L) < 0$ , and

$$\begin{aligned}
\Delta b_2(x_2(k_L)) &= b_2(x_2(k_L + 1)) - b_2(x_2(k_L)) \\
&= b'_2(\cdot) a_2(\mathbf{x}(k_L)) g_2(k_L) \\
&\leq -d_2 g_2(k_L) \\
&\leq -g_2(k_L).
\end{aligned}$$

Thus,  $b_2(x_2(k_{L+1})) = b_2(x_2(k_L)) + \Delta b_2(x_2(k_L)) < b_2(x_2(k_L)) - g_2(k_L) = c(\mathbf{x}(k_L))$ . Hence  $\hat{b}(k_L + 1) = b_2(x_2(k_L + 1)) < c(\mathbf{x}(k_L)) < \hat{B}$ . It is impossible, since  $\{b_{\max}(k)\} \downarrow \hat{B}$ .

From the above discussions, we conclude that  $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) = \hat{B}$ .

The second part of the lemma asserts that  $\lim_{k \rightarrow \infty} \check{b}(k)$  exists, denoted by  $\check{B}$ , and  $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) = \check{B}$ . The proof for the assertion resembles the first part. Let us elaborate.

Case (i): If there exist finitely many jumps of type-2, there exists some  $K_5 \in \mathbb{N}$ , some  $i$ , say 1, such that  $g_{\min}(k) = g_1(k) \leq 0$ , for all  $k \geq K_5$ , then  $\{x_1(k)\}$  will be decreasing as  $k \uparrow \infty$ . By **Lemma 1**,  $\{x_1(k)\}$  are bounded below, therefore  $\lim_{k \rightarrow \infty} x_1(k)$  exists, hence  $\lim_{k \rightarrow \infty} b_1(x_1(k))$  exists, denoted by  $\hat{b}$ . (Namely,  $\lim_{k \rightarrow \infty} b_{\min}(k) = \hat{b}$ .)

**claim :**  $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) = \hat{b}$ .

Because  $g_{\min}(k) = g_1(k) \leq 0$ , for all  $k \geq K_5$ ,  $b_1(x_1(k)) \leq c(\mathbf{x}(k))$ , for all  $k \geq K_5$ .

**Assume otherwise:**  $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) \neq \hat{b}$ .

There exists some  $\varepsilon > 0$ , and subsequence  $\{k_j\}_{j=1}^{\infty}$  of positive integer number with  $k_1 > K_5$  such that  $|c(\mathbf{x}(k_j)) - \hat{b}| > \varepsilon$ , for all  $j \in \mathbb{N}$ . Because  $\lim_{\ell \rightarrow \infty} b_1(x_1(\ell)) = \hat{b}$ , for such  $\varepsilon$ , there exists  $K_6 \in \mathbb{N}$ , such that  $|b_1(x_1(\ell)) - \hat{b}| \leq \frac{\varepsilon}{2}$ , for all  $\ell \geq K_6$ . Therefore  $g_1(k_j) = b_1(x_1(k_j)) - c(\mathbf{x}(k_j)) < -\frac{\varepsilon}{2}$ , for all  $k_j \geq K_6$ . We find that  $\{x_1(k)\}$  are always decreasing after  $K_6$ -th iteration, especially at  $k_j$ -th iteration,  $\Delta x_1(k_j) = a_1(\mathbf{x}(k_j))[b_1(x_1(k_j)) - c(\mathbf{x}(k_j))] < -\rho \frac{\varepsilon}{2}$ . Hence,  $\{x_1(k)\}$  will decrease unboundedly, and make contradiction to **Lemma 1**.

**case (ii):** If there exist infinitely many jumps of type-2.

**claim :**  $\{b_{\min}(k)\}$  increase as  $\{k\} \uparrow \infty$ .

For any  $k$ ,

**subcase (i):** no jump of type-2 occurs at  $k$ -th iteration.

Say  $J(k) = J(k+1) = i$ , therefore  $g_i(k) \leq 0, g_i(k+1) \leq 0$ .  $b_{\min}(k+1) = b_i(x_i(k+1)) \geq b_i(x_i(k))$  (by (2.5), and  $\Delta x_i(k) = a_i(\mathbf{x}(k))g_i(k) \leq 0$ )  $= b_{\min}(k)$ .

**subcase (ii):** jump of type-2 occurs at  $k$ -th iteration.

Say  $J(k) = i \neq J(k+1) = j$ ,

**subcase (ii-a):**  $g_i(k) \leq 0, g_j(k) \leq 0$ ,

$$\begin{aligned} b_{\min}(k+1) &= b_j(x_j(k+1)) \\ &\geq b_j(x_j(k)) \text{ (by (2.5), } \Delta x_j(k) = a_j(\mathbf{x}(k))g_j(k) \leq 0) \\ &> b_i(x_i(k)) \text{ (by } J(n) = i \neq j) \\ &= b_{\min}(k) \end{aligned}$$

**subcase (ii-b):**  $g_i(k) \leq 0, g_j(k) > 0$ ,

At first, we check

$$b_j(x_j(k+1)) - b_j(x_j(k)) \geq b_i(x_i(k)) - b_j(x_j(k)). \quad (3.16)$$

$$\begin{aligned} LHS &= b_j'(\cdot) \Delta x_j(k) \\ &= b_j'(\cdot) a_j(\mathbf{x}(k)) g_j(k) \\ &\geq b_j'(\cdot) g_j(k) \text{ (by (2.1))} \\ &\geq -d_j g_j(k) \text{ (by (2.5), and } g_j(k) > 0) \\ &\geq g_i(k) - g_j(k) \text{ (by } (1 - d_j)g_j(k) > 0 \geq g_i(k)) \\ &= b_i(x_i(k)) - b_j(x_j(k)) \\ &= RHS \text{ (where “} \cdot \text{” defined as before).} \end{aligned}$$

Hence,

$$\begin{aligned} b_{\min}(k+1) &= b_j(x_j(k+1)) \\ &\geq b_i(x_i(k)) \text{ (by (3.16))} \\ &= b_{\min}(k) \end{aligned}$$

After considering all cases, we can find that  $\{b_{\min}(k)\}$  increase forever as  $\{k\} \uparrow \infty$ . By **Lemma 1**,  $\{\mathbf{x}(k)\}$  are attracted into some compact cube contained in  $\mathbb{R}^n$ , therefore  $\{b_i(x_i(k))\}$  are bounded above, and so are  $\{b_{\min}(k)\}$ . Hence  $\{b_{\min}(k)\}$  increase and converge to some number, say  $\hat{b}$  (Denoted by  $\{b_{\min}(k)\} \uparrow \hat{b}$ ).

**claim :**  $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) = \hat{b}$ .

**Assume otherwise:**  $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) \neq \hat{b}$ .

There exist some positive  $\varepsilon$ , subsequence  $\{k_j\}_{j=1}^{\infty}$  of positive integer number such that

$$|c(\mathbf{x}(k_j)) - \hat{b}| > \frac{\varepsilon}{\epsilon \rho}. \quad (3.17)$$

Because  $\{b_{\min}(k)\} \uparrow \hat{b}$ , for  $\varepsilon' := \min\{\frac{\varepsilon}{\epsilon \rho}, \varepsilon\} > 0$ , there exists  $L \in \mathbb{N}$ , such that

$$\hat{b} \geq b_{J(k_j)}(x_{J(k_j)}(k_j)) \geq \hat{b} - \varepsilon', \text{ for all } j \geq L. \quad (3.18)$$

Moreover

$$g_{\min}(\ell) = b_{J(\ell)}(x_{J(\ell)}(\ell) - c(\mathbf{x}(\ell))) \leq 0, \text{ for all } \ell \in \mathbb{N}. \quad (3.19)$$

By (3.17), (3.18), and (3.19)

$$g_{\min}(k_L) = b_1(x_1(k_L)) - c(\mathbf{x}(k_L)) < -\frac{\varepsilon}{\epsilon \rho} \text{ (For convenience, we say } J(k_L)=1 \text{ without}$$

loss of generality).

At  $k_L - th$  iteration, either jump of type-2 occurs or not

**In latter case,**

$$\begin{aligned}
|\Delta b_{\min}(k_L)| &= |b_{\min}(k_L + 1) - b_{\min}(k_L)| \\
&= |b_1(x_1(k_L + 1)) - b_1(x_1(k_L))| \\
&= |b'_1(\cdot)| |x_1(k_L + 1) - (x_1(k_L))| \\
&= |b'_1(\cdot)| |a_1(\mathbf{x}(k_L))| |g_1(k_L)| \\
&= |b'_1(\cdot)| |a_1(\mathbf{x}(k_L))| |g_{\min}(k_L)| \\
&> \epsilon \rho \frac{\varepsilon}{\epsilon \rho} \\
&= \varepsilon
\end{aligned}$$

But it is impossible, since (3.18).

**In former case,** we say  $J(k_L + 1) = 2$ .

Below we consider three different cases for  $b_2(x_2(k_L))$  as follows:

**case (a):**  $\hat{b} \geq b_2(x_2(k_L)) > b_1(x_1(k_L))$ , then

$g_2(k_L) < -\frac{\varepsilon}{\epsilon \rho}$ ,  $|\Delta b_2(x_2(k_L))| = |b'_2(\cdot)| |a_2(\mathbf{x}(k_L))| |g_2(k_L)| > \epsilon \rho \frac{\varepsilon}{\epsilon \rho} = \varepsilon$ . It is impossible, since (3.18).

**case (b):**  $\hat{b} < b_2(x_2(k_L)) \leq c(\mathbf{x}(k_L))$ , then

$g_2(k_L) \leq 0$ ,  $x_2(k_L + 1) \leq x_2(k_L)$ , therefore  $b_{\min}(k_L + 1) = b_2(x_2(k_L + 1)) \geq b_2(x_2(k_L)) > \hat{b}$ . It is impossible, since  $\{b_{\min}(k)\} \uparrow \hat{b}$ .

**case (c):**  $b_2(x_2(k_L)) > c(\mathbf{x}(k_L))$ , then

$g_2(k_L) > 0$ ,  $\Delta b_2(x_2(k_L)) = b_2(x_2(k_L + 1)) - b_2(x_2(k_L)) = b'_2(\cdot) a_2(\mathbf{x}(k_L)) g_2(k_L) \geq -d_2 g_2(k_L) \geq -g_2(k_L)$ .  $b_2(x_2(k_{L+1})) = b_2(x_2(k_L)) + \Delta b_2(x_2(k_L)) > b_2(x_2(k_L)) - g_2(k_L) = c(\mathbf{x}(k_L))$ . Hence  $b_{\min}(k_L + 1) = b_2(x_2(k_L + 1)) > c(\mathbf{x}(k_L)) > \hat{b}$ . It is impossible, since  $\{b_{\min}(k)\} \uparrow \hat{b}$ .

From above three cases, we can find that “ $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) \neq \hat{b}$ ” will make contradiction, therefore  $\lim_{k \rightarrow \infty} c(\mathbf{x}(k)) = \hat{b}$ .

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