

Transactions Briefs

An Alternative Approach for Designing Maximally Flat Low-Pass Filters with Multiple Order Imaginary-Axis Zeros

CHE-HO WEI

Abstract—A systematic procedure is introduced to design a maximally flat low-pass filter with multiple order imaginary-axis zeros. The filter characteristics is specified by the passband and stopband edge frequencies, maximum loss in passband and minimum loss in stopband. This approach can lead to an optimal design of this class of filters.

I. INTRODUCTION

Most of the previous studies on designing a maximally flat lowpass filter with multiple-order imaginary-axis zeros [1]–[3] are based on the filter's cutoff slope and minimum loss in the stopband. The filter so designed often depends on the choice of the position of the imaginary-axis zeros. Recently, a procedure for evaluating all the parameters of this kind of filters from the given specifications in the passband and stopband has been introduced by Varanasi and Dutta Roy [4].

In this brief, an alternative approach is introduced to design this kind of filters, specified by the passband's maximum loss $A_{\max}$, stopband's minimum loss $A_{\min}$, passband edge frequency ω_p and stopband edge frequency ω_s . The transfer function is derived from the modified Butterworth filter of Rakovich and Popovich [5]. It is similar to the derivation of the inverse Chebyshev low-pass function from the classical Chebyshev low-pass function [6]–[8]. By this approach, an optimal design of this kind of filters can be obtained.

II. MODIFIED BUTTERWORTH FILTER

The classical Butterworth low-pass filter has maximally flat attenuation response at $\omega = 0$ [8]. A class of low-pass filters with zero loss at $\omega = 0$ and at some frequency within the passband, known as modified Butterworth filter, has been studied [5]. In general, the magnitude squared function can be written in the form

$$|T(j\omega)|^2 = \frac{1}{1 + \epsilon^2 \frac{(\omega^2)^{n-2m} (\omega_0^2 \omega^2 - 1)^{2m}}{(\omega_0^2 - 1)^{2m}}} \quad (1)$$

where m is an integer such that $2m < n$ and $\omega_0 > 1$. ϵ^2 is a constant controlling the maximum passband loss. The loss of the filter in decibels is given by

$$A(\omega) = 10 \log [1 + \epsilon^2 L(\omega^2)] \quad (2)$$

where

$$L(\omega^2) = \frac{(\omega^2)^{n-2m} (\omega_0^2 \omega^2 - 1)^{2m}}{(\omega_0^2 - 1)^{2m}} \quad (3)$$

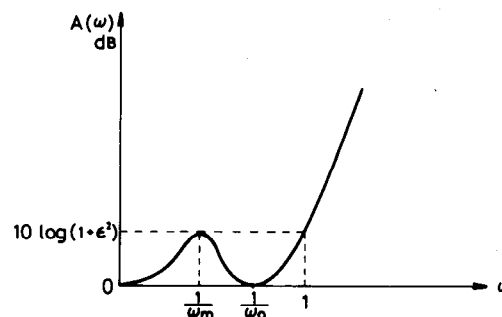


Fig. 1. Loss characteristics of a normalized modified Butterworth low-pass filter.

TABLE I
ZERO-LOSS FREQUENCY $1/\omega_0$ AND MAXIMUM PASSBAND-LOSS FREQUENCY $1/\omega_m$ OF THE NORMALIZED MODIFIED BUTTERWORTH FILTERS FOR $n = 4-12$

n	m	$\frac{1}{\omega_0}$	$\frac{1}{\omega_m}$
3	1	0.8660	0.5000
4	1	0.9102	0.6436
5	1	0.9323	0.7221
	2	0.8207	0.3670
6	1	0.9456	0.7721
	2	0.8660	0.5000
7	1	0.9545	0.8067
	2	0.8926	0.5843
	3	0.7969	0.3012
8	1	0.9609	0.8322
	2	0.9102	0.6436
	3	0.8391	0.4195
9	1	0.9657	0.8517
	2	0.9227	0.6878
	3	0.8660	0.5000
	4	0.7820	0.2607
10	1	0.9695	0.8671
	2	0.9323	0.7221
	3	0.8550	0.5597
	4	0.8207	0.3670
11	1	0.9725	0.8797
	2	0.9396	0.7496
	3	0.8992	0.6062
	4	0.8468	0.4423
	5	0.7717	0.2327
12	1	0.9750	0.8900
	2	0.9456	0.7721
	3	0.9102	0.6436
	4	0.8660	0.5000
	5	0.8073	0.3296

The filter has zero loss at $\omega = 0$ and $\omega = 1/\omega_0$. The first $n-2m-1$ derivatives of $L(\omega^2)$ with respect to ω^2 at $\omega = 0$ and the first $2m-1$ derivatives of $L(\omega^2)$ with respect to ω^2 at $\omega = 1/\omega_0$ are all equal to zero.

The loss characteristics of (2) is shown in Fig. 1. The frequency $1/\omega_m$, corresponding to an analytical maximum of passband loss, obtained from the condition $(dL(\omega^2))/d\omega = 0$ at $\omega = 1/\omega_m$, is given by

$$\frac{1}{\omega_m} = \left(\frac{n-2m}{n} \right)^{1/2} \frac{1}{\omega_0} \quad (4)$$

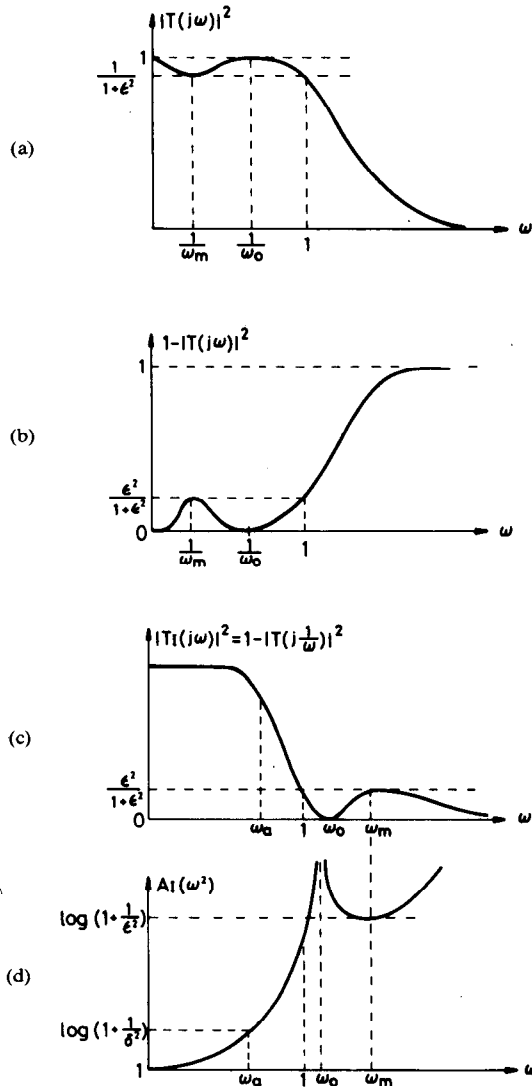


Fig. 2. Pictorial derivation of a normalized maximally flat filter with multiple-order imaginary-axis zeros from a normalized modified Butterworth filter.

The loss at $\omega = 1/\omega_m$ is then obtained as

$$A\left(\frac{1}{\omega_m}\right) = 10 \log \left[1 + \epsilon^2 \frac{(2m)^{2m} (n-2m)^{n-2m}}{n^n} \cdot \frac{1}{(\omega_0^2 - 1)^{2m} (\omega_0^2)^{n-2m}} \right]. \quad (5)$$

The parameter ω_0 can be chosen such that $A(1) = A(1/\omega_m) = 10 \log(1 + \epsilon^2)$. If the maximum passband loss is equal to $10 \log(1 + \epsilon^2)$, then the passband edge frequency ω_p is equal to unity. The normalized value of $1/\omega_0$ can be found from the roots of the following equation:

$$\left(\frac{1}{\omega_0^2}\right)^{2n} - \frac{n^n}{(n-2m)^{n-2m} (2m)^{2m}} \cdot \left(1 - \frac{1}{\omega_0^2}\right)^{2m} = 0. \quad (6)$$

The values of $1/\omega_0$ and $1/\omega_m$ for $n = 3 \sim 12$ and $2m < n$ are given in Table I.

TABLE II
SELECTIVITY PARAMETER k OF THE MAXIMALLY FLAT LOW-PASS FUNCTION WITH MULTIPLE-ORDER IMAGINARY AXIS ZEROS (A), BUTTERWORTH APPROXIMATION (B) AND INVERSE CHEBYSHEV APPROXIMATION (C) FOR $A_{I,\max} = 3$ dB AND $A_{I,\min} = 30$ dB

(A)			(B)		(C)	
n	m	$k=\omega_a$	n	$k=\omega_a$	n	$k=\omega_a$
4	1	0.5998	3	0.3160	3	0.4720
5	1	0.6794	4	0.4217	4	0.6297
	2	0.6768	5	0.5012	5	0.7329
6	1	0.7331	6	0.5623	6	0.8008
	2	0.7457	7	0.6105	7	0.8468
7	1	0.7716	8	0.6494	8	0.8791
	2	0.7905	9	0.6813	9	0.9024
	3	0.7688	10	0.7079	10	0.9197
8	1	0.8005	11	0.7305	11	0.9328
	2	0.8220	12	0.7499	12	0.9431
	3	0.8116	13	0.7667		
9	1	0.8229	14	0.7814		
	2	0.8451	15	0.7943		
	3	0.8408	16	0.8058		
	4	0.8198	17	0.8161		
10	1	0.8408	18	0.8254		
	2	0.8631	19	0.8338		
	3	0.8621	20	0.8414		
	4	0.8492	21	0.8483		
11	1	0.8555	22	0.8547		
	2	0.8773	23	0.8606		
	3	0.8783	24	0.8660		
	4	0.8700				
	5	0.8521				
12	1	0.8677				
	2	0.8889				
	3	0.8911				
	4	0.8856				
	5	0.8737				

III. DERIVATION OF MAXIMALLY FLAT LOW-PASS FUNCTION WITH MULTIPLE-ORDER IMAGINARY-AXIS ZEROS FROM MODIFIED BUTTERWORTH FILTER

Similar to the derivation of the inverse Chebyshev low-pass function from the classical Chebyshev low-pass function [6], maximally flat low-pass function with multiple-order imaginary-axis zeros can be derived from $|T(j\omega)|^2$, as shown in Fig. 2.

The magnitude squared function $|T_I(j\omega)|^2$ of the derived low-pass filter is given by

$$|T_I(j\omega)|^2 = 1 - \left| T\left(j\frac{1}{\omega}\right) \right|^2 = \left[1 + \frac{1}{\epsilon^2} \cdot \frac{\omega^{2n} (\omega_0^2 - 1)^{2m}}{(\omega^2 - \omega_0^2)^{2m}} \right]^{-1} \quad (7)$$

and then

$$A_I(\omega) = 10 \log \left[1 + \frac{1}{\epsilon^2} \cdot \frac{\omega^{2n} (\omega_0^2 - 1)^{2m}}{(\omega^2 - \omega_0^2)^{2m}} \right]. \quad (8)$$

For a given set of $(n, m, \omega_0, \epsilon^2, \delta^2)$, the passband edge frequency can be obtained by solving the following equation:

$$\omega_a^{2n} - \frac{(\epsilon^2/\delta^2)(\omega_a^2 - \omega_0^2)^{2m}}{(\omega_0^2 - 1)^{2m}} = 0. \quad (9)$$

The values ω_a , for $n = 4 \sim 12$, $2m < n$, $A_{I,\min} = 30$ dB, $A_{I,\max} = 3$ dB, are calculated and given in Table II. The values of ω_a for the classical Butterworth low-pass filter with $n = 3 \sim 24$ and the inverse Chebyshev low-pass filter with $n = 3 \sim 12$ are also given in this table for comparison.

The value $k = \omega_a/\omega_s$ is known as the selectivity of a low-pass filter [9]. The larger k is, the more selectivity, i.e., steeper, the

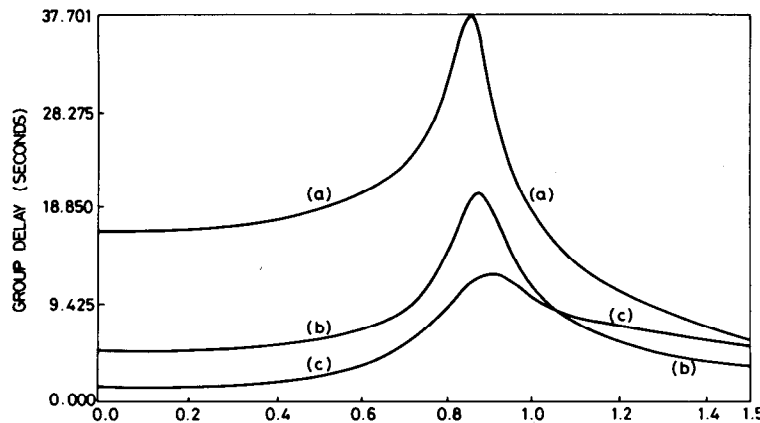


Fig. 3. Time-delay characteristics of the three low-pass filters. (a) Butterworth filter, $n = 22$. (b) Maximally flat low-pass filter with multiple order imaginary axis zeros $n = 10$, $m = 2$. (c) Inverse Chebyshev filter, $n = 8$.

response. For $\omega_s = 1$, $k = \omega_a$. It is interesting to note that, for a given n , the largest selectivity occurs when m is equal to $n/2$ or an integer close to $n/2$. It may be pointed out that the number of finite imaginary-axis zeros in the n th order inverse Chebyshev low-pass filter is $n/2$ or $(n-1)/2$, depending on n is even or odd number.

IV. DESIGN PROCEDURE AND EXAMPLE

Suppose it is required to design a maximally flat low-pass filter with multiple order imaginary-axis zeros, given that

$$A_{I,\max} = 10 \log \left(1 + \frac{1}{\delta^2} \right) \quad \text{and} \quad A_{I,\min} = 10 \log \left(1 + \frac{1}{\epsilon^2} \right).$$

The specified passband edge frequency and stopband edge frequency are ω_p and ω_s , respectively. Then

$$\frac{1}{\epsilon^2} = 10^{0.1 A_{I,\min}} - 1 \quad \text{and} \quad \frac{1}{\delta^2} = 10^{0.1 A_{I,\max}} - 1.$$

The desired selectivity k is equal to ω_p/ω_s . The loss function will be of the form given by (8).

The first step is to find the corresponding normalized filter, i.e., ω_s is set to unity. Equation (9) can be used to find the smallest (n, m) such that ω_a is equal to or larger than k . To find the optimal values of (n, m) , arbitrary initial values of (n, m) are first given. The value ω_0 is found from Table I or by solving (6). Then (9) is solved to obtain the corresponding ω_a . Check if this ω_a is equal to or larger than k . By several iterations, optimal values of (n, m) can be obtained. After the normalized filter function has been found, frequency transformation [8] can be applied to obtain the required filter function.

For example, $A_{I,\max} = 3$ dB, $A_{I,\min} = 30$ dB and $k = \omega_p/\omega_s = 0.85$, the optimal values of n and m are $n = 10$, $m = 2$. To design a low-pass filter with the same specifications by Butterworth function and inverse Chebyshev function, the orders of the required filter function are 22 and 8, respectively. This is also shown in Table II. The Q -factors Q_c of the critical pole pair (the nearest to the imaginary axis) [10] of the corresponding filter functions are obtained as

$$\begin{aligned} Q_c &= 7.01 && \text{for Butterworth function } n = 22, \\ &= 5.30 && \text{for inverse Chebyshev function } n = 8, \\ &= 5.49 && \text{for maximally flat low-pass function with} \\ &&& \text{multiple-order imaginary-axis zeros } n = 10, m = 2. \end{aligned}$$

The required Q_c of the maximally flat low-pass filter with multiple-order imaginary-axis zeros is only slightly larger than that of the corresponding inverse Chebyshev filter, but smaller than that of the Butterworth filter. The phase linearity of a filter can be illustrated by the corresponding time-delay performance of its transfer function. From Fig. 3, it can be seen that the phase linearities of these three filters in the passband are comparable.

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Multifrequency Measurement of Testability with Application to Large Linear Analog Systems

G. IUCULANO, A. LIBERATORE, S. MANETTI, AND M. MARINI

Abstract—With increasing electronic circuit complexity, assessing the testability features becomes a necessity during the design, implementation, and operational or maintenance phases of an analog system.

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The authors are with the Department of Electronic Engineering, University of Florence, 50139 Florence, Italy.
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