

Transfer matrix method for deriving transfer functions of LTI systems

S. L. Jeng^{*a}, B. H. Lue^b, W. H. Chieng^b

^aDepartment of Electrical and Electronic Engineering, Ta Hua University of Science and Technology, Taiwan, R.O.C. ; ^bDepartment of Mechanical Engineering National Chiao Tung University, Taiwan, R.O.C.

ABSTRACT

This paper presents a new method, called the transfer matrix method for obtaining transfer functions among the state variables of a linear time- invariant (LTI) system defined in either a block diagram or the corresponding signal flow graph. The procedures introduced in this paper for obtaining the transfer function require only knowledge on matrix operations, which are easy to explain and remember. Advanced control engineers can automate the symbolic matrix operation by using existing software such as Mathematica.

Keywords: Block diagram, linear time-invariant, signal flow graph, transfer function

1. INTRODUCTION

The block diagram and the signal flow graph are the most common tools used to describe a control system and to derive the gain or the transfer function of the control system [1, 2, 3]. Block diagram reduction is an obvious and direct method used to simplify control systems. The corresponding shifting rule [4] greatly helped the simplification of block diagrams. However, block diagram reduction has not been systematized and deriving the transfer function from a multiple-input multiple-output system remains difficult. Mason's gain formula [5] was used to derive the system transfer function for complex block diagrams. Mason's gain formula provides a systematized method that involves dividing a signal flow graph into several independent loop gains and analyzing the input-output transfer function as well as the transfer functions of other signal states. However, Mason's gain formula is not straightforward, and the system must be separated into several independent loops and forward paths. Neither determining the exact number of independent loop gains that should be separated nor recognizing the touching loops in a complex system is easy.

This paper presents a systematic method for determining the transfer function. This method, which is based on the transfer matrix derived from a signal flow graph is called the transfer matrix method.

2. TRANSFER MATRIX METHOD

Single-input systems are commonly described schematically by using a signal flow graph [5, 6]. The signal flow graph is a directed graph (or digraph) that contains the set of nodes X and the directed edge set E . Figure 1 depicts a general SI system with n vertices. There are $n-1$ tail nodes in the signal flow graph. The various node signals are related according to the associated equation [7] as follows:

$$\begin{aligned} a_{1,2}x_1 + a_{2,2}x_2 + \dots + a_{n,2}x_n &= x_2, \\ a_{1,3}x_1 + a_{2,3}x_2 + \dots + a_{n,3}x_n &= x_3, \\ &\dots \\ a_{1,n}x_1 + a_{2,n}x_2 + \dots + a_{n,n}x_n &= x_n, \end{aligned} \tag{1}$$

where $a_{i,j}$ denotes the gain from x_i to x_j .

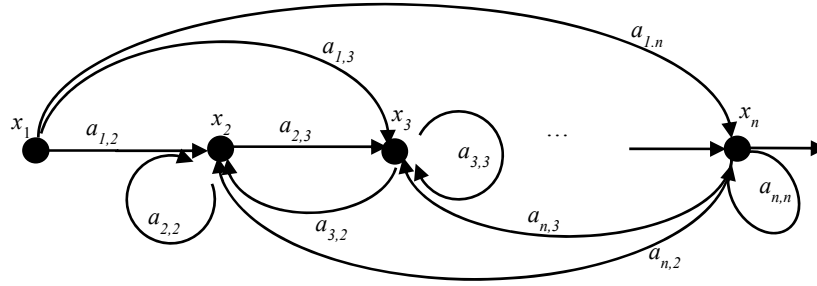


Figure 1. The signal flow graph with n vertices.

Let $\mathbf{A}^{*T}$ be a nodal adjacency matrix, $\mathbf{X}$ be the node vector, and $\mathbf{X}^*$ be the tail-node vector. This system of nodal equations in (1) may be rewritten in matrix form as follows:

$$\mathbf{A}^{*T} \mathbf{X} = \mathbf{X}^* \quad (2)$$

where

$$\mathbf{A}^{*T} = \begin{bmatrix} a_{1,2} & a_{2,2} & \cdots & a_{n,2} \\ a_{1,3} & a_{2,3} & \cdots & a_{n,3} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} \end{bmatrix},$$

$$\mathbf{X}^* = [x_2 \quad x_3 \quad \cdots \quad x_n]^T,$$

$$\mathbf{X} = [x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_n]^T.$$

Adding a self-sustaining equation to (1), yields an augmented nodal adjacency matrix (or called the transfer matrix) producing

$$\mathbf{A}^T \mathbf{X} = \mathbf{X} \quad (3)$$

where

$$\mathbf{A}^T = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_{1,2} & a_{2,2} & \cdots & a_{(n-1),2} & a_{n,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1,(n-1)} & a_{2,(n-1)} & \cdots & a_{(n-1),(n-1)} & a_{n,(n-1)} \\ a_{1,n} & a_{2,n} & \cdots & a_{(n-1),n} & a_{n,n} \end{bmatrix}.$$

The transfer function of a LTI system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero [7]. The

gain $a_{i,j} = a_{i,j}(s)$ can be determined based on the transfer function of $\frac{x_j(s)}{x_i(s)}$ derived from the Laplace transformation.

The transfer matrix $\mathbf{A}^T$ is divided into submatrices as follows:

$$\mathbf{A}^T(s) = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{C}(s) & \mathbf{D}(s) \end{bmatrix},$$

where $\mathbf{C}(s) = [a_{1,2}(s) \quad a_{1,3}(s) \quad \cdots \quad a_{1,n}(s)]^T$ must be a non-zero vector to transfer the state from $x_1(s)$ to other states, and

$$\mathbf{D}(s) = \begin{bmatrix} a_{2,2}(s) & \dots & a_{n,2}(s) \\ \vdots & \dots & \vdots \\ a_{2,n}(s) & \dots & a_{n,n}(s) \end{bmatrix}.$$

Theorem 1:

If the control system is a relaxed system, then $\det[\mathbf{D}(s) - \mathbf{I}] \neq 0$.

Proof:

The nodal adjacency matrix $\mathbf{A}^{*T}(s)$ can be divided into submatrices as follows:

$$\mathbf{A}^{*T}(s) = \begin{bmatrix} \mathbf{C}(s) & \mathbf{D}(s) \end{bmatrix}$$

Based on (2),

$$\mathbf{A}^{*T}(s)\mathbf{X} = \begin{bmatrix} \mathbf{C}(s) & \mathbf{D}(s) \end{bmatrix} \begin{bmatrix} x_1 \\ \mathbf{X}^* \end{bmatrix} = \mathbf{X}^*$$

It yields that

$$x_1\mathbf{C}(s) + (\mathbf{D}(s) - \mathbf{I})\mathbf{X}^* = \mathbf{0}$$

The solution of this equation is

$$x_1 = \frac{-\mathbf{C}^T(s)(\mathbf{D}(s) - \mathbf{I})\mathbf{X}^*}{\mathbf{C}^T(s)\mathbf{C}(s)}$$

When $(\mathbf{D}(s) - \mathbf{I})\mathbf{X}^* = \mathbf{0}$ for a non-trivial $\mathbf{X}^*$, the first element x_1 of the node vector $\mathbf{X}$ can be zero, thus contradicting the condition of a relaxed system, that is, all states are zero. Q.E.D.

Theorem 2:

Let $\mathbf{Q}(s) = [q_1(s) \ q_2(s) \ \dots \ q_n(s)]^T$ be the eigenvector of the transfer matrix $\mathbf{A}^T(s)$ associated with a single unity eigenvalue. The transfer function between nodes x_i and x_1 is $\frac{q_i(s)}{q_1(s)}$, $i = 2, 3, \dots, n$.

Proof:

Let $\hat{\mathbf{Q}}(s) = \begin{bmatrix} 1 & \frac{q_2(s)}{q_1(s)} & \dots & \frac{q_n(s)}{q_1(s)} \end{bmatrix}^T$ be a nonzero $1/q_1(s)$ multiple of the eigenvector $\mathbf{Q}(s)$. Based on the eigenvalue theorem that $(\mathbf{A}^T(s) - \mathbf{I})\hat{\mathbf{Q}}(s) = \mathbf{0}$, the non-repeated unity eigenvalue only exists one corresponding eigenvector. The output at the i -th element in $\hat{\mathbf{Q}}(s)$ attributable to the unit impulse input on the first element in $\hat{\mathbf{Q}}(s)$ satisfying the equilibrium of the state transfer given in (3) must be the transfer function between the output and the input. Q.E.D.

The unique eigenvector $\mathbf{Q}(s)$ of the transfer matrix $\mathbf{A}^T(s)$ associated with a unity eigenvalue is also the null space of $(\mathbf{A}^T(s) - \mathbf{I})$ [8]. In the sequel, the procedures used to derive the null space and the eigenvector are introduced. The derivation is conducted in two steps. First, the first row of $(\mathbf{A}^T(s) - \mathbf{I})$ is replaced with a row vector $[\hat{x}_1 \ \hat{x}_2 \ \dots \ \hat{x}_n]$ and, second, the determinant of the resulting matrix is calculated to obtain the normal vector as follows:

$$\begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \dots & \hat{x}_n \\ a_{1,2}(s) & -1 & \dots & a_{n,2}(s) \\ \vdots & \vdots & \dots & \vdots \\ a_{1,n}(s) & a_{2,n}(s) & \dots & -1 \end{vmatrix} \\
= \alpha_1(s)\hat{x}_1 + \alpha_2(s)\hat{x}_2 + \dots + \alpha_n(s)\hat{x}_n$$
(4)

The eigenvector $\mathbf{Q}(s)$ of $\mathbf{A}^T(s)$ associated with a single unity can be expressed as follows:

$$\mathbf{Q}(s) = [\alpha_1(s) \quad \alpha_2(s) \quad \dots \quad \alpha_n(s)]^T$$

Based on Theorem 2, the transfer function between nodes $x_n(s)$ and $x_1(s)$ is

$$\frac{x_n(s)}{x_1(s)} = \frac{\alpha_n(s)}{\alpha_1(s)} \quad (5)$$

Example 1

A block diagram of a single-input single-output system is shown in Figure 2(a) and the corresponding signal flow graph is shown in Figure 2(b).

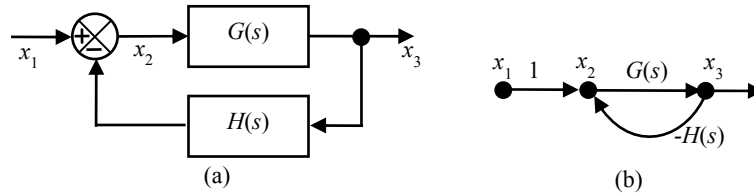


Figure2. (a) The block diagram and (b) the signal flow graph of the single-input single-output system.

The node vector and the tail node vector are $\mathbf{X}(s) = [x_1(s) \quad x_2(s) \quad x_3(s)]^T$ and $\mathbf{X}^*(s) = [x_2(s) \quad x_3(s)]^T$, respectively. The transfer matrix $\mathbf{A}^T(s)$ according to (3) yields

$$\mathbf{A}^T(s) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -H(s) \\ 0 & G(s) & 0 \end{bmatrix}.$$

The first row of $(\mathbf{A}^T(s) - \mathbf{I})$ is replaced with a row vector $[\hat{x}_1 \quad \hat{x}_1 \quad \hat{x}_3]$, and the determinant yields

$$\begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ 1 & -1 & -H(s) \\ 0 & G(s) & -1 \end{vmatrix} = (1 + G(s)H(s))\hat{x}_1 + \hat{x}_2 + G(s)\hat{x}_3.$$

The eigenvector $\mathbf{Q}(s)$ of $\mathbf{A}^T(s)$ associated with a single unity eigenvalue is

$$\mathbf{Q}(s) = [1 + G(s)H(s) \quad 1 \quad G(s)]^T.$$

According to (5), the transfer function is then obtained as follows:

$$\frac{x_3(s)}{x_1(s)} = \frac{G(s)}{1 + G(s)H(s)}.$$

Example 2:

Consider the observable system and the controllable system[9] shown in Figure 3.

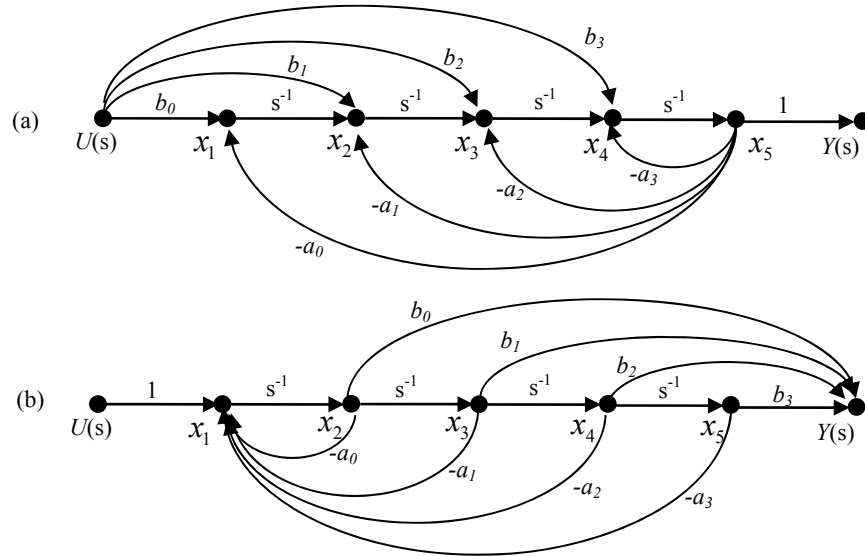


Figure 3. (a) The signal flow graph of the Observable system and (b) the signal flow graph of the controllable system.

According to the given signal flow graph in Figure 3(a), the transfer matrix $\mathbf{A}^T$ is

$$\mathbf{A}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_0 & 0 & 0 & 0 & 0 & -a_0 & 0 \\ b_1 & s^{-1} & 0 & 0 & 0 & -a_1 & 0 \\ b_2 & 0 & s^{-1} & 0 & 0 & -a_2 & 0 \\ b_3 & 0 & 0 & s^{-1} & 0 & -a_3 & 0 \\ 0 & 0 & 0 & 0 & s^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad (6)$$

and the node vector $\mathbf{X}$ is

$$\mathbf{X} = [U(s) \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad Y(s)].$$

According to (4),

$$\begin{vmatrix} \hat{U} & \hat{x}_1 & \hat{x}_2 & \hat{x}_3 & \hat{x}_4 & \hat{x}_5 & \hat{Y} \\ b_0 & -1 & 0 & 0 & 0 & -a_0 & 0 \\ b_1 & s^{-1} & -1 & 0 & 0 & -a_1 & 0 \\ b_2 & 0 & s^{-1} & -1 & 0 & -a_2 & 0 \\ b_3 & 0 & 0 & s^{-1} & -1 & -a_3 & 0 \\ 0 & 0 & 0 & 0 & s^{-1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{vmatrix} \\ = \alpha_1(s)\hat{U} + \alpha_2(s)\hat{x}_1 + \alpha_3(s)\hat{x}_2 + \alpha_4(s)\hat{x}_3 + \alpha_5(s)\hat{x}_4 + \alpha_6(s)\hat{x}_5 + \alpha_7(s)\hat{Y},$$

where

$$\alpha_1(s) = \frac{s^4 + a_3s^3 + a_2s^2 + a_1s + a_0}{s^4},$$

$$\alpha_7(s) = \frac{b_3s^3 + b_2s^2 + b_1s + b_0}{s^4}.$$

The transfer function of the observable system is

$$\frac{\alpha_7(s)}{\alpha_1(s)} = \frac{b_3s^3 + b_2s^2 + b_1s + b_0}{s^4 + a_3s^3 + a_2s^2 + a_1s + a_0}.$$

The first column in (6) denotes the gain from the input $U(s)$ to the other nodes and the sixth column denotes the gain from x_1 which is equal to the output $Y(s)$ to the other nodes. The elements in the matrix are 0 or s^{-1} except the first and the sixth columns. According to the matrix, the system's output is possible to determine the behavior of the entire system in Fig. 3(a).

According to the given signal flow graph in Fig. 3(b), the transfer matrix is

$$\mathbf{A}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -a_0 & -a_1 & -a_2 & -a_3 & 0 \\ 0 & s^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s^{-1} & 0 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & 0 \end{bmatrix}, \quad (7)$$

According to (4),

$$\begin{vmatrix} \hat{U} & \hat{x}_1 & \hat{x}_2 & \hat{x}_3 & \hat{x}_4 & \hat{x}_5 & \hat{Y} \\ 1 & -1 & -a_0 & -a_1 & -a_2 & -a_3 & 0 \\ 0 & s^{-1} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & s^{-1} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^{-1} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s^{-1} & -1 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & -1 \end{vmatrix}$$

$$= \alpha_1(s)\hat{U} + \alpha_2(s)\hat{x}_1 + \alpha_3(s)\hat{x}_2 + \alpha_4(s)\hat{x}_3 + \alpha_5(s)\hat{x}_4 + \alpha_6(s)\hat{x}_5 + \alpha_7(s)\hat{Y},$$

where

$$\alpha_1(s) = \frac{s^4 + a_0s^3 + a_1s^2 + a_2s + a_3}{s^4},$$

$$\alpha_7(s) = \frac{b_0s^3 + b_1s^2 + b_2s + b_3}{s^4}.$$

The transfer function of the controllable system is

$$\frac{\alpha_7(s)}{\alpha_1(s)} = \frac{b_0s^3 + b_1s^2 + b_2s + b_3}{s^4 + a_0s^3 + a_1s^2 + a_2s + a_3}.$$

According to the second row in (7), the state of x_1 is determined by the input $U(s)$ and the differential of itself. According to the seventh row in (7), the state of output $Y(s)$ is determined by the n -order differential of x_1 . Thus, the output $Y(s)$ can be controlled by the input $U(s)$.

3. CONCLUSIONS

This paper presents a systematic, symbolic, matrix operational method for analyzing the system gain and continuing design of control systems. The procedures for deriving the transfer function described in this paper consist of only three steps: 1) derive the transfer matrix $\mathbf{A}^T$ from the signal flow graph; 2) calculate the eigenvector $\mathbf{X}$ of $\mathbf{A}^T$ associated with unity eigenvalue (for this purpose a cross-product derivation method for hyperspace is proposed); and 3) determine the transfer function based on the eigenvector $\mathbf{X}$ obtained in Step 2. Compared with conventional block diagram reduction, this method is systematic. Compared with Mason's gain formula method, this method is easier to explain and the result is easier to visualize.

ACKNOWLEDGMENTS

The authors would like to thank the National Science Council of Taiwan, the Republic of China, for financially supporting this research under Contract No. NSC103-2221-E-009 -123 –

REFERENCES

- [1] Xiong, Q., Cai, W.J., "Effective transfer function method for decentralized control system design of multi-input multi-output processes," *Journal of Process Control* 16, 773-784 (2006).
- [2] Xiong, Q., Cai, W.J., He, M.J., "Equivalent transfer function method for PI/PID controller design of MIMO processes," *Journal of Process Control* 17, 665-673 (2007).
- [3] Chang, C.T., Lin, Y.S., Georgakis, C., "A simple graphic approach for observer decomposition," *Journal of Process Control* 12, 857-873 (2002).
- [4] Mei, C., "On Teaching the Simplification of Block Diagrams," *International Journal of Engineering Education* 18, 697-703 (2002).
- [5] Mason, S. J., "Feedback Theory-Some Properties of Signal Flow Graphs," *Proceedings of the I.R.E.* 41, 1144-1156 (1953).
- [6] Ogata K., [Modern Control Engineering: International Edition], Prentice Hall, Upper Saddle River, NJ, chapter 2 (2010).
- [7] Mason, S. J., "Feedback Theory-Further Properties of Signal Flow Graphs," *Proceedings of the I.R.E.* 44, 920-926 (1956).
- [8] Agnarsson, G., Greenlaw, R., [Graph theory: Modeling, applications, and algorithms], Prentice Hall, Upper Saddle River, NJ, chapter 2 (2006).
- [9] Ghosh, A. K., [Introduction to Control Systems], Prentice Hall, India, 604 – 608 (2014).