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摘 要

在本論文中，我們探討一維與二維的細胞類神經網路的花樣與其複雜性。首先，我們根據合適局部花樣來切割參數空間。參數空間可被切割成有限個不相交的區域。在同一區域參數的細胞類神經網路，其可能花樣的集合是一樣的。在此同時，我們可完全解決細胞類神經網路的直接學習問題。關於花樣的複雜性，我們分別對一維與二維的細胞類神經網路來考慮。在一維的細胞類神經網路，我們提出一個演算法來計算花樣的空間熵的確切值。這個演算法是利用合適局部花樣所決定的轉移矩陣，其最大特徵值取自然對數後，即是空間熵。但在二維的細胞類神經網路，因為垂直方向與水平方向的轉移矩陣有相容性的問題，我們無法求出確切的花樣空間熵。我們可取而代之地求花樣空間熵的上界與下界。我們提除一個演算法，利用貼的技巧，來求在有限格子點上的花樣總數，上界因此而得到。另外，我們求出最小尺寸的幅貼花樣磚的總數，來計算下界。

Abstract

In this thesis, We investigate the mosaic patterns and spatial entropy of cellular neural networks(CNNs). First, we partition the parameter space into finitely many disjoint regions such that in each region the CNN has the same mosaic patterns. The direct problem is then completely solved. After that, the spatial entropy is investigated for one-dimensional case and two-dimensional case respectively. In a one-dimensional case, we propose an algorithm to compute the exact spatial entropy of mosaic patterns for general templates. This algorithm is based on the feasible local pattern and the corresponding transition matrix. By computing the maximum eigenvalue of the transition matrix, the exact spatial entropy is then obtained. As for the two-dimensional case, for which we cannot obtain the exact spatial entropy in general, is markedly different with the one-dimensional case because of the compatibility problem between horizontal and vertical transition matrices. Instead of the exact spatial entropy, we give the upper and lower bound. We present an algorithm to attain the mosaic patterns in finite lattice and the upper bound is then obtained. On the other hand, the lower bound is obtained by constructing the maximal set of patching blocks of minimal size(depend on the characteristic of the global patterns).

Introduction

The cellular neural network(CNN), as designed by Chua and Yang[1988a,b], is an array of an identical system of cells that are locally coupled. Each C_{ij} has input u_{ij} , state x_{ij} , and output y_{ij} . The state equation of a cell C_{ij} is

$$\dot{x}_{ij}(t) = -x_{ij}(t) + \sum_{|k|,|l| \leq d} a_{kl}y_{i+k,j+l}(t) + z, \quad i, j \in \mathbb{Z}^2, \quad (\text{StateEq})$$

where z is a threshold and d is a positive integer which represent the coupling distance. The output y_{ij} is a piecewisely linear function of x_{ij} given by the following output equation:

$$y_{ij} = f(x_{ij}) = \frac{1}{2}(|x_{ij} + 1| - |x_{ij} - 1|). \quad (\text{OutEq})$$

The coupling parameters a_{kl} 's are assumed to be space-invariant, which is arranged in a $(2d + 1) \times (2d + 1)$ matrix A called a *template*.

The stationary solutions $\mathbf{x} = (x_{ij})$ of (StateEq) are prerequisite for understanding the CNN dynamics. Indeed, when studying the long time behavior of a dynamic system, the stationary solutions are the simplest objects to be considered. In the case of CNN with finite cells and symmetric template A , Chua and Yang[1988a], Lin and Shih[1999] demonstrated that every trajectory of (StateEq) tends to an equilibrium as time proceeds. Multiple stationary solutions typically induce complicated phenomena

involving the dynamic system. The stationary solutions $x = (x_{ij})$ of (StateEq) satisfy

$$x_{ij} = z + \sum_{|k|, |l| \leq d} a_{kl} f(x_{i+k, j+l}), \quad i, j \in \mathbb{Z}. \quad (0.0.1)$$

A stationary x solution is called a *mosaic solution* if $|x_{ij}| > 1$ for all $i, j \in \mathbb{Z}$. The corresponding output $y = (f(x_{ij}))$ is called a *mosaic pattern*. The mosaic solution of (StateEq) is always stable [Juang and Lin, 1997a] (and the references given therein). Among all stationary solutions, the stable mosaic solutions, which have been studied before [Juang and Lin, 1997a], is the most fundamental and important applications (finite cells) in image-processing [Chua, 1988b; Chua, 1988]. Those investigations addressed two problems directly related to the mosaic patterns: (a) the directed problem, in which the parameter space is partitioned into finitely many regions so that in each region the CNNs have the same patterns and (b) the complexity of global patterns.

The complexity of mosaic solutions can be examined according to its entropy. For completeness, the following discussion introduces some definitions and results concerning spatial entropy. Further details can be found in [Chow et al, 1996a]. Denote by $\{-1, 1\}^{\mathbb{Z}^2}$ the set of all $y : \mathbb{Z}^2 \rightarrow \{-1, 1\}$ i.e. the set of all mosaic patterns. Let $\mathcal{U}$ be a translation-invariant subset of $\{-1, 1\}^{\mathbb{Z}^2}$ and $\mathbb{Z}_{mn} = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$. The number of distinct patterns observed among the elements of $\mathcal{U}$ when observation is restricted to subset $\mathbb{Z}_{mn}$, is denoted by $\Gamma_{mn}(\mathcal{U})$. The spatial entropy $h(\mathcal{U})$ of $\mathcal{U}$ is defined by

$$h(\mathcal{U}) = \lim_{m, n \rightarrow \infty} \frac{\ln \Gamma_{mn}(\mathcal{U})}{mn}.$$

$\mathcal{U}$ is called *spatial chaos* if the spatial entropy $h(\mathcal{U})$ is positive. Otherwise, $\mathcal{U}$ is called *pattern formation*.

In chapter 1 we consider the case of a CNN with one-dimensional template, i.e.

$$A = [a_{-d} \dots a_{-1} \ a_0 \ a_1 \ \dots \ a_d].$$

The parameter is partitioned according the feasible local patterns [Hsu et al, 1999], the directed problem is then solved. For simplicity, we illustrated this process for the special 1×5 symmetric template $A = [r \ s \ p \ s \ r]$. To construct the global patterns, we introduce the transition matrix associated with the feasible local patterns. Moreover, By computing the largest eigenvalue of the transition matrix, we obtain the exact spatial entropy of mosaic patterns [Lin and Yang, 1999].

A CNN of a two-dimensional template is discuss in chapter 2. The directed problem is also solved by the method used in a one-dimensional case. However, unlike a one-dimensional case, the global patterns as well as the spatial entropy cannot be obtained in general by a transition matrix. Up to now, no efficient and general method is available for two-dimensional templates. In [Juang et al, 2000], horizontal and vertical transition matrices are defined to compute the spatial entropy for some special case of the square cross template. [Juang and Lin, 1997] and [Shih, 1998] discuss the symmetric and asymmetric square cross templates, respectively. In those papers, the global patterns are classified by building blocks and the corresponding compatible conditions. In addition, the spatial chaos is confirmed to occur by identifying the blocks which can be adjacent to each other. In contrast to the above developments, we presents a novel algorithm to generate the patterns in the finite lattice, which gives the upper bound of the spatial entropy. Moreover, the lower bound of the spatial entropy is estimated by finding the maximal set of the patching blocks of some size(depends on the characteristic of global patterns). This investigation concentrates

mainly on the simplest two-dimensional template, L-shaped liked, i.e.

$$A = \begin{pmatrix} 0 & r & 0 \\ 0 & p & s \\ 0 & 0 & 0 \end{pmatrix},$$

although the tools developed herein can also be applied to the general case.

Chapter 1

One-dimensional CNNs

1.1 Introduction

In this chapter, we consider the CNN with one-dimensional template,

$$A = [a_{-d} \dots a_{-1} \ a_0 \ a_1 \ \dots \ a_d].$$

In the case, the state equation and output equation become

$$\dot{x}_i(t) = -x_i(t) + \sum_{|k| \leq d} a_k y_{i+k}(t) + z, \quad i \in \mathbb{Z}, \quad (1.1.1)$$

and

$$y_i = f(x_i) = \frac{1}{2}(|x_i + 1| - |x_i - 1|). \quad (1.1.2)$$

, respectively. Therefore, the stationary solutions $\mathbf{x} = (x_i)$ of Eq. (1.1.1) satisfy

$$x_i = z + \sum_{|k| \leq d} a_k f(x_{i+k}), \quad i \in \mathbb{Z}. \quad (1.1.3)$$

Definition 1.1. 1. A stationary solution $\mathbf{x} = (x_i)$ of (1.1.1) is called a mosaic solution if $|x_i| > 1$ for all $i \in \mathbb{Z}$. In this case, the corresponding output $\mathbf{y} = (f(x_i))$ is called a mosaic pattern.

2. Let $\mathbb{Z}_N = \{1, 2, \dots, N\}$ and $\mathcal{U}$ be the set of mosaic patterns. Denote by $\Gamma_N(\mathcal{U})$ the number of distinct patterns observed among the elements of $\mathcal{U}$ when observation is restricted to subset $\mathbb{Z}_N$. The spatial entropy of mosaic patterns, $h(\mathcal{U})$, is defined by

$$h(\mathcal{U}) = \lim_{N \rightarrow \infty} \frac{\ln \Gamma_N(\mathcal{U})}{N}.$$

Moreover, $\mathcal{U}$ is called *spatial chaos* if $h(\mathcal{U})$ is positive. Otherwise, $\mathcal{U}$ is called *pattern formation*.

In this chapter we completely solve the "direct problem" for 1×5 symmetric and anti-symmetric template. Moreover, we give a systematical procedure to find the global pattern and the spatial entropy. This procedure initially investigates the local patterns for a general template[Hsu, et al 1999]. Then, the global patterns can be constructed by using the transition matrix associated with the local patterns. The spatial entropy is then obtained by computing the largest eigenvalue of the transition matrix.

The rest of this chapter is organized as follows. Sec. 2 presents a novel means of partitioning the parameter space. Sec. 3 constructs the global patterns from local patterns. We illustrate this process with the symmetric template of size 5.

$$A_{ss} = \begin{bmatrix} r & s & p & s & r \end{bmatrix}$$

Finally, in Sec. 4 the spatial entropy of 1×5 symmetric and anti-symmetric templates is computed.

1.2 Partitioning the Parameter Space

Firstly, we introduce the notation of local pattern [Juang and Lin, 1997a; Hsu et al, 1999]. For a mosaic solution $\mathbf{x}$, the output at cell C_i is $+$, i.e., $x_i > 1$, if and only if

$$\sum_{|k| \leq d, k \neq 0} a_k y_{i+k} + a_0 + z - 1 > 0, \quad (1.2.1)$$

where $a = a_0$. Similarly, the output at cell C_i is $-$, i.e., $x_i < -1$, if and only if

$$\sum_{|k| \leq d, k \neq 0} a_k y_{i+k} - a_0 + z - 1 < 0. \quad (1.2.2)$$

(1.2.1) and (1.2.2) can be written in a much more compact form by introducing the following notation.

Denote by

$$\alpha = (a_{-d}, \dots, a_{-1}, a_1, \dots, a_d)$$

and

$$V = (v_{-d}, \dots, v_{-1}, v_1, \dots, v_d), \quad |v_j| = 1$$

the off-center template and off-center outputs, respectively. Denote the configuration space, i.e. the set of all possible local patterns, by

$$X^{2d} = \{V = (v_{-d}, \dots, v_{-1}, v_1, \dots, v_d) : |v_j| = 1 \text{ for } |j| \leq d \text{ and } j \neq 0\} \quad (1.2.3)$$

Then, for given template A and threshold z , the family of feasible local patterns with $+$ in the center is defined by

$$\mathcal{F}(+, A, z) = \{(v_{-d}, \dots, v_{-1}, 1, v_1, \dots, v_d) : V \in X^{2d}, \alpha \cdot V + a_0 + z - 1 > 0\}. \quad (1.2.4)$$

Similarly, the family of feasible local patterns with $-$ in the center is defined by

$$\mathcal{F}(-, A, z) = \{(v_{-d}, \dots, v_{-1}, -1, v_1, \dots, v_d) : V \in \mathbf{X}^{2d}, \alpha \cdot V - a_0 + z - 1 < 0\}. \quad (1.2.5)$$

The set of feasible local patterns is therefore defined by

$$\mathcal{F}(A, z) = \mathcal{F}(+, A, z) \cup \mathcal{F}(-, A, z). \quad (1.2.6)$$

The parameter space is

$$\mathcal{P}^{2d+2} = \{(a_{-d}, \dots, a_d, z) : a_{-d}, \dots, a_d, z \in \mathbb{R}\}. \quad (1.2.7)$$

According to Hsu et al[1999], the parameter space can be divided into finitely many regions such that in each region (1.1.1) has the same mosaic patterns. In distinction from that, we presents an effective means of partitioning the parameter space. For clarity, we demonstrate with the symmetric template of size 5, represented as follows

$$A_{ss} = \begin{bmatrix} r & s & p & s & r \end{bmatrix},$$

then apply to the general one-dimensional template. In the case of A_{ss} , the parameter is $\mathcal{P}^4 = \{(r, s, p, z) | z, p, r, s \in \mathbb{R}\}$ and the stationary solution $\mathbf{x} = (x_i)$ satisfies

$$x_i = z + py_i + 2kr + 2ms, \quad i \in \mathbb{Z} \quad (1.2.8)$$

where $2m = y_{i-1} + y_{i+1}$ and $2k = y_{i-2} + y_{i+2}$, $k, m \in I[-1, 1]$. Here $I[j, l]$ denotes the set of all integers which are not smaller than j and not greater than l .

Let $\tilde{\mathbf{X}}^2 = \{(k, m) : k, m \in I[-1, 1]\}$. Each $(k, m) \in \tilde{\mathbf{X}}^2$ is naturally associated with two sets of local patterns

$$P_{k,m} = \{(y_{-2}, y_{-1}, 1, y_1, y_2) | y_{-1} + y_1 = 2m, y_{-2} + y_2 = 2k\},$$

$$N_{k,m} = \{(y_{-2}, y_{-1}, -1, y_1, y_2) \mid y_{-1} + y_1 = 2m, y_{-2} + y_2 = 2k\}.$$

Then, the elements in $P_{k,m}$ and $N_{k,m}$ are feasible if and only if z and p satisfies (1.2.9) and (1.2.10), respectively.

$$z + (p - 1) + 2kr + 2ms > 0, \quad (1.2.9)$$

$$z - (p - 1) + 2kr + 2ms < 0. \quad (1.2.10)$$

To partition the r - s plane, we first arrange the elements of $\bar{X}^2$ according to the value of $kr + ms$. Indeed, given a (r, s) , the (r, s) -arrangement of $\bar{X}^2$ is a function from $\bar{X}^2$ into $I[1, 9]$, which assign the integer i to (k_0, m_0) provided that $k_0r + m_0s$ is the i th largest among all of $kr + ms$, $(k, m) \in \bar{X}^2$. As easily verified, the following half lines, $L_1, \dots, L_{16}$:

$$\begin{aligned} L_1 : r > 0, s = 0, & \quad L_2 : r > 0, r = 2s, & \quad L_3 : r > 0, r = s, \\ L_4 : r > 0, r = \frac{1}{2}s, & \quad L_5 : r = 0, s > 0, & \quad L_6 : r < 0, r = -\frac{1}{2}s, \\ L_7 : r < 0, r = -s, & \quad L_8 : r < 0, r = -2s, & \quad L_9 : r < 0, s = 0, \\ L_{10} : r < 0, r = 2s, & \quad L_{11} : r < 0, r = s, & \quad L_{12} : r < 0, r = \frac{1}{2}s, \\ L_{13} : r = 0, s > 0, & \quad L_{14} : r > 0, r = -\frac{1}{2}s, & \quad L_{15} : r > 0, r = -s, \\ L_{16} : r > 0, r = -2s \end{aligned}$$

divide the r - s plane into 16 open regions: $R_1, \dots, R_{16}$ with R_i the open region being bounded by L_i and L_{i+1} , for $1 \leq i \leq 15$ and R_{16} the open region being bounded by L_{16} and L_1 , so that only one (r, s) -arrangement of $\bar{X}^2$ is induced in L_i or R_i (see Fig. 1.2). Denote by $J_{R_i}(J_{L_i})$ the (r, s) -arrangement of $\bar{X}^2$ induced on $R_i(L_i)$.

Fig. 1.2 is illustrated as follows: Each element (k, m) of $\bar{X}^2$ naturally corresponds to a position in a 3×3 lattice as shown in Fig. 1.1. In each $R_i(L_i)$, there is also

a 3×3 lattice and the values of $J_{R_i}(k, m)(J_{L_i}(k, m))$ is put into the position of this lattice, which (k, m) occupies

Comment. J_{R_i} is one-to-one but J_{L_i} is not. We refer to (r, s) as degenerate if (r, s) lies in some L_i .

Now, we demonstrate the partitioning of the z - p plane for (r, s) in R_i . Let ℓ_j^+ and ℓ_j^- denote the lines on z - p plane with equation:

$$z + (p - 1) + 2kr + 2ms = 0, \quad (1.2.11)$$

$$z - (p - 1) + 2kr + 2ms = 0, \quad (1.2.12)$$

respectively provided that $J_{R_i}(k, m) = j$. Then $\{\ell_j^+\}_{j=1}^9$ and $\{\ell_j^-\}_{j=1}^9$ form two sets of parallel lines in the z - p plane with ℓ_{j+1}^+ and ℓ_j^- lying above ℓ_j^+ and ℓ_{j+1}^- , respectively. Denote by $[\mu, \nu]_{R_i}$, $0 \leq \mu, \nu \leq 9$, the open region bounded by ℓ_μ^+ , $\ell_{\mu+1}^+$, $\ell_{10-\nu}^-$, $\ell_{9-\nu}^-$ with ℓ_0^+ , ℓ_0^- , ℓ_{10}^+ , ℓ_{10}^- being the empty lines, as observed in Fig. 1.3. Hence, the z - p plane is partitioned into finitely many disjoint regions, namely, $[\mu, \nu]_{R_i}$, $0 \leq \mu, \nu \leq 9$.

As for $(r, s) \in L_i$, the process goes on similarly but there appear less parallel lines since J_{L_i} is not one-to-one. Assume that q_{L_i} is the maximum value of $J_{L_i}(\cdot)$ and denote by $[\mu, \nu]_{L_i}$, $0 \leq \mu, \nu \leq q_{L_i}$, the open region bounded by ℓ_μ^+ , $\ell_{\mu+1}^+$, $\ell_{q_{L_i}-\nu}^-$, $\ell_{q_{L_i}-\nu-1}^-$ with ℓ_0^+ , ℓ_0^- , $\ell_{q_{L_i}+1}^+$, $\ell_{q_{L_i}+1}^-$ being the empty lines. Then, we also have a partition on the z - p plane.

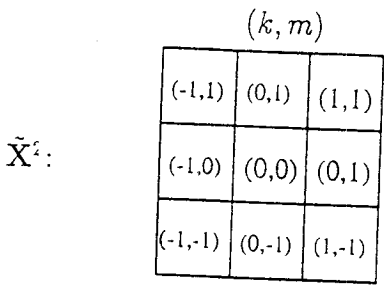


Fig. 1.1

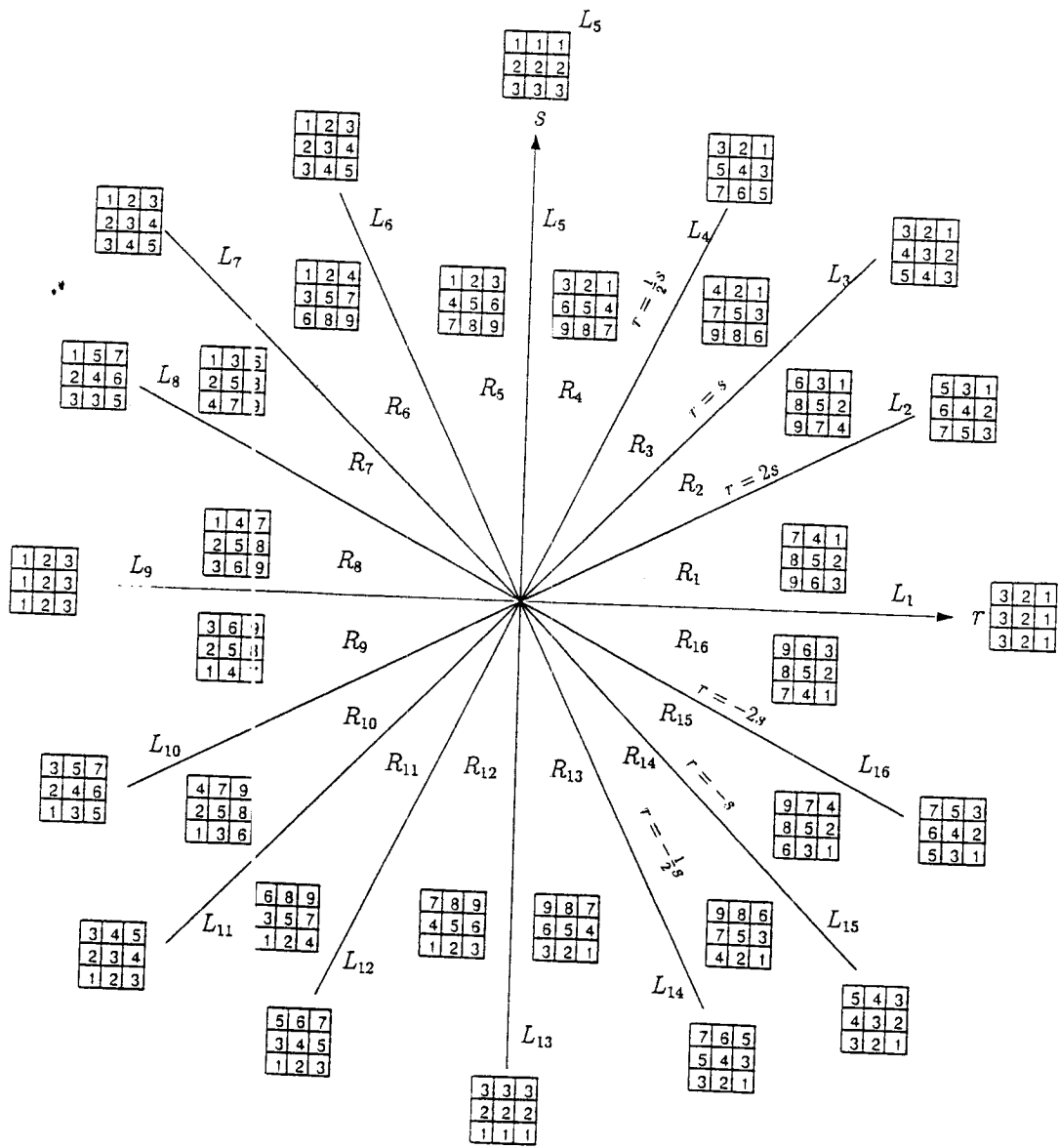


Fig. 1.2

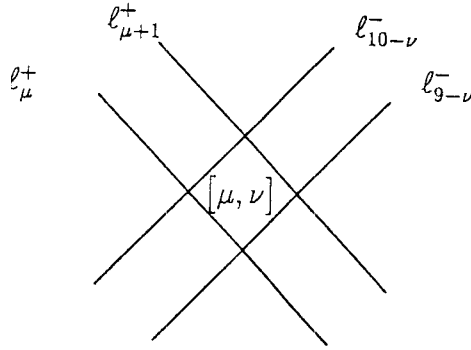


Fig. 1.3

After partitioning the parameter space, Theorem 1.1 indicates how the feasible local patterns are determined on $[\mu, \nu]_{R_i}$. (Similar results hold in $[\mu, \nu]_{L_i}$).

Theorem 1.1. For $(z, p) \in [\mu, \nu]_{R_i}$, the feasible local patterns are exactly the union of $\bigcup_{J_{R_i}(k, m) \leq \mu} P_{k, m}$ and $\bigcup_{J_{R_i}(k, m) \geq 10 - \nu} N_{k, m}$.

Proof. Assume $J_{R_i}(k_1, m_1) = \mu$, $J_{R_i}(k_2, m_2) = \mu + 1$. Let $(z, p) \in [\mu, \nu]_{R_i}$. According to the definition of $[\mu, \nu]_{R_i}$, z and p satisfy

$$z + (p - 1) + 2(k_1 r + m_1 s) > 0, \quad (1.2.13)$$

$$z + (p - 1) + 2(k_2 r + m_2 s) < 0, \quad (1.2.14)$$

which implies that

$$z + (p - 1) + 2(kr + ms) > 0 \quad \text{if } J_{R_i}(k, m) \leq J_{R_i}(k_1, m_1) = \mu, \quad (1.2.15)$$

$$z + (p - 1) + 2(kr + ms) < 0 \quad \text{if } J_{R_i}(k, m) \geq J_{R_i}(k_2, m_2) = \mu + 1. \quad (1.2.16)$$

In addition, there is no (k, m) such that $\mu < J_{R_i}(k, m) < \mu + 1$. Hence, $\bigcup_{(k, m) \leq \mu} P_{k, m}$ are exactly the feasible local patterns with $+1$ as the center. A similar argument can

show that $\bigcup_{J_{R_1}(k,m) \geq 10-\nu} N_{k,m}$ are exactly the feasible local patterns with -1 as the center. The proof is complete. $\square$

The following example demonstrates the partition of the z - p plane when (r, s) are given.

Example 1.1. Assume that $(r, s) \in R_1$. The z - p plane is partitioned by two sets of lines, $\{\ell_j^+\}_{j=1}^9$ and $\{\ell_j^-\}_{j=1}^9$, Fig. 1.4 provides further details. For each $(k, m) \in \bar{X}^2$, the associated set of feasible local patterns are listed below. Herein, we have identified 1 with $+$ and -1 with $-$ respectively.

$$P_{1,1} = \{++\oplus++\}$$

$$P_{1,0} = \{++\oplus+-, -+\oplus++\}$$

$$P_{1,-1} = \{-+\oplus+-\}$$

$$P_{0,1} = \{++\oplus-+, +- \oplus ++\}$$

$$P_{0,0} = \{++\oplus--, +- \oplus +-, -+\oplus-+, --\oplus++\}$$

$$P_{0,-1} = \{-+\oplus--, --\oplus+-\}$$

$$P_{-1,1} = \{+- \oplus -+\}$$

$$P_{-1,0} = \{+- \oplus --, --\oplus -+\}$$

$$P_{-1,-1} = \{--\oplus --\}$$

$$N_{1,1} = \{++\ominus++\}$$

$$N_{1,0} = \{+-\ominus+-, -+\ominus++\}$$

$$N_{1,-1} = \{-+\ominus+-\}$$

$$N_{0,1} = \{+-\ominus-+, +- \ominus ++\}$$

$$N_{0,0} = \{+-\ominus--, +-++-, -+\ominus-+, --\ominus++\}$$

$$N_{0,-1} = \{-+\ominus--, --\ominus+-\}$$

$$N_{-1,1} = \{+-\ominus-+\}$$

$$N_{-1,0} = \{+-\ominus--, --\ominus-+\}$$

$$N_{-1,-1} = \{--\ominus--\}$$

The feasible local patterns for a region $[\mu, \nu]_{R_1}$ can be easily determined by Theorem 1.1. For example, the set of feasible local patterns for on $[5, 5]_{R_1}$ is

$$\{P_{1,1}, P_{1,0}, P_{1,-1}, P_{0,-1}, P_{0,0}, N_{-1,-1}, N_{-1,0}, N_{-1,1}, N_{0,1}, N_{0,0}\}.$$

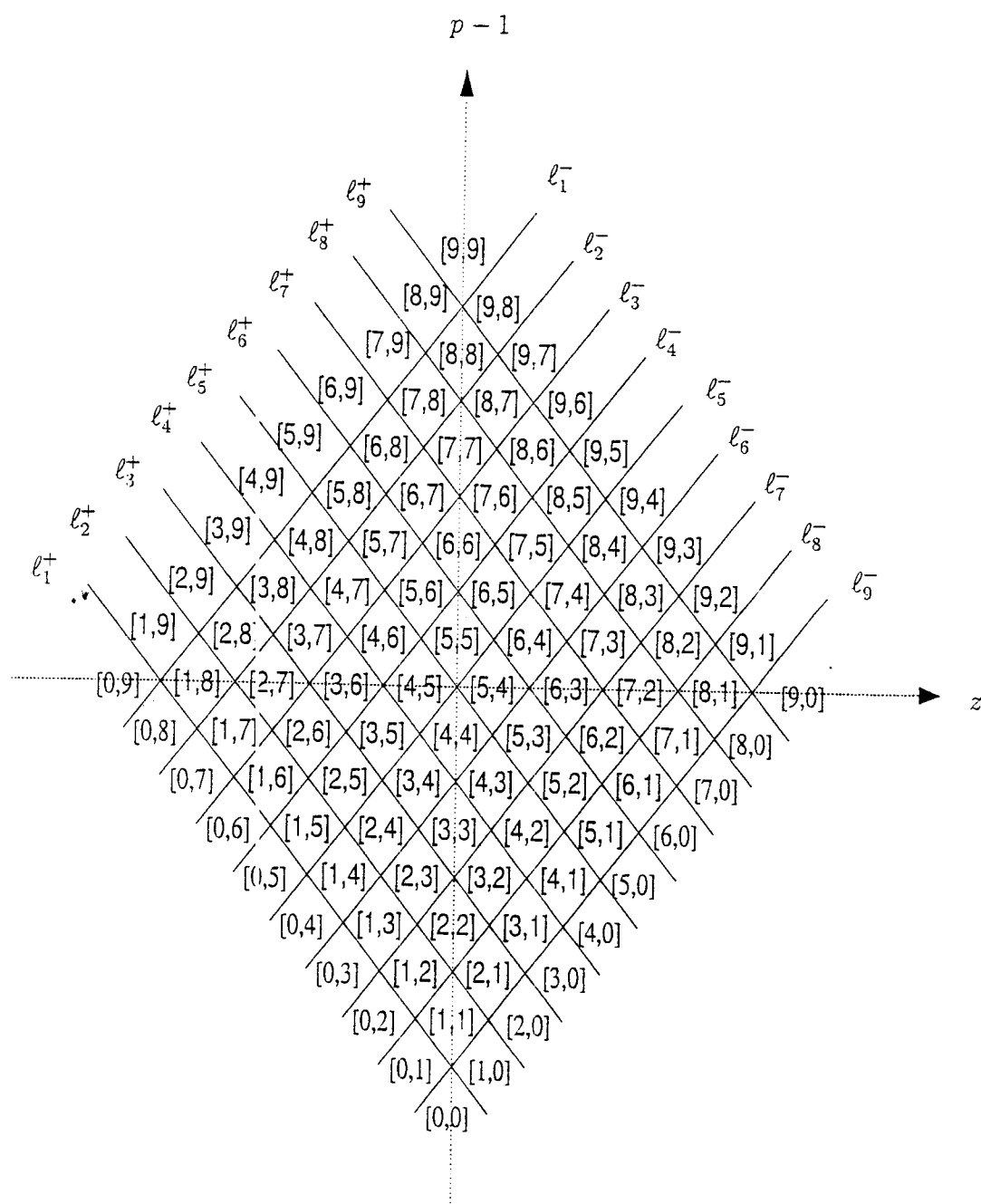


Fig. 1.4

The process to partition the parameter space for (A_{ss}, z) can be applied to partition the parameter space of one-dimensional CNN with a general template of a larger size, despite the fact that the relevant calculation is much more complicated. For completeness, we sketch the method for the general template

$$A = [a_{-d} \ \dots \ a_{-1} \ a_0 \ a_1 \ \dots \ a_d].$$

A mosaic solution $\mathbf{x} = (x_i)$ satisfies

$$0 = z - x_i + a_0 y_i + \sum_{k=1}^d (a_{-k} y_{i-k} + a_k y_{i+k}) \quad (1.2.17)$$

$$= z - x_i + a_0 y_i + \alpha \cdot V \quad (1.2.18)$$

where $\alpha = (a_{-d}, \dots, a_{-1}, a_1, \dots, a_d)$ and

$$V = (v_{-d}, \dots, v_{-1}, v_1, \dots, v_d) = (y_{-d}, \dots, y_{-1}, y_1, \dots, y_d).$$

Each V is naturally associated with two local patterns: one has central output +1 and the other has central output -1.

$$P_V = (v_{-d}, \dots, v_{-1}, +1, v_1, \dots, v_d),$$

$$N_V = (v_{-d}, \dots, v_{-1}, -1, v_1, \dots, v_d)$$

Then, P_V and N_V are feasible if and only if (1.2.19) and (1.2.20) hold, respectively.

$$\alpha \cdot V + a_0 + z - 1 > 0. \quad (1.2.19)$$

$$\alpha \cdot V - a_0 + z - 1 < 0. \quad (1.2.20)$$

The elements of X^{2d} are arranged according the value of $\alpha \cdot V$. Given α , the α -arrangement of X^{2d} is an function $J_\alpha : X^{2d} \rightarrow I[1, 2^n]$, defined by $J_\alpha(V_0) = i$

provided that $\alpha \cdot V_0$ is the i th largest among all of $\alpha \cdot V$, $V \in X^{2d}$. Unlike to the case of A_{ss} , it is complicate to partition the α -space in such the way that each subregion induce only one α -arrangement of X^{2d} . Nevertheless, we can still do the partition on the z - a_0 plane. Denote by ℓ_V^+ , ℓ_V^- the lines on the z - a_0 plane with equations

$$\alpha \cdot V + a_0 + z - 1 = 0, \quad (1.2.21)$$

$$\alpha \cdot V - a_0 + z - 1 = 0, \quad (1.2.22)$$

respectively. For convenience, identify ℓ_V^+ and ℓ_V^- with ℓ_j^+ and ℓ_j^- , respectively, provided that $J_\alpha(V) = j$. Suppose q_α is the maximum value of $J_\alpha(\cdot)$. Denote by $[\mu, \nu]_\alpha$, $0 \leq \mu, \nu \leq q_\alpha$ the interior of the region bounded by ℓ_μ^+ , $\ell_{\mu+1}^+$, $\ell_{q_\alpha-\nu}^-$, $\ell_{q_\alpha-\nu-1}^-$ with ℓ_0^+ , ℓ_0^- , $\ell_{q_\alpha+1}^+$, $\ell_{q_\alpha+1}^-$ being the empty lines. Then the z - a_0 plane has been partitioned by two set of parallel lines, $\{\ell_j^+\}_{j=1}^{q_\alpha}$, $\{\ell_j^-\}_{j=1}^{q_\alpha}$. The same argument shows that in each region $[\mu, \nu]_\alpha$, the feasible local patterns are determined according to the theorem below:

Theorem 1.2. *On $[\mu, \nu]_\alpha$, the feasible local patterns are exactly the union of $\bigcup_{J_\alpha(V) \leq \mu} P_V$ and $\bigcup_{J_\alpha(V) \geq q_\alpha + 1 - \nu} N_V$.*

1.3 Global Patterns and Spatial Entropy

In this section we present a method to evaluate the spatial entropy of mosaic patterns for a given (A, z) . This method is based on the transition matrix associated with the feasible local patterns and can be easily implemented as a computer program. The idea is very intuitive. For example, when constructing a global pattern for A_{ss} , a feasible local pattern can glue to another feasible local pattern with overlapping 4

cells. The transition matrix records the rule which allows a feasible local pattern to glue to another one. More precisely, suppose that $\mathcal{F}(A, z) = \{U^1, U^2, \dots, U^N\}$ is the set of feasible patterns for (A, z) . The *transition matrix* T associated with $\mathcal{F}(A, z)$ is an $N \times N$ matrix, defined by

$$t_{ij} = \begin{cases} 1 & \text{if } U_{k+1}^i = U_k^j, k = -d, \dots, d-1, \\ 0 & \text{otherwise.} \end{cases} \quad (1.3.1)$$

i.e. $t_{ij} = 1$ if and only if U^i can glue U^j on right with overlapping $2d$ cells. Without loss of generality, we can assume that each U_i can glue some U_j on either right or left, otherwise removing those useless local pattern can make $\mathcal{F}(A, z)$ satisfy this assumption. Hence, each row and each column of T has at least one nonzero entry. An global pattern therefore corresponds to a infinite sequence which records the process of gluing.

Definition 1.2. Given an $N \times N$ transition matrix T , Let

$$\Sigma_T = \{s = (\dots, s_{-1}, s_0, s_1, \dots) : s_i \in \{1, 2, \dots, N\} \text{ and } t_{s_i s_{i+1}} = 1, \forall i \in \mathbb{Z}\}.$$

The map $\sigma_T : \Sigma_T \rightarrow \Sigma_T$, defined by $\sigma(s) = s'$ with $s'_i = s_{i+1}$, is called the subshift of finite type for T .

A "word" $(s_0, s_1, \dots, s_{n-1})$ of length n from "letters" $s_i \in \{1, 2, \dots, N\}$ is said to be admissible if it is a restriction of an element in Σ_T . Denote by K_n the number of admissible words of length n . By introducing a suitable metric on Σ_T , the topological entropy h of σ_T is equal to

$$h = \lim_{m \rightarrow \infty} \frac{\ln K_m}{m}.$$

According to Perron-Frobenius theorem, this limit is equal to the maximum eigenvalue of the transition matrix T [Robinson, 1995]. Denote by $\mathcal{U} \subset \{-1, 1\}^{\mathbb{Z}}$ the set of mosaic

patterns for (A, z) :

$$\mathcal{U} = \{ \mathbf{w} = (\dots w_{-1}, u_0, w_1, \dots) : w_i = 1 \text{ or } -1, \mathbf{w} \text{ is a mosaic pattern for } (A, z) \}.$$

We want to show that the global patterns in $\mathcal{U}$ are one-to-one corresponding to the sequences in Σ_T . Then the number of admissible words, K_m , is equal to $\Gamma_m(\mathcal{U})$, the number of the restrictions of global patterns. Hence, the spatial entropy of global patterns is equal to the maximum eigenvalue of the corresponding transition matrix.

Theorem 1.3. *There is an one-to-one correspondence between $\mathcal{U}$ and Σ_T*

Proof. To prove this theorem, we need to find a bijective map from $\mathcal{U}$ to Σ_T . Consider a map h from $\mathcal{U}$ into Σ_T defined by $h(\mathbf{w}) = \mathbf{s}$ with $(w_{i-d}, \dots, w_{i+d}) = U^{s_i}$, $i \in \mathbb{Z}$. Then, h is one-to-one. Indeed, assume that $\mathbf{w} \neq \mathbf{w}'$ with $w_i \neq w'_i$ for some i . By doing so, we have $(h(\mathbf{w}))_i \neq (h(\mathbf{w}'))_i$, which implies $h(\mathbf{w}) \neq h(\mathbf{w}')$. On the other hand, given a $\mathbf{s} \in \Sigma_T$, take $\mathbf{w} \in \mathcal{U}$ satisfies $(w_{i-d}, \dots, w_{i+d}) = U^{s_i}$ for all $i \in \mathbb{Z}$ (Such $\mathbf{w}$ exists since $T_{s_i s_{i+1}} = 1$, for all $i \in \mathbb{Z}$). This shows that h is onto. The proof is complete. $\square$

The computation of spatial entropy of mosaic patterns can be implemented by writing a spatial entropy program, via a high-level language, to execute the flow chart shown in the following:

SPATIAL ENTROPY PROGRAM

1. Generate the configuration space X^{2d} , i.e. the set of all possible local patterns.
2. Generate the α -arrangement of X^{2d} , i.e. $J_\alpha(\cdot)$.
3. Determine the feasible local patterns $\mathcal{F}(A, z)$ according to Theorem 1.2.
4. Form the transition matrix T according to $\mathcal{F}(A, z)$.
5. Evaluate the maximum eigenvalue $\lambda_{\max}$ of T and $\ln(\lambda_{\max})$ is the spatial entropy.

Example 1.2, continue with Example 1.1, demonstrate the computation of spatial entropy.

Example 1.2. In Example 1.1, under the assumption $(r, s) \in R_1$, we have listed the feasible local patterns for (z, p) in the region $[5, 5]_{R_1}$. To compute the spatial entropy $h([5, 5]_{R_1})$, we numerate them as follows:

$$\begin{array}{ccccc}
 \underbrace{++\oplus++}_{1} & \underbrace{++\oplus+-}_{2} & \underbrace{-+\oplus++}_{3} & \underbrace{-+\oplus+-}_{4} & \underbrace{++\oplus-+}_{5} \\
 \underbrace{+-\oplus++}_{6} & \underbrace{++\oplus--}_{7} & \underbrace{+-\oplus+-}_{8} & \underbrace{-+\oplus-+}_{9} & \underbrace{--\oplus++}_{10} \\
 \underbrace{--\ominus--}_{1'} & \underbrace{++\ominus-+}_{2'} & \underbrace{+-\ominus--}_{3'} & \underbrace{+-\ominus-+}_{4'} & \underbrace{--\ominus+-}_{5'} \\
 \underbrace{-+\ominus--}_{6'} & \underbrace{--\ominus++}_{7'} & \underbrace{-+\ominus-+}_{8'} & \underbrace{+-\ominus+-}_{9'} & \underbrace{++\ominus--}_{10'}
 \end{array}$$

The transition rules for these local patterns, shown in Fig. 1.5., are represented by a number-edge graph each of whose directed edge from i to j denote that the local pattern " i " can glue to the local pattern " j " on the right with overlapping 4 cells. The

number whose corresponding local pattern can not be constructed as a global pattern does not appear in Fig. 1.5. Thereafter, the transition matrix T , obtained according the transition rule, is given (1.3.2). The maximum eigenvalue is about 1.4656.

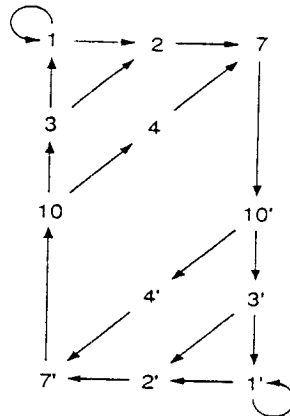


Fig. 1.5

$$T: \begin{cases} \begin{array}{cccccccccccccc} & 1 & 2 & 3 & 4 & 7 & 10 & 1' & 2' & 3' & 4' & 7' & 10' \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 10 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1' & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3' & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 4' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 7' & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array} \end{cases} \quad (1.3.2)$$

Similar to the region $[5, 5]_{R_1}$, we calculate the spatial entropy for all regions $[\mu, \nu]_{R_1}$, $0 \leq \mu, \nu \leq 9$ and show the regions which give positive spatial entropy, see Fig. 1.6

Example 1.2 demonstrates that for fixed r and s , the regions which give spatial chaos has a boundary shaped like a sawtooth. These "convex corners" of the boundary characterize the region entirely. Restated, the regions which gives the spatial chaos can be determined from the knowledge of the convex corners. For example, the region of spatial chaos in Fig. 1.6 can be determined by giving $[5, 5]_{R_1}$, $[4, 9]_{R_1}$, $[4, 9]_{R_1}$. Furthermore, the feasible local patterns of $[\mu, \nu]$ and those of $[\nu, \mu]$ have opposite sign. Hence it is sufficient to find the corners $[\mu, \nu]$ with $\mu \leq \nu$. For (r, s) in various regions, $R_1, \dots, R_{16}$, we list these convex corners, the corresponding spatial entropy and the characteristic polynomial for the symmetric template A_{ss} , the anti-symmetric templates $A_{sa} = [-r \ s \ p \ s \ r]$, $A_{as} = [r \ -s \ p \ s \ r]$, and $A_{aa} = [-r \ -s \ p \ s \ r]$, respectively. Table 1.1 provides further details. The complete list of spatial entropy of each region can be found at <http://www.math.nctu.edu.tw/~sslin>.

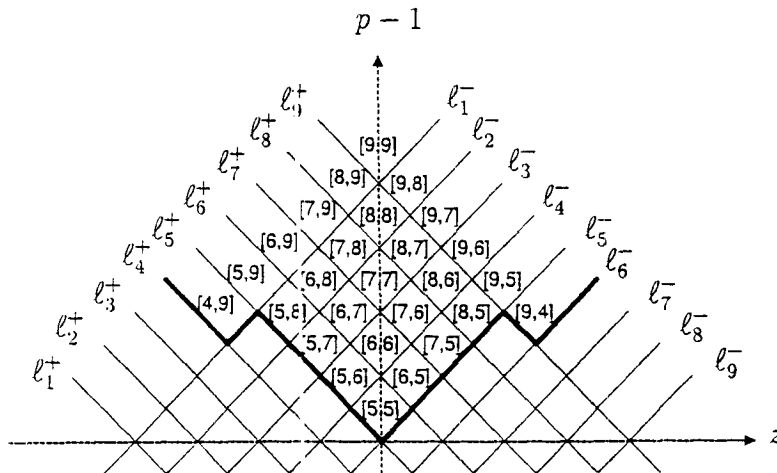


Fig. 1.6

Table 1.1: The corners of spatial chaos region for the templates A_{ss} , A_{sa} , A_{as} , and A_{aa} .

(r, s)	A_{ss}	P_i	A_{sa}	P_i	A_{sa}	P_i	A_{aa}	P_i
R_1	$[5, 5], [6, 4]$	P_3, P_4	$[5, 8], [6, 7]$	P_3	$[5, 5]$	P_1	$[5, 7]$	P_{12}
R_2	$[5, 5], [4, 7]$	P_3, P_4	$[5, 8], [6, 7]$	P_3	$[5, 5]$	P_1	$[5, 7]$	P_{12}
	$[3, 9]$	P_5						
R_3	$[5, 5], [3, 9]$	P_1, P_3	$[5, 7]$	P_3	$[5, 6]$	P_4	$[6, 7], [5, 8]$	P_1, P_5
R_4	$[5, 5], [4, 9]$	P_1, P_3	$[5, 6]$	P_3	$[5, 7]$	P_2	$[6, 7], [5, 8]$	P_1, P_5
R_5	$[4, 5]$	P_3	$[5, 6]$	P_3	$[4, 9], [5, 8]$	P_1, P_{13}	$[5, 8], [2, 7]$	P_3, P_1
R_6	$[3, 5]$	P_3	$[5, 7]$	P_3	$[5, 8], [3, 9]$	P_{13}, P_1	$[5, 8], [6, 7]$	P_3, P_7
R_7	$[2, 5]$	P_{10}	$[5, 8], [6, 7]$	P_3	$[2, 9], [4, 7]$	P_1, P_3	$[5, 8], [6, 7]$	P_3, P_{10}
R_8	$[2, 5]$	P_{10}	$[5, 8], [6, 7]$	P_3	$[3, 6], [2, 9]$	P_3, P_1	$[5, 6], [6, 7]$	P_3, P_6
R_9	$[2, 5], [1, 8]$	P_8, P_5	$[6, 7], [4, 8]$	P_6, P_{12}	$[3, 6], [2, 9]$	P_5, P_1	$[5, 7]$	P_{12}
R_{10}	$[2, 5], [1, 8]$	P_8, P_5	$[6, 7], [3, 8]$	P_9, P_{12}	$[4, 7], [2, 9]$	P_3, P_1	$[5, 7]$	P_{12}
R_{11}	$[2, 5], [1, 7]$	P_{13}, P_{12}	$[4, 7], [2, 8]$	P_5, P_{12}	$[5, 8], [3, 9]$	P_{12}, P_1	$[5, 8], [6, 7]$	P_1, P_5
R_{12}	$[2, 5], [1, 7]$	P_{13}, P_{12}	$[3, 6], [2, 8]$	P_5, P_{12}	$[5, 8], [4, 9]$	P_{12}, P_1	$[5, 8], [6, 7]$	P_1, P_5
R_{13}	$[2, 5]$	P_{11}	$[3, 6], [2, 8]$	P_5, P_{12}	$[5, 7]$	P_2	$[7, 7], [5, 8]$	P_1, P_3
R_{14}	$[2, 5]$	P_{11}	$[4, 7], [2, 8]$	P_5, P_{12}	$[5, 6]$	P_4	$[6, 7], [5, 8]$	P_7, P_3
R_{15}	$[3, 5]$	P_{11}	$[6, 7], [3, 8]$	P_9, P_{12}	$[5, 5]$	P_1	$[6, 7], [5, 8]$	P_{10}, P_3
R_{16}	$[4, 5]$	P_{11}	$[6, 7], [4, 8]$	P_9, P_{12}	$[5, 5]$	P_1	$[6, 7], [5, 8]$	P_6, P_3

characteristic polynomial	maximaum eigenvalue	characteristic polynomial	maximaum eigenvalue
$P_1(x) = x^3 - x^2 - 1$	1.4656	$P_2(x) = x^6 - x^5 - x^4 + x^3 - 1$	1.3931
$P_3(x) = x^4 - x^3 - 1$	1.3803	$P_4(x) = x^7 - x^6 - x^5 + x^4 - 1$	1.3549
$P_5(x) = x^3 - x + 1$	1.3247	$P_6(x) = x^4 - x^3 + x^2 - x - 1$	1.2906
$P_7(x) = x^6 - x^5 - 1$	1.2852	$P_8(x) = x^8 - x^5 - x^4 - x^3 + 1$	1.2806
$P_9(x) = x^4 - x^2 - 1$	1.2720	$P_{10}(x) = x^5 - x^4 + x^3 - x^2 - 1$	1.2499
$P_{11}(x) = x^5 - x^3 - 1$	1.2365	$P_{12}(x) = x^4 - x - 1$	1.2207
$P_{13}(x) = x^5 - x^2 - 1$	1.1939		

Table 1.2:

Chapter 2

Two-dimensional CNNs

2.1 Introduction

In this chapter, we mainly concentrate on the CNN with the simplest two-dimensional template .

$$A = \begin{pmatrix} 0 & r & 0 \\ 0 & p & s \\ 0 & 0 & 0 \end{pmatrix},$$

although the tools developed herein can also be applied to the general case. The direct problem is solved like in sec. 1.1, i.e. the parameter space is partitioned according the feasible local patterns. Based on the feasible local patterns, we presents a novel algorithm to generate the patterns in the finite lattice, which gives the upper bound of the spatial entropy. Moreover, the lower bound of the spatial entropy is estimated by obtaining the maximal set of the patching blocks of some size(depends on the characteristic of global patterns).

2.2 Partitioning the Parameter Space

First, we introduce the notation of local pattern [Juang and Lin, 1997a; Hsu et al, 1999] for the template

$$A = \begin{pmatrix} 0 & r & 0 \\ 0 & p & s \\ 0 & 0 & 0 \end{pmatrix}.$$

The state equation (StateEq) now becomes

$$\dot{x}_{ij}(t) = z + ry_{i-1,j} + py_{ij} + sy_{i,j+1}. \quad (2.2.1)$$

For a mosaic solution $\mathbf{x} = (x_{ij})$, the output at cell C_{ij} is +1, i.e. $x_{ij} > 1$ if and only if

$$ry_{i-1,j} + sy_{i,j+1} + p + z - 1 > 0, \quad (2.2.2)$$

and, similarly, the output at cell C_{ij} is -1, i.e. $x_{ij} < -1$ if and only if

$$ry_{i-1,j} + sy_{i,j+1} - p + z + 1 < 0. \quad (2.2.3)$$

Denoted by $X^2 = \{-1, 1\}^2$ the set of all possible $(y_{i-1,j}, y_{i,j+1})$. Each $(k, m) \in X^2$ is naturally associated with two local patterns $P_{k,m}$ and $N_{k,m}$, which has y_{ij} '+1' and y_{ij} '-1', respectively. Visually, the '+1' and '-1' are represented by black and white squares, respectively.

$$\begin{aligned} P_{1,1} &= \begin{array}{|c|c|} \hline \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare \\ \hline \end{array}, & N_{1,1} &= \begin{array}{|c|c|} \hline \blacksquare & \square \\ \hline \square & \blacksquare \\ \hline \end{array}, \\ P_{1,-1} &= \begin{array}{|c|c|} \hline \blacksquare & \square \\ \hline \blacksquare & \square \\ \hline \end{array}, & N_{1,-1} &= \begin{array}{|c|c|} \hline \blacksquare & \square \\ \hline \square & \square \\ \hline \end{array}, \\ P_{-1,1} &= \begin{array}{|c|c|} \hline \square & \blacksquare \\ \hline \blacksquare & \blacksquare \\ \hline \end{array}, & N_{-1,1} &= \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \blacksquare \\ \hline \end{array}, \\ P_{-1,-1} &= \begin{array}{|c|c|} \hline \square & \square \\ \hline \blacksquare & \square \\ \hline \end{array}, & N_{-1,-1} &= \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}. \end{aligned}$$

Obviously, for a given template A and threshold z , $P_{k,m}$ and $N_{k,m}$ are feasible if and only if (2.2.4) and (2.2.5) hold, respectively.

$$z + (p - 1) + 2kr + 2ms > 0, \quad (2.2.4)$$

$$z - (p - 1) + 2kr + 2ms < 0. \quad (2.2.5)$$

The set of the feasible local patterns, denoted by $\mathcal{F}(A, z)$, is defined to be the collection of all feasible $P_{k,m}$'s and $N_{k,m}$'s.

The parameter space, $\mathcal{P}^4 = \{(r, s, p, z) | z, p, r, s \in \mathbb{R}\}$, can be partitioned according to the feasible local patterns. To partition the r - s plane, we first arrange the elements of X^2 according to the value of $kr + ms$. Denoting by $I[i, j]$ the set of the integers not smaller than j and not larger than j . Given a (r, s) , the (r, s) -arrangement of X^2 is a function from X^2 into $I[1, 4]$, which assigns the integer i to (k_0, m_0) provided that $k_0 r + m_0 s$ is the i th largest among all of $kr + ms$, $(k, m) \in X^2$. It can be verified that the following half lines, $L_1, \dots, L_8$,

$$\begin{aligned} L_1 : r > 0, s = 0, & \quad L_2 : r > 0, r = s, & \quad L_3 : r = 0, s > 0, \\ L_4 : r > 0, r = -s & \quad L_5 : r < 0, s = 0, & \quad L_6 : r < 0, r = s, \\ L_7 : r = 0, s < 0, & \quad L_8 : r > 0, r = -s, \end{aligned}$$

divide the r - s plane into 8 open regions, $R_1, \dots, R_8$. Here R_i is the open region bounded by L_i and L_{i+1} , for $1 \leq i \leq 7$ and R_8 is the open region bounded by L_8 and L_1 such that only one (r, s) -arrangement of $\tilde{X}^2$ is induced in each R_i (see Fig. 2.2). Denote by J_i , the (r, s) -arrangement of X^2 induced in R_i .

(k, m)

$$X^2: \begin{array}{|c|c|} \hline (-1,1) & (1,1) \\ \hline (-1,-1) & (1,-1) \\ \hline \end{array}$$

Fig. 2.1

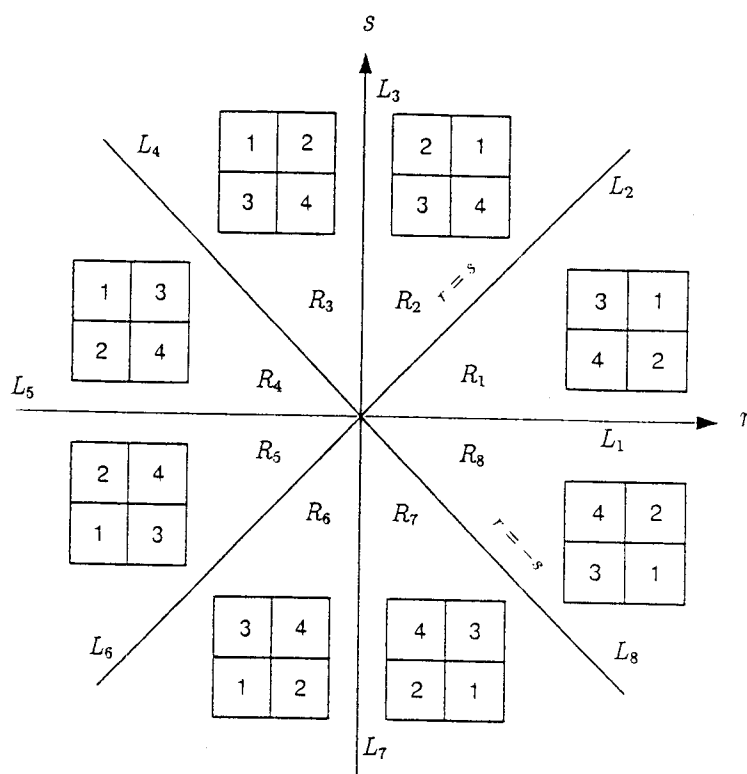


Fig. 2.2

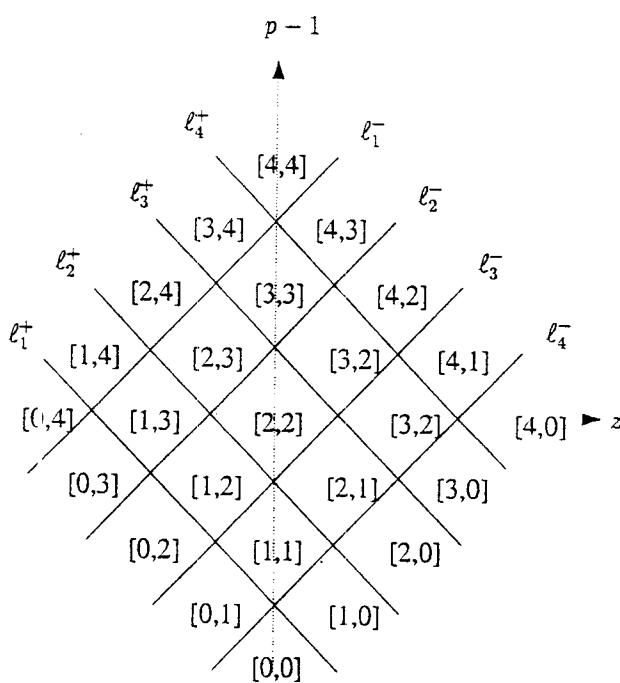


Fig. 2.3

Fig. 2.2 is illustrated as follows: Each element (k, m) of X^2 corresponds to a position in a 2×2 lattice as shown in Fig. 2.1. On each R_i , there is also a 2×2 lattice and the value of $J_i(k, m)$ are put into the position of this lattice, which corresponds to (k, m) .

Now, we partition z - p plane when (r, s) in R_i . if $J_i(k, m) = j$, let ℓ_j^+ and ℓ_j^- be the lines on z - p plane with equations:

$$z + (p - 1) + 2kr + 2ms = 0, \quad (2.2.6)$$

and

$$z - (p - 1) + 2kr + 2ms = 0, \quad (2.2.7)$$

respectively. Therefore $\{\ell_j^+\}_{j=1}^4$ and $\{\ell_j^-\}_{j=1}^4$ form two sets of parallel lines in the z - p plane with ℓ_{j+1}^+ and ℓ_j^- lying above ℓ_j^+ and ℓ_{j+1}^- , respectively. let $[\mu, \nu]_i$, $0 \leq \mu, \nu \leq 4$, denote the open region bounded by ℓ_μ^+ , $\ell_{\mu+1}^+$, $\ell_{5-\nu}^-$, $\ell_{4-\nu}^-$ with ℓ_0^+ , ℓ_0^- , ℓ_5^+ , ℓ_5^- being the empty lines, as observed in Fig. 2.3. Hence, the z - p plane is partitioned into finitely many disjoint regions, i.e., $[\mu, \nu]_i$, $0 \leq \mu, \nu \leq 4$. After partitioning the parameter space, Theorem 2.1 indicates how the feasible local patterns are determined for each region $[\mu, \nu]_i$.

Theorem 2.1. For $(z, p) \in [\mu, \nu]_i$, the feasible local patterns are exactly the union of

$$\bigcup_{J_i(k, m) \leq \mu} P_{k, m} \text{ and } \bigcup_{J_i(k, m) \geq 5 - \nu} N_{k, m}.$$

Proof. Assume $J_i(k_1, m_1) = \mu$, $J_i(k_2, m_2) = \mu + 1$ and $(z, p) \in [\mu, \nu]_i$. According to the definition of $[\mu, \nu]_i$, z and p satisfy

$$z + (p - 1) + 2(k_1 r + m_1 s) > 0, \quad (2.2.8)$$

and

$$z + (p - 1) + 2(k_2r + m_2s) < 0, \quad (2.2.9)$$

which implies that

$$z + (p - 1) + 2(kr + ms) > 0 \quad \text{if } J_i(k, m) \leq J_i(k_1, m_1) = \mu, \quad (2.2.10)$$

and

$$z + (p - 1) + 2(kr + ms) < 0 \quad \text{if } J_i(k, m) \geq J_i(k_2, m_2) = \mu + 1. \quad (2.2.11)$$

In addition, there is no (k, m) such that $\mu < J_i(k, m) < \mu + 1$. Hence, $\bigcup_{(k, m) \leq \mu} P_{k, m}$ are exactly the feasible local patterns whose left-lower cell is $+1$. A similar argument shows that, $\bigcup_{J_i(k, m) \geq \mu + 1} N_{k, m}$ are exactly the feasible local patterns whose left-lower cell is -1 . The proof is complete. $\square$

2.3 An Algorithm to Generate the Patterns in the Finite Lattice

Herein, the patterns in the finite lattice can be constructed based on the feasible local patterns. Indeed, assume that we have the set of feasible local patterns $\mathcal{F}(A, z) = \{U_1, \dots, U_k\}$ where U_i has the shape like $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$. We dilate U_i to the minimal rectangular block containing U_i by adding a black or white block in the upper right corner. By doing so, we obtain the set of patterns of size 2×2 corresponding $\mathcal{F}(A, z)$, say $\mathcal{G} = \{V_1, \dots, V_{2k}\}$. Starting from $\mathcal{G}$, we use the technique of by gluing right or below to obtain the patterns of larger size. The patterns in finite lattice can be obtained by executing the following algorithm.

Algorithm:

Initially set $\mathcal{G} = \{V_1, \dots, V_{2k}\}$. Repeat executing (A) and (B) in suitable orders. The patterns in the lattice of any finite size are then attained.

(A) One step horizontal extension.

- (i) Assume that each pattern in $\mathcal{G}$ is of size $m \times n$. From $\mathcal{G}$, take all pairs (V_i, V_j) such that V_i can glue to V_j in the right with overlapping $m - 1$ columns, i.e. the last $m - 1$ columns of V_i must equal to the first $m - 1$ columns of V_j , respectively. Such a pair is referred to herein as a *horizontal compatible pair of patterns*.
- (ii) Refresh $\mathcal{G}$ as the set of the patterns obtained by gluing the horizontal compatible pairs of patterns in (i) together with overlapping $m - 1$ columns. Then, $\mathcal{G}$ is now the set of patterns of size $m \times (n + 1)$.

(B) One step vertical extension.

- (i) Assume that each pattern in $\mathcal{G}$ is of size $m \times n$. From $\mathcal{G}$, take all pairs (V_i, V_j) such that V_i can glue to V_j below with overlapping $m - 1$ rows, i.e. the last $m - 1$ rows of V_i are equal to the first $m - 1$ rows of V_j , respectively. Such a pair are referred to herein as a *vertical compatible pair of patterns*.
- (ii) Refresh $\mathcal{G}$ as the set of the patterns obtained by gluing the vertical compatible pairs of patterns in (i) together with overlapping $m - 1$ rows. Then, $\mathcal{G}$ is now the set of patterns of size $(m + 1) \times n$.

For fixed A and z , let $\bar{h}_{m,n} = \frac{\ln \Gamma(m,n)}{mn}$, which can be obtained by executing (A) m times and (B) n times. Prop. 2.2 shows the properties of $\bar{h}_{m,n}$.

Proposition 2.2. $\{\bar{h}_{2^k m, 2^k n}\}_{k=1}^{\infty}$ is a decreasing sequence and converge to the spatial entropy of mosaic patterns.

Proof. First, we claim that $\bar{h}_{2m, 2n} \leq \bar{h}_{m,n}$. In a $2m \times 2n$ lattice, there are 2×2 disjoint blocks of size $m \times n$. When constructing the patterns in the lattice, each block has at most $\Gamma(m, n)$ choices i.e. $\Gamma(2m, 2n) \leq (\Gamma(m, n))^{2 \times 2}$. Hence $\bar{h}_{2m, 2n} \leq \bar{h}_{m,n}$ follows immediately. The standard induction argument shows that $\{\bar{h}_{2^k m, 2^k n}\}_k$ is decreasing and its limit follows from the definition of the spatial entropy of the mosaic patterns. The proof is complete. $\square$

Prop. 2.2 suggests the upper bound of the spatial entropy $\bar{h}_{m,n}$. However, the computation of $\Gamma(m, n)$ consumes lots of times and memories as the spatial chaos happens. Therefore we need a procedure to confirm the spatial chaos for some regions.

2.4 Global Patterns and Spatial Chaos

In this section, we construct the global patterns for each region $[\mu, \nu]_i$ of the parameter space. According to the feasible local patterns for each region, we induce and describe the class of the global patterns. However, once the spatial chaos occurs, the number of restrictions of the global patterns in the finite lattice grows too fast to compute even though the class of the global patterns can be characterized. Instead of computing the exact spatial entropy, we give the maximal set of the *patching blocks* of some size (depends on the characteristic of the global patterns). Restated, the set contains the largest number of the blocks of some size, which can be adjacent to each other. Though this process, we give the lower bound of the spatial entropy.

First, we introduce some special subsets of $\mathbb{Z}^2$, which are necessary to describe the global patterns.

- Definition 2.1.**
1. A subset U of $\mathbb{Z}^2$ is called semi-connected if $(i, j) \in U$ implies one of its nearest cells also belong to U .
 2. A semi-connected subset of $\mathbb{Z}^2$ is called connected if the followings hold:
 - (a) if (i, j) and $(i + 1, j + 1)$ belong to U , then at least one of $(i + 1, j)$ and $(i, j + 1)$ belong to U .
 - (b) if (i, j) and $(i - 1, j + 1)$ belong to U , then at least one of $(i - 1, j)$ and $(i, j + 1)$ belong to U .
 3. The subset U defined by $U = \{(i, j_0) : i \in S \subset \mathbb{Z}\}$ is called a horizontal line, half line, segment if $S = \mathbb{Z}$, $S = I[i_1, \infty]$ or $I[-\infty, i_2]$ and $S = I[i_1, i_2]$ for some $i_1, i_2 \in \mathbb{Z}$, respectively.

4. The subset U defined by $U = \{(i_0, j) : j \in S \subset \mathbb{Z}\}$ is called a vertical line, half line, segment if $S = \mathbb{Z}$, $S = I[i_1, \infty]$ or $I[-\infty, i_2]$ and $S = I[i_1, i_2]$ for some $i_1, i_2 \in \mathbb{Z}$, respectively.
5. A semi-connected subset U of $\mathbb{Z}^2$ is called a path if U does not contain a 2×2 lattice i.e. with width one. if, in addition, (i, j) belongs to U implies $(i+1, j+1)$ does not belong to U , then U is said to be a non-increasing path. Similarly, U is said to be a non-decreasing path if (i, j) belongs to U implies $(i+1, j-1)$ does not belong to U .

Since the patterns for $[\mu, \nu]_i$ and for $[\nu, \mu]_i$ have opposite color, it is sufficient to discuss only the cases of $[\nu, \mu]_i$ with $\mu \geq \nu$ and $1 \leq i \leq 4$. Herein, we first state the result of this section.

Theorem 2.3. (2.2.1) *has spatial chaos if and only if r, s, p and z belong to the following regions of $\mathcal{P}^4$:*

1. $[4, 4]_i$, $1 \leq i \leq 8$.
2. $[4, 3]_i$, $1 \leq i \leq 8$.
3. $[4, 2]_i$, $i = 4, 5, 6, 7$.
4. $[4, 1]_i$, $i = 5, 6$.
5. $[3, 3]_i$, $1 \leq i \leq 8$.
6. $[3, 2]_i$, $i = 4, 5, 6, 7$.

Furthermore, we observe that $[\mu, \nu]_i$ with $\min(\mu, \nu) \geq 3$ always has spatial chaos for each i .

The proof of theorem 2.3 is proceeded as follows. The class of global patterns for each case of theorem 2.3 is first described. The lower bound of the spatial entropy is then obtained by finding the the maximal set of patching blocks of minimal size. Next, for the remaining regions of the parameter space, only finite types of global patterns appear. Therefore, it is easily seen that they are pattern formation.

case 1: $[4, 4]_i, 1 \leq i \leq 8$.

All possible local patterns are feasible and, therefore, each position can have two choices: $\blacksquare$ or $\square$. Hence, the spatial entropy is $\ln 2$.

case 2a: $[4, 3]_{1,2}$.

Only $\begin{smallmatrix} \blacksquare & \blacksquare \\ \square & \blacksquare \end{smallmatrix}$ is infeasible, The global patterns are obtained by allowing some connected path of $\mathbb{Z}^2$ to be white and the reminders are black.

Patching blocks of size 2×2 :

$$\begin{smallmatrix} \square & \blacksquare \\ \square & \square \end{smallmatrix}, \begin{smallmatrix} \blacksquare & \blacksquare \\ \square & \square \end{smallmatrix}, \begin{smallmatrix} \square & \blacksquare \\ \square & \blacksquare \end{smallmatrix}.$$

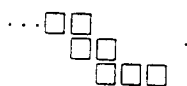
Hence, on the $2n \times 2n$ lattice, we have n^2 disjoint blocks of size 2×2 and each block have 3 choice. Therefore,

$$h[4, 3]_{1,2} \geq \lim_{n \rightarrow \infty} \frac{\ln 3^{n \times n}}{2n \times 2n} = \frac{\ln 3}{4}.$$

case 2b: $[4, 3]_{3,4}$.

Only $\begin{smallmatrix} \square & \blacksquare \\ \square & \blacksquare \end{smallmatrix}$ is not feasible. The global patterns is obtained by allowing some semi-connected, non-increasing paths U be white, whose each downward step is at most

one, i.e. if (i, j) and $(i - 1, j)$ belong to U then $(i, j + 1)$ must belong to U . For example,



And the remainders are black.

Patching blocks of size 2×2 :



Hence, a similar argument shows that

$$h[4, 3]_{3,4} \geq \frac{\ln 4}{4}.$$

case 2c: $[4, 3]_{5,6}$.

In this case, only $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ is not feasible. The global pattern is obtained by letting some semi-connected paths of $\mathbb{Z}^2$, containing no $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$, to be white and the remainders to be black.

Patching blocks of size 2×2 :



Spatial entropy:

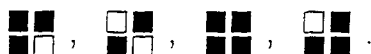
$$h[4, 3]_{5,6} \geq \frac{\ln 5}{4}.$$

case 2d: $[4, 3]_{7,8}$.

Only $\begin{smallmatrix} \blacksquare & \square \\ \square & \square \end{smallmatrix}$ is not feasible. It is the reflection of the $\begin{smallmatrix} \square & \square \\ \square & \blacksquare \end{smallmatrix}$, the only infeasible local pattern for $[4, 3]_{1,2}$, about the axis $\{(i, i) : i \in \mathbb{Z}\}$. Hence, the global patterns are

obtained by reflecting those of the case 2a with respect to the axis $\{(i, i) : i \in \mathbb{Z}\}$.

Patching blocks of size 2×2 :



Spatial entropy:

$$h[4, 3]_{7,8} \geq \frac{\ln 4}{4}.$$

case 3a: $[4, 2]_{4,5}$.

$\begin{smallmatrix} \square & \square \\ \square & \blacksquare \end{smallmatrix}$ and $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ are infeasible, i.e. more than one consecutively vertical white cells are forbidden. Hence, the global patterns are obtained by letting some horizontal lines, half lines or segments to be white and the remainders to be black.

Patching blocks of size 2×2 :



Spatial entropy:

$$h[4, 2]_{4,5} \geq \frac{\ln 4}{4}.$$

case 3b: $[4, 2]_{6,7}$.

$\begin{smallmatrix} \blacksquare & \square \\ \square & \square \end{smallmatrix}$ and $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ are not feasible, i.e. more than one consecutively horizontal white cells are forbidden. Hence, The global patterns are obtained by allowing some vertical lines, half lines or segments to be white and the remainders are black.

Patching blocks of size 2×2 :



Spatial entropy:

$$h[4, 2]_{6,7} \geq \frac{\ln 4}{4}.$$

case 4: $[4, 1]_{5,6}$.

$\left\{ \begin{smallmatrix} \blacksquare & \square \\ \square & \square \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ \square & \blacksquare \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \right\}$ are not feasible. This is the union of the infeasible local patterns of the case 2d and case 3a. Hence the global patterns are the intersection of those of the case case 2d and case 3a. Indeed, the global patterns are obtained by allowing some semi-connected non-increasing paths whose each downward and rightward step is at most one.

Patching blocks of size 2×2 :

$$\begin{smallmatrix} \square & \blacksquare \\ \blacksquare & \square \end{smallmatrix}, \begin{smallmatrix} \square & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}, \begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \square \end{smallmatrix}, \begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}.$$

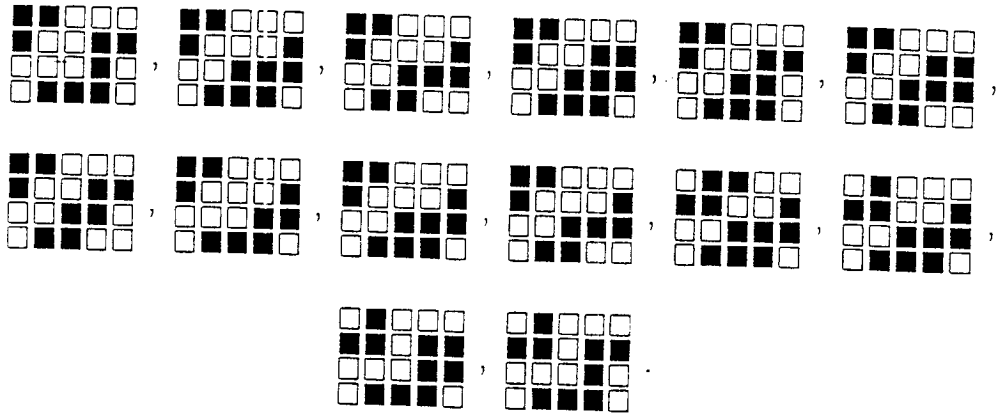
In each case $[3, 3]_i$, $1 \leq i \leq 8$, the set of feasible local patterns has the property: if a local pattern is feasible then its color-reversed pattern is also feasible. Therefore we say that global pattern for $[\mu, \nu]_i$ has the *symmetric property* in $[\mu, \nu]_i$ if its color-reversed pattern is also for $[\mu, \nu]_i$. Obviously, every global pattern of $[3, 3]_i$ inherits the symmetric property from its feasible local patterns.

case 5a: $[3, 3]_{1,2}$.

$\begin{smallmatrix} \blacksquare & \blacksquare \\ \square & \blacksquare \end{smallmatrix}$ and $\begin{smallmatrix} \square & \square \\ \blacksquare & \square \end{smallmatrix}$ are not feasible. Due to the one more infeasible local pattern $\begin{smallmatrix} \square & \square \\ \blacksquare & \square \end{smallmatrix}$ than in the case 2a. The global patterns are obtained by taking those from case 2a, which have the symmetric property in $[4, 3]_{1,2}$.

Patching block of size 4×5 :

$$\begin{smallmatrix} \blacksquare & \blacksquare & \blacksquare & \square & \square \\ \blacksquare & \blacksquare & \square & \square & \square \\ \square & \square & \square & \blacksquare & \blacksquare \\ \square & \blacksquare & \blacksquare & \blacksquare & \square \end{smallmatrix}, \begin{smallmatrix} \blacksquare & \blacksquare & \square & \square & \square \\ \blacksquare & \blacksquare & \square & \square & \blacksquare \\ \square & \square & \square & \blacksquare & \blacksquare \\ \square & \blacksquare & \blacksquare & \blacksquare & \square \end{smallmatrix}, \begin{smallmatrix} \blacksquare & \blacksquare & \square & \square & \square \\ \blacksquare & \blacksquare & \square & \square & \blacksquare \\ \square & \square & \square & \blacksquare & \blacksquare \\ \square & \blacksquare & \blacksquare & \blacksquare & \square \end{smallmatrix}, \begin{smallmatrix} \blacksquare & \blacksquare & \square & \square & \square \\ \blacksquare & \blacksquare & \square & \square & \blacksquare \\ \square & \square & \square & \blacksquare & \blacksquare \\ \square & \blacksquare & \blacksquare & \blacksquare & \square \end{smallmatrix}, \begin{smallmatrix} \blacksquare & \blacksquare & \square & \square & \square \\ \blacksquare & \blacksquare & \square & \square & \blacksquare \\ \square & \square & \square & \blacksquare & \blacksquare \\ \square & \blacksquare & \blacksquare & \blacksquare & \square \end{smallmatrix}, \begin{smallmatrix} \blacksquare & \blacksquare & \square & \square & \square \\ \blacksquare & \blacksquare & \square & \square & \blacksquare \\ \square & \square & \square & \blacksquare & \blacksquare \\ \square & \blacksquare & \blacksquare & \blacksquare & \square \end{smallmatrix},$$



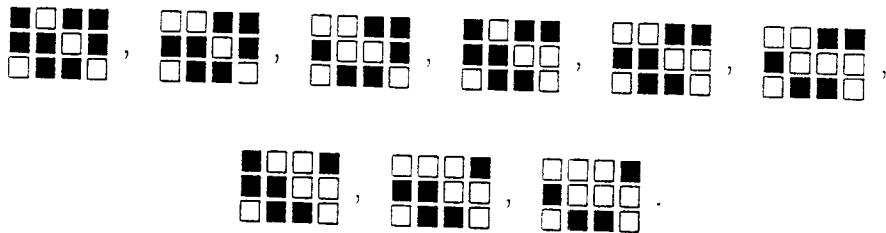
Spatial entropy:

$$h[3, 3]_{1,2} \geq \frac{\ln 20}{20}.$$

case 5b: $[3, 3]_{3,4}$.

$\begin{smallmatrix} \square & \square \\ \square & \blacksquare \end{smallmatrix}$ and $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \square \end{smallmatrix}$ are not feasible. Due to the one more infeasible local pattern $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \square \end{smallmatrix}$ than in the case 2b. The global patterns are obtained by taking those from case 2b, which have the symmetric property in $[4, 3]_{3,4}$.

Patching block: Patching blocks of size 3×4 :



Spatial entropy:

$$h[3, 3]_{3,4} \geq \frac{\ln 9}{20}.$$

case 5c: $[3, 3]_{5,6}$.

$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ and $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ are not feasible. Due to the one more infeasible local pattern $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ than in the case 2c. The global patterns are obtained by taking those from case 2c,

which have the symmetric property in $[4, 3]_{5,6}$.

Patching block of size 3×3 :



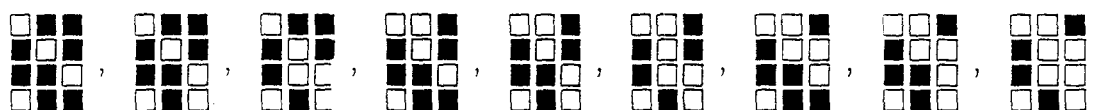
Spatial entropy:

$$h[3, 3]_{5,6} \geq \frac{\ln 9}{9}.$$

case 5d: $[3, 3]_{7,8}$.

and are not feasible. Due to the one more infeasible local pattern than in the case 2d. The global patterns are obtained by taking of case 2d those which have the symmetric property in $[4, 3]_{7,8}$.

Patching blocks of size 4×3 :



Spatial entropy:

$$h[3, 3]_{7,8} \geq \frac{\ln 9}{20}.$$

case 6a: $[3, 2]_4$.

$\left\{ \begin{smallmatrix} \blacksquare & \square \\ \square & \blacksquare \end{smallmatrix}, \begin{smallmatrix} \blacksquare & \blacksquare \\ \square & \square \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \right\}$ are not feasible. This case has one more infeasible local pattern than in the case 5b. Therefore, the global patterns are obtained by taking those of the case 5b such that each semi-connected white subset is a white path containing no .

Patching block of size 3×4 :



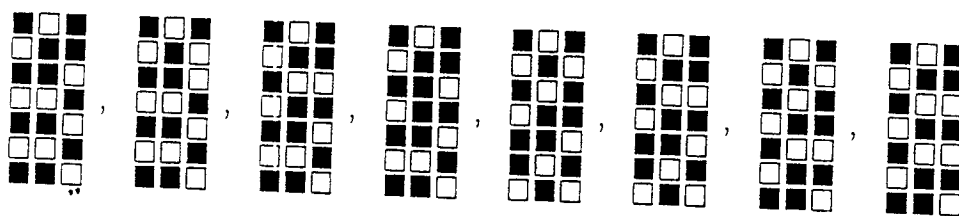
Spatial entropy:

$$h[4, 3]_4 \geq \frac{\ln 3}{15}.$$

case 6b: $[3, 2]_5$.

$\left\{ \begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \begin{smallmatrix} \square & \blacksquare \\ \square & \blacksquare \end{smallmatrix} \right\}$ are not feasible. This case has one more infeasible local pattern $\begin{smallmatrix} \square & \square \\ \square & \blacksquare \end{smallmatrix}$ than in the case 5c, so the global patterns are obtained by taking those from the case 5c, which contain no $\begin{smallmatrix} \square & \square \\ \square & \blacksquare \end{smallmatrix}$.

Patching blocks of size 7×3 :



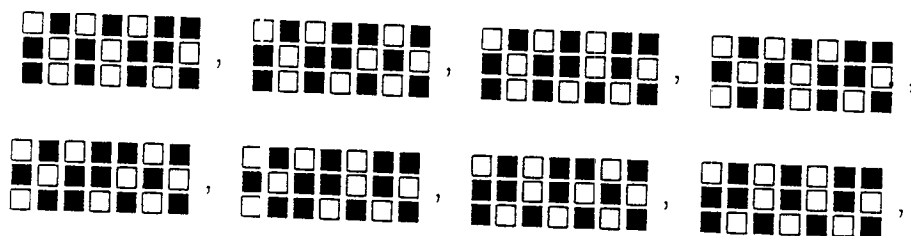
Spatial entropy:

$$h[3, 2]_5 \geq \frac{\ln 8}{21}.$$

case 6c: $[3, 2]_6$.

$\left\{ \begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \begin{smallmatrix} \blacksquare & \square \\ \square & \square \end{smallmatrix} \right\}$ are not feasible. The global patterns are obtained by reflecting those patterns for $[3, 2]_5$ about the axis $\{(i, i) : i \in \mathbb{Z}\}$.

Patching block of size 3×7 :



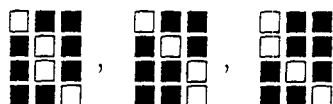
Spatial entropy:

$$h[3, 2]_6 \geq \frac{\ln 8}{21}.$$

case 6d: $[3, 2]_7$.

$\left\{ \begin{smallmatrix} \square & \\ \blacksquare & \blacksquare \end{smallmatrix}, \begin{smallmatrix} \blacksquare & \\ \square & \square \end{smallmatrix}, \begin{smallmatrix} \square & \\ \square & \square \end{smallmatrix} \right\}$ are not feasible. The global patterns are obtained by reflecting those patterns for $[3, 2]_4$ about the axis $\{(i, i) : i \in \mathbb{Z}\}$.

Patching block of size 4×3 :



Spatial entropy:

$$h[3, 2]_7 \geq \frac{\ln 3}{15}.$$

The remaining regions of the parameter space are all pattern formation. Only finite types of patterns appear. Therefore Table 2.1 lists the results for each case, where $\times$ means no global patterns exist. The Appendix contains the patterns which P_i' s represent.

2.5 Discussion

In Sec.3, we use the maximal set of patching block of minimal size to estimate the lower bound of the spatial entropy. The reason for using the minimal size is that it requires less time to find them. However, the set of patching blocks of a larger size will give the more precise estimate of lower bound. Indeed, denote by $B(m, n)$ the number of elements in the maximal set of the patching blocks of size $m \times n$. Define

$$\underline{h}_{m,n} = \frac{\ln B(m, n)}{mn},$$

the lower bound of the spatial entropy given by the maximal set of patching blocks.

Prop. 2.4 indicates the superiority of the patching blocks of large size.

Proposition 2.4. $\{\bar{h}_{2^k m, 2^k n}\}_{k=1}^{\infty}$ is an increasing sequence.

Proof. First, we claim that $\bar{h}_{m,n} \leq \bar{h}_{2m, 2n}$. In a $2m \times 2n$ lattice, there are 2×2 disjoint blocks of size $m \times n$. When constructing the patching blocks in the lattice, each block has at least $B(m, n)$ choices i.e. $B(m, m) \leq (B(m, n))^{2 \times 2}$. Hence $\bar{h}_{m,n} \leq \bar{h}_{2m, 2n}$ follows immediately. The standard induction argument shows that $\{\bar{h}_{2^k m, 2^k n}\}_k$ is decreasing. The proof is complete. $\square$

Combining the Prop 2.2 and 2.4 we get the inequality:

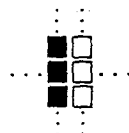
$$\underline{h}_{m,n} \leq \underline{h}_{2m, 2n} \leq \underline{h}_{2^2 m, 2^2 n} \leq \dots \leq h \leq \dots \leq \bar{h}_{2^2 m, 2^2 n} \leq \bar{h}_{2m, 2n} \leq \bar{h}_{m,n}.$$

Appendix A

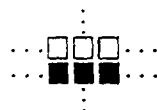
P1: the pattern of all black.

P2: the pattern of all white.

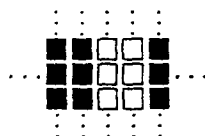
P3: The $\mathbb{Z}^2$ is divided into two parts: all black and all white. The boundary of the two parts is a vertical line.



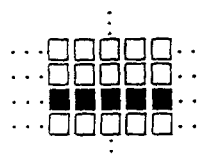
P4: The $\mathbb{Z}^2$ is divided into two parts: all black and all white. The boundary of the two parts is a horizontal line.



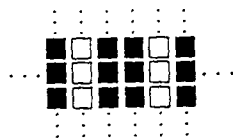
P5: Each vertical line is either all black or all white.



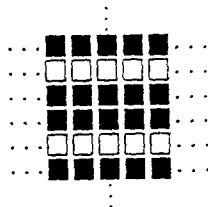
P6: Each horizontal line is either all black or all white.



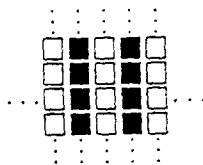
P7: Each vertical line is either all black or all white and the vertical white line is adjacent to only black lines.



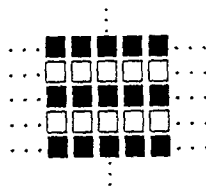
P8: Each horizontal line is either all black or all white and the horizontal white line is adjacent to only black lines.



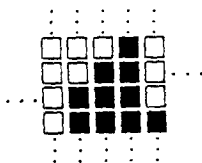
P9: Each vertical line is either all black or all white and every two adjacent vertical lines have different colors.



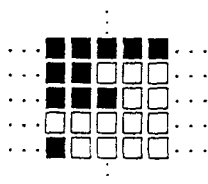
P10: Each horizontal line is either all black or all white and every two adjacent horizontal lines have different colors.



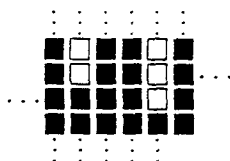
P11: The vertical lines has only two types: (a) whole black or white and (b) upper half white and lower half black.



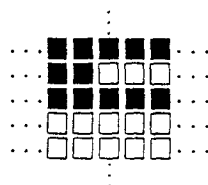
P12: The horizontal lines has only two types: (a) whole black or white and (b) left half black and right half white.



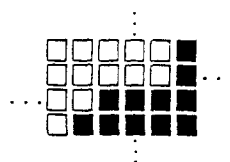
P13: The vertical lines have two types: (a) all black or white. (b) upper half white and lower half black. And (b) is adjacent to vertical black lines.



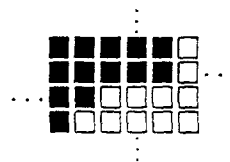
P14: The horizontal lines has two types: (a) all black or white. (b) upper half black and lower half black. And (b) is adjacent to the horizontal white lines.



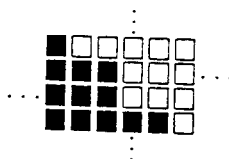
P15: The $\mathbb{Z}^2$ is divided into two parts: the left part is whole white and the right part is whole black. The boundary of the two parts is a non-decreasing path.



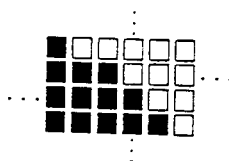
P16: The $\mathbb{Z}^2$ is divided into two parts: the left part is all black and the right part is all white. The boundaries of the two parts are a non-decreasing path.



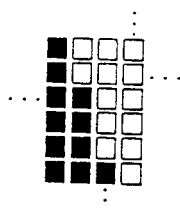
P17: The $\mathbb{Z}^2$ is divided into two part. The left part is all black and the right part is all white. The boundaries of the two parts are a non-increasing path.



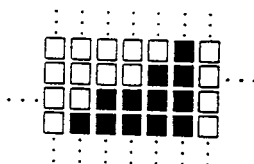
P18: The $\mathbb{Z}^2$ is divided into two part. The left part is all white and the right part is all black. The boundary of the two parts is a strictly decreasing path whose each downward step is at most one.



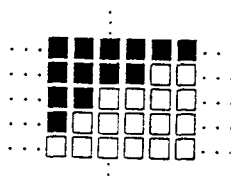
P19: The $\mathbb{Z}^2$ is divided into two parts. The left part is all white and the right part is all black. The boundaries of the two parts are a strictly decreasing path whose each righthward step is at most one.



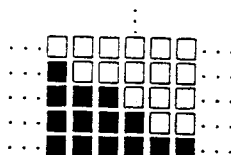
P20: This is a mixed type of *P15* and *P5*. Indeed, the pattern consists of (a) *P15* with finite width and (b) vertical lines of all black or all white. And (a) is adjacent to a vertical white line in the right and a white or black vertical line in the left, respectively.



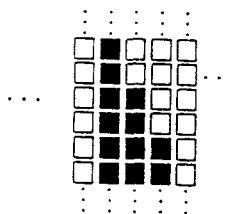
P21: This is a mixed type of *P16* and *P6*. Indeed, the pattern consist of (a) *P16* with finite height and horizontal lines of all black or all white. And (a) is adjacent above to a horizontal black line and below a horizontal white line, respectively.



P22: This is a mixed type of *P17* and *P6*. Indeed, the pattern consists of (a) *P17* with finite height and (b) horizontal lines of all black or all white. And (a) is adjacent above to a horizontal white line and below a white or black horizontal line, respectively.

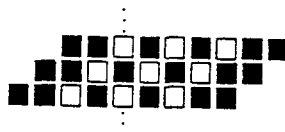


P23: This is a mixed type of *P18* and *P5*. Indeed, the pattern consist of (a) *P18* with finite width and (b) vertical lines of all black or all white. And (a) is adjacent to vertical lines.



P24: Checkerboard.

P25:



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