

QUADRATIC MODEL UPDATING WITH SYMMETRY, POSITIVE DEFINITENESS, AND NO SPILL-OVER*

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Abstract. Updating a system modeled as a real symmetric quadratic eigenvalue problem to match observed spectral information has been an important task for practitioners in different disciplines. It is often desirable in the process to match only the newly measured data without tampering with the other unmeasured and often unknown eigenstructure inherent in the original model. Such an updating, known as no spill-over, has been critical yet challenging in practice. Only recently, a mathematical theory on updating with no spill-over has begun to be understood. However, other imperative issues such as maintaining positive definiteness in the coefficient matrices remain to be addressed. This paper highlights several theoretical aspects about updating that preserves both no spill-over and positive definiteness of the mass and the stiffness matrices. In particular, some necessary and sufficient conditions for the solvability conditions are established in this investigation.

Key words. quadratic model, inverse eigenvalue problem, model updating, eigenstructure assignment, spill-over, positive definiteness

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1. Introduction. By a *real symmetric quadratic model*, we refer in this paper to any system that leads to a quadratic λ -matrix of the form

$$(1.1) \quad Q(\lambda) = \lambda^2 M + \lambda C + K,$$

where M , C , and K are $n \times n$ real symmetric matrices and, additionally, *both M and K are positive definite*. Real symmetric quadratic models arise frequently in areas such as applied mechanics, electrical oscillation, vibro-acoustics, fluid dynamics, signal processing, and the finite element model of some critical partial differential equations [21, 32]. The specifications of the underlying physical system are embedded (in certain structural ways) in the matrix coefficients (M, C, K) . For example, in a vibrating system, M , C , and K often represent the mass, damping, and stiffness, respectively.

A quadratic eigenvalue problem (QEP) involves, given (M, C, K) , finding scalars $\lambda \in \mathbb{C}$ and nonzero vectors $\mathbf{x} \in \mathbb{C}^n$, called the eigenvalues and eigenvectors of the system, respectively, to satisfy the algebraic equation $Q(\lambda)\mathbf{x} = 0$. The spectral information is essential for deducing the dynamical behavior of the underlying physical system. The theoretical framework for matrix polynomials in general and quadratic pencils in particular can be found in the seminal books by Lancaster [21] and by Gohberg, Lancaster, and Rodman [15, 16]. A good survey of applications, mathematical

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properties, and a variety of numerical algorithms for the QEP can be found in the treatise by Tisseur and Meerbergen [32].

As the forward problem characterizes the dynamical behavior of a system in terms of its physical parameters, an equally important topic is the inverse problem of expressing the physical parameters in terms of the dynamical behavior. A quadratic inverse eigenvalue problem (QIEP) is concerned with determining coefficients M , C , and K of the system (1.1) from its observed or expected eigeninformation. There are different ways to formulate a QIEP, depending on the type of eigeninformation available and the conditions imposed upon the matrix coefficients. We mention, for instance, the QIEP where only partial eigenstructure is prescribed [6, 20] and the QIEP where the complete spectral information is given but M and K are positive definite [22, 24, 23, 30]. Model updating is another class of QIEP of practical importance, where the purpose is to correct the coefficient matrices in an existent model so that the updated model will have a behavior closely matching the experimental data [14, 27, 28].

The notion of model updating emerged in the 90's as an important tool for the design, construction, and maintenance of mechanical systems. It has attracted extensive research interests ever since. To be specific, a real symmetric quadratic model updating problem can be stated as follows.

(MUP) *Given a real symmetric quadratic model (M_0, C_0, K_0) and a few of its associated eigenpairs $\{(\lambda_j, \mathbf{x}_j)\}_{j=1}^k$, $k \leq n$, where λ_j 's are distinct simple eigenvalues, assume that new eigenpairs $\{(\sigma_j, \mathbf{y}_j)\}_{j=1}^k$ have been measured, where σ_j 's are distinct and simple. Update the quadratic model (M_0, C_0, K_0) to a new real symmetric quadratic model (M, C, K) such that*

- (i) *The newly measured $\{(\sigma_j, \mathbf{y}_j)\}_{j=1}^k$ form k eigenpairs of the new model (M, C, K) .*
- (ii) *The remaining $2n - k$ eigenpairs of (M, C, K) are kept the same as those of the original (M_0, C_0, K_0) .*

Other than updating the eigeninformation $\{(\lambda_j, \mathbf{x}_j)\}_{j=1}^k$ to $\{(\sigma_j, \mathbf{y}_j)\}_{j=1}^k$, we stress two inherent conditions in the above statement. First, the matrices M and K must remain symmetric and positive definite. Second, the remaining $2n - k$ eigenpairs must remain invariant, a property known as the *no spill-over phenomenon* [9].

Some less stringent versions of the MUP have been discussed in the literature. Far from being complete, we mention articles [2, 3, 4, 13, 33, 34, 35] as references for the undamped case, where $C = C_0 = 0$, and [1, 13, 19, 29] for the damped problem. Some low-rank updating tactics for C and K are discussed in [12, 18, 28, 36, 37]. As it stands to date, the theory developed thus far is fragmentary and the techniques are certainly inadequate. One major difficulty is that all these methods can maintain the symmetry and reproduce measured data, but cannot guarantee no spill-over after the update. Another difficulty is that neither of these methods can warrant positive definiteness of the mass matrix M and stiffness matrix K , which often is critical in applications.

A mathematical theory for the no spill-over phenomenon has been established only recently in the work [9, 10]. This paper represents another step of advancement. We generalize the no spill-over theory to include the preservation of positive definiteness for M and K . The main thrust in this study is to develop necessary and sufficient conditions for the solvability of the MUP. We believe that our results are innovative in the field.

This paper is organized as follows. We begin in section 2 with the preliminaries, introducing notations and some fundamental facts. In section 3 we derive a solvability

condition under a parameterized structure. This parametric representation has been known to be generic in the literature. By saying something is “generically true” we mean that those which fail to be true form a set of measure zero in the ambient space. So our emphasis is on the choice of the parameter to ensure positive definiteness and no spill-over. Being generic, our results in this section serve to answer the MUP in the most practical setting. In section 4 we tackle the possibility of solving the MUP without assuming the parametric form. This is a more general and much harder problem. Still, we offer a complete answer for the case when half of the entire spectrum is given. Without assuming the parametric form, it remains an open question if fewer or more than half of the eigenvalues are to be updated. Some interesting numerical examples will be given in the course of discussion to demonstrate our points.

2. Preliminaries. To set the notation for later discussion, let

$$\lambda_1, \dots, \lambda_s, \quad \lambda_{s+1}, \dots, \lambda_t, \quad \bar{\lambda}_{s+1}, \dots, \bar{\lambda}_t$$

denote the portion ($k = 2t - s$) of the spectrum to be replaced, where $\lambda_1, \dots, \lambda_s \in \mathbb{R}$ are the distinct real eigenvalues and $\lambda_{s+1}, \dots, \lambda_t \in \mathbb{C}$ are the distinct complex eigenvalues. Let the corresponding eigenvectors be denoted by

$$\mathbf{x}_1, \dots, \mathbf{x}_s, \quad \mathbf{x}_{s+1}, \dots, \mathbf{x}_t, \quad \bar{\mathbf{x}}_{s+1}, \dots, \bar{\mathbf{x}}_t.$$

For $j = s+1, \dots, t$, write $\lambda_j = \alpha_j + \iota\beta_j$ and $\mathbf{x}_j = \mathbf{x}_{jR} + \iota\mathbf{x}_{jI}$ with $\iota = \sqrt{-1}$, $\alpha_j, \beta_j \in \mathbb{R}$ and $\mathbf{x}_{jR}, \mathbf{x}_{jI} \in \mathbb{R}^n$. Upon introducing

$$\Lambda_i := \begin{cases} \lambda_i, & i = 1, \dots, s, \\ \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}, & i = s+1, \dots, t, \end{cases}$$

and

$$X_i := \begin{cases} \mathbf{x}_i, & i = 1, \dots, s, \\ \begin{bmatrix} \mathbf{x}_{iR} & \mathbf{x}_{iI} \end{bmatrix}, & i = s+1, \dots, t, \end{cases}$$

we see that the equation

$$(2.1) \quad M_0 X \Lambda^2 + C_0 X \Lambda + K_0 X = 0$$

is satisfied by the matrices

$$(2.2) \quad \Lambda := \text{diag}\{\Lambda_1, \dots, \Lambda_s, \Lambda_{s+1}, \dots, \Lambda_t\} \in \mathbb{R}^{k \times k},$$

$$(2.3) \quad X := [X_1, \dots, X_s, X_{s+1}, \dots, X_t] \in \mathbb{R}^{n \times k}.$$

Without causing ambiguity, we shall say that the pair of real matrices (Λ, X) represents k eigenpairs of $Q_0(\lambda) = \lambda^2 M_0 + \lambda C_0 + K_0$. In a similar way, let $(\Sigma, Y) \in \mathbb{R}^{k \times k} \times \mathbb{R}^{n \times k}$ and $(\Upsilon, Z) \in \mathbb{R}^{(2n-k) \times (2n-k)} \times \mathbb{R}^{n \times (2n-k)}$ denote the newly measured k eigenpairs and the remaining $2n - k$ eigenpairs of $Q_0(\lambda)$, respectively. Keep in mind that the pair (Υ, Z) may not be known at all.

The MUP is equivalent to finding $n \times n$ real symmetric matrices $(\Delta M, \Delta C, \Delta K)$ such that

$$(2.4) \quad (M_0 + \Delta M)Y\Sigma^2 + (C_0 + \Delta C)Y\Sigma + (K_0 + \Delta K)Y = 0,$$

$$(2.5) \quad (M_0 + \Delta M)Z\Upsilon^2 + (C_0 + \Delta C)Z\Upsilon + (K_0 + \Delta K)Z = 0,$$

and $M_0 + \Delta M$ and $K_0 + \Delta K$ are positive definite. It has been shown in [10, Theorem 4.1] that the newly measured eigenvector Y cannot be arbitrary. In fact, a necessary condition for (2.4) to hold is that

$$(2.6) \quad Y = XT$$

for some $T \in \mathbb{R}^{k \times k}$. Furthermore, if Y is of full rank as is assumed hereafter, then T is nonsingular. In the subsequent discussion, we shall always assume that (2.6) is satisfied.

3. Solvability under a general structure. Observe that Λ in (2.2) is nonsingular because of K_0 being positive definite, and $Q_0(\lambda)$ has no zero eigenvalue. It is known that the product

$$S := \begin{bmatrix} X & Z \\ X\Lambda & Z\Upsilon \end{bmatrix}^\top \begin{bmatrix} C_0 & M_0 \\ M_0 & 0 \end{bmatrix} \begin{bmatrix} X & Z \\ X\Lambda & Z\Upsilon \end{bmatrix}$$

is necessarily block diagonal (indeed, with the same block structure as that of $\text{diag}\{\Lambda, \Upsilon\}$) [10, 11]. Since

$$\begin{bmatrix} X & Z \\ X\Lambda & Z\Upsilon \end{bmatrix}^\top \begin{bmatrix} -K_0 & 0 \\ 0 & M_0 \end{bmatrix} \begin{bmatrix} X & Z \\ X\Lambda & Z\Upsilon \end{bmatrix} = S \begin{bmatrix} \Lambda & 0 \\ 0 & \Upsilon \end{bmatrix},$$

it follows that

$$\Lambda^{-T} X^T K_0 Z = X^T M_0 Z \Upsilon.$$

It thus can be checked that the incremental matrices $(\Delta M, \Delta C, \Delta K)$ of the form

$$(3.1) \quad \begin{cases} \Delta M &:= M_0 X \Phi X^\top M_0, \\ \Delta C &:= -M_0 X \Phi \Lambda^{-\top} X^\top K_0 - K_0 X \Lambda^{-1} \Phi X^\top M_0, \\ \Delta K &:= K_0 X \Lambda^{-1} \Phi \Lambda^{-\top} X^\top K_0, \end{cases}$$

where $\Phi \in \mathbb{R}^{k \times k}$ is an arbitrary symmetric matrix, are sufficient for solving (2.5). This updating formula therefore would fulfill the requirement of no spill-over. Note that the information of (Υ, Z) is not needed in the definition (3.1). It has been shown in [10] that, under some very mild conditions on (Υ, Z) which generically are true, this form is also necessary for solving (2.5). We shall assume this parametric form on $(\Delta M, \Delta C, \Delta K)$ in this section and derive a necessary and sufficient condition for the solvability of the MUP. The focus, of course, should be on satisfying (2.4) and maintaining positive definiteness. The case when $(\Delta M, \Delta C, \Delta K)$ is not in this generic form will be considered in the next section.

The following three lemmas, already established in the literature, are useful for facilitating our main discussion.

LEMMA 3.1 (see [25]). *Given $\mathcal{A} \in \mathbb{R}^{n \times n}$ and $\mathcal{B} \in \mathbb{R}^{m \times m}$, the equation*

$$\mathcal{A}H - H\mathcal{B} = 0$$

has only the trivial solution $H = 0$ if and only if $\mathcal{A}$ and $\mathcal{B}$ have no common eigenvalues.

LEMMA 3.2 (see [20]). *Suppose that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C} \in \mathbb{R}^{k \times k}$ are symmetric and that all eigenvalues of $\Omega \in \mathbb{R}^{k \times k}$ are distinct. Then the equation*

$$\mathcal{A}\Omega^2 + \mathcal{B}\Omega + \mathcal{C} = 0$$

holds if and only if

$$\begin{aligned}\mathcal{B} &= \Pi - \mathcal{A}\Omega - \Omega^\top \mathcal{A}, \\ \mathcal{C} &= \Omega^\top \mathcal{A}\Omega - \Pi\Omega\end{aligned}$$

for some symmetric matrix $\Pi \in \mathbb{R}^{k \times k}$ satisfying $\Omega^\top \Pi = \Pi\Omega$. Moreover, if the matrix Ω is block diagonal with the same structure as that of Λ described in (2.2), then the matrix Π is also block diagonal of the same structure except that its 2×2 diagonal blocks are of the particular form $\begin{bmatrix} \mu & \nu \\ \nu & -\mu \end{bmatrix}$.

LEMMA 3.3 (see [7]). Suppose that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C} \in \mathbb{R}^{n \times n}$, $\mathcal{E} = [\mathcal{E}_1, \mathcal{E}_2]$, and $\Omega = \text{diag}\{\Omega_1, \Omega_2\}$ satisfy the equation

$$\mathcal{A}\mathcal{E}\Omega^2 + \mathcal{B}\mathcal{E}\Omega + \mathcal{C}\mathcal{E} = 0,$$

where $\mathcal{E}_1 \in \mathbb{R}^{n \times m}$, $\mathcal{E}_2 \in \mathbb{R}^{n \times (2n-m)}$, $\Omega_1 \in \mathbb{R}^{m \times m}$, and $\Omega_2 \in \mathbb{R}^{(2n-m) \times (2n-m)}$ with $m \leq n$. If the two matrices Ω_1 and Ω_2 have no common eigenvalues, then it is true that

$$(\mathcal{E}_1\Omega_1)^\top \mathcal{A}\mathcal{E}_2\Omega_2 - \mathcal{E}_1^\top \mathcal{C}\mathcal{E}_2 = 0.$$

From (3.1), it is obvious that if Φ is positive definite, then we will have the desired positive definiteness for $M_0 + \Delta M$ and $K_0 + \Delta K$, albeit that (2.4) is yet to be satisfied. Our goal is to characterize the more general symmetric matrix Φ for the MUP. Toward that end, let

$$X = [Q_1, Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

be the QR factorization of X , where $[Q_1, Q_2] \in \mathbb{R}^{n \times n}$ is orthogonal, $Q_1 \in \mathbb{R}^{n \times k}$, and $R_1 \in \mathbb{R}^{k \times k}$ is nonsingular. We shall use the congruence transformation via the nonsingular matrix $[Q_1 R_1, Q_2]$ to examine the three equations (2.1), (2.4), and (2.5) in what follows. This would provide us with a handle to set apart the conditions that the parameter matrix Φ must satisfy in order to solve the MUP.

First, write

$$(3.2) \quad \begin{aligned} \begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix} &:= [Q_1 R_1, Q_2]^\top M_0 [Q_1 R_1, Q_2], \\ \begin{bmatrix} C_1 & C_2 \\ C_2^\top & C_3 \end{bmatrix} &:= [Q_1 R_1, Q_2]^\top C_0 [Q_1 R_1, Q_2], \\ \begin{bmatrix} K_1 & K_2 \\ K_2^\top & K_3 \end{bmatrix} &:= [Q_1 R_1, Q_2]^\top K_0 [Q_1 R_1, Q_2], \end{aligned}$$

where the partitioning is such that the symmetric matrices M_1 , C_1 , and K_1 are all of size $k \times k$. Then (2.1) is equivalent to

$$\begin{bmatrix} M_1 \\ M_2^\top \end{bmatrix} \Lambda^2 + \begin{bmatrix} C_1 \\ C_2^\top \end{bmatrix} \Lambda + \begin{bmatrix} K_1 \\ K_2^\top \end{bmatrix} = 0.$$

By Lemma 3.2, if we define

$$(3.3) \quad \Gamma := C_1 + M_1 \Lambda + \Lambda^\top M_1,$$

then Γ is symmetric, satisfies

$$(3.4) \quad \Lambda^\top \Gamma = \Gamma \Lambda,$$

and is block diagonal with a similar (but symmetric) structure as that of Λ . We also have the relationships that

$$(3.5) \quad \begin{cases} C_1 &= \Gamma - M_1 \Lambda - \Lambda^\top M_1, \\ K_1 &= \Lambda^\top M_1 \Lambda - \Gamma \Lambda, \\ K_2 &= -(\Lambda^\top)^2 M_2 - \Lambda^\top C_2. \end{cases}$$

Likewise, apply the same congruence transformation to $(\Delta M, \Delta C, \Delta K)$, and write

$$(3.6) \quad \begin{cases} \begin{bmatrix} \Delta M_1 & \Delta M_2 \\ (\Delta M_2)^\top & \Delta M_3 \end{bmatrix} &:= \begin{bmatrix} Q_1 R_1 & Q_2 \end{bmatrix}^\top \Delta M \begin{bmatrix} Q_1 R_1 & Q_2 \end{bmatrix}, \\ \begin{bmatrix} \Delta C_1 & \Delta C_2 \\ (\Delta C_2)^\top & \Delta C_3 \end{bmatrix} &:= \begin{bmatrix} Q_1 R_1 & Q_2 \end{bmatrix}^\top \Delta C \begin{bmatrix} Q_1 R_1 & Q_2 \end{bmatrix}, \\ \begin{bmatrix} \Delta K_1 & \Delta K_2 \\ (\Delta K_2)^\top & \Delta K_3 \end{bmatrix} &:= \begin{bmatrix} Q_1 R_1 & Q_2 \end{bmatrix}^\top \Delta K \begin{bmatrix} Q_1 R_1 & Q_2 \end{bmatrix}. \end{cases}$$

By construction, it follows from (3.1) that

$$(3.7) \quad \begin{cases} \Delta M_1 &= M_1 \Phi M_1, \\ \Delta M_2 &= M_1 \Phi M_2, \\ \Delta K_1 &= K_1 \Lambda^{-1} \Phi \Lambda^{-\top} K_1. \end{cases}$$

We next turn our attention to (2.4) which pertains to the updated eigeninformation. Upon substituting (2.6) into (2.4) and defining

$$\tilde{\Sigma} = T \Sigma T^{-1},$$

we see that

$$(3.8) \quad (M_0 + \Delta M) X \tilde{\Sigma}^2 + (C_0 + \Delta C) X \tilde{\Sigma} + (K_0 + \Delta K) X = 0,$$

which is equivalent to

$$\begin{bmatrix} M_1 + M_1 \Phi M_1 \\ (M_2 + M_1 \Phi M_2)^\top \end{bmatrix} \tilde{\Sigma}^2 + \begin{bmatrix} C_1 + \Delta C_1 \\ (C_2 + \Delta C_2)^\top \end{bmatrix} \tilde{\Sigma} + \begin{bmatrix} K_1 + \Delta K_1 \\ (K_2 + \Delta K_2)^\top \end{bmatrix} = 0.$$

By Lemma 3.2 again, there exists a symmetric matrix Ξ which satisfies

$$(3.9) \quad \tilde{\Sigma}^\top \Xi = \Xi \tilde{\Sigma},$$

and such that the equations

$$(3.10) \quad \begin{cases} C_1 + \Delta C_1 &= \Xi - (M_1 + M_1 \Phi M_1) \tilde{\Sigma} - \tilde{\Sigma}^\top (M_1 + M_1 \Phi M_1), \\ K_1 + \Delta K_1 &= \tilde{\Sigma}^\top (M_1 + M_1 \Phi M_1) \tilde{\Sigma} - \Xi \tilde{\Sigma} = \tilde{\Sigma}^\top (M_1 + M_1 \Phi M_1) \tilde{\Sigma} - \tilde{\Sigma}^\top \Xi, \\ K_2 + \Delta K_2 &= -(\tilde{\Sigma}^\top)^2 (M_2 + M_1 \Phi M_2) - \tilde{\Sigma}^\top (C_2 + \Delta C_2) \end{cases}$$

hold. Combining (3.5) and (3.10), we conclude that

$$(3.11) \quad \begin{cases} \Delta C_1 &= \Xi - M_1 \tilde{\Sigma} - \tilde{\Sigma}^\top M_1 - M_1 \Phi M_1 \tilde{\Sigma} - \tilde{\Sigma}^\top M_1 \Phi M_1 - \Gamma + M_1 \Lambda + \Lambda^\top M_1, \\ \Delta K_1 &= \tilde{\Sigma}^\top M_1 \tilde{\Sigma} + \tilde{\Sigma}^\top M_1 \Phi M_1 \tilde{\Sigma} - \tilde{\Sigma}^\top \Xi - \Lambda^\top M_1 \Lambda + \Lambda^\top \Gamma, \\ \Delta K_2 &= -(\tilde{\Sigma}^\top)^2 M_2 - \tilde{\Sigma}^\top C_2 - (\tilde{\Sigma}^\top)^2 M_1 \Phi M_2 - \tilde{\Sigma}^\top \Delta C_2 + (\Lambda^\top)^2 M_2 + \Lambda^\top C_2. \end{cases}$$

For the convenience of later reference, we summarize the updates developed thus far as follows:

$$\begin{aligned}
 (3.12) \quad & \begin{bmatrix} \Delta M_1 & \Delta M_2 \end{bmatrix} = M_1 \Phi \begin{bmatrix} M_1 & M_2 \end{bmatrix}, \\
 & \begin{bmatrix} \Delta C_1 & \Delta C_2 \end{bmatrix} = M_1 \Phi \begin{bmatrix} \Gamma - M_1 \Lambda & \Lambda^\top M_2 + C_2 \end{bmatrix} \\
 & \quad + (\Lambda^\top - \tilde{\Sigma}^\top - \tilde{\Sigma}^\top M_1 \Phi) \begin{bmatrix} M_1 & M_2 \end{bmatrix} \\
 & \quad + \begin{bmatrix} (M_1 + M_1 \Phi M_1)(\Lambda - \tilde{\Sigma}) + \Xi - \Gamma - M_1 \Phi \Gamma & \Delta W \end{bmatrix}, \\
 & \begin{bmatrix} \Delta K_1 & \Delta K_2 \end{bmatrix} = (\Lambda^\top - \tilde{\Sigma}^\top - \tilde{\Sigma}^\top M_1 \Phi) \begin{bmatrix} \Gamma - M_1 \Lambda & \Lambda^\top M_2 + C_2 \end{bmatrix} \\
 & \quad - \tilde{\Sigma}^\top \begin{bmatrix} (M_1 + M_1 \Phi M_1)(\Lambda - \tilde{\Sigma}) + \Xi - \Gamma - M_1 \Phi \Gamma & \Delta W \end{bmatrix},
 \end{aligned}$$

with

$$\Delta W := \Delta C_2 - [M_1 \Phi (\Lambda^\top M_2 + C_2) + (\Lambda^\top - \tilde{\Sigma}^\top - \tilde{\Sigma}^\top M_1 \Phi) M_2].$$

Finally, we look into (2.5) which is needed for the no spill-over to the unmeasured or unknown eigeninformation. By defining

$$\mathcal{Z} = \begin{bmatrix} Q_1 R_1 & Q_2 \end{bmatrix}^{-1} Z,$$

the equation

$$M_0 \begin{bmatrix} X & Z \end{bmatrix} \begin{bmatrix} \Lambda & \Upsilon \end{bmatrix}^2 + C_0 \begin{bmatrix} X & Z \end{bmatrix} \begin{bmatrix} \Lambda & \Upsilon \end{bmatrix} + K_0 \begin{bmatrix} X & Z \end{bmatrix} = 0$$

is reduced to

$$\begin{aligned}
 & \begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix}, & Z \end{bmatrix} \begin{bmatrix} \Lambda & \Upsilon \end{bmatrix}^2 + \begin{bmatrix} C_1 & C_2 \\ C_2^\top & C_3 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix}, & Z \end{bmatrix} \begin{bmatrix} \Lambda & \Upsilon \end{bmatrix} \\
 & \quad + \begin{bmatrix} K_1 & K_2 \\ K_2^\top & K_3 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix}, & Z \end{bmatrix} = 0.
 \end{aligned}$$

By Lemma 3.3 and (3.4) and (3.5), we obtain the relationship

$$\Lambda^\top \{ \begin{bmatrix} M_1 & M_2 \end{bmatrix} \mathcal{Z} \Upsilon + \begin{bmatrix} \Gamma - M_1 \Lambda & \Lambda^\top M_2 + C_2 \end{bmatrix} \mathcal{Z} \} = 0,$$

which implies that

$$(3.13) \quad \begin{bmatrix} M_1 & M_2 \end{bmatrix} \mathcal{Z} \Upsilon + \begin{bmatrix} \Gamma - M_1 \Lambda & \Lambda^\top M_2 + C_2 \end{bmatrix} \mathcal{Z} = 0.$$

An algebraic manipulation of (2.5) with the aid of (3.2), (3.4)–(3.6), (3.12), and (3.13) yields the following facts:

$$\begin{aligned}
 & (M_0 + \Delta M) \mathcal{Z} \Upsilon^2 + (C_0 + \Delta C) \mathcal{Z} \Upsilon + (K_0 + \Delta K) \mathcal{Z} = 0 \\
 \Rightarrow & \begin{bmatrix} M_1 + \Delta M_1 & M_2 + \Delta M_2 \end{bmatrix} \mathcal{Z} \Upsilon^2 + \begin{bmatrix} C_1 + \Delta C_1 & C_2 + \Delta C_2 \end{bmatrix} \mathcal{Z} \Upsilon \\
 & \quad + \begin{bmatrix} K_1 + \Delta K_1 & K_2 + \Delta K_2 \end{bmatrix} \mathcal{Z} = 0 \\
 \Leftrightarrow & \begin{bmatrix} (M_1 + M_1 \Phi M_1)(\Lambda - \tilde{\Sigma}) + \Xi - \Gamma - M_1 \Phi \Gamma & \Delta W \end{bmatrix} \mathcal{Z} \Upsilon \\
 (3.14) \quad & - \tilde{\Sigma}^\top \begin{bmatrix} (M_1 + M_1 \Phi M_1)(\Lambda - \tilde{\Sigma}) + \Xi - \Gamma - M_1 \Phi \Gamma & \Delta W \end{bmatrix} \mathcal{Z} = 0.
 \end{aligned}$$

We now have all the tools needed to characterize the parameter matrix Φ . We begin with the claim that, as (Λ, X) is being replaced by (Σ, Y) and $Y = XT$, the

two seemingly unassociated matrices Γ and Ξ employed in (3.5) and (3.10) are related in a special way.

LEMMA 3.4. *Assume that the unmeasured eigenvectors Z are of full row rank and that the unmeasured eigenvalues Υ and the newly measured eigenvalues Σ are disjoint. Then it is true that $\Xi = \Gamma$.*

Proof. By assumption, Υ and $\tilde{\Sigma} = T\Sigma T^{-1}$ have no common eigenvalues. By Lemma 3.1 we see from (3.14) that

$$[(M_1 + M_1\Phi M_1)(\Lambda - \tilde{\Sigma}) + \Xi - \Gamma - M_1\Phi\Gamma, \quad \Delta W] \mathcal{Z} = 0.$$

Since $\mathcal{Z}$ is of full row rank, we also see that

$$(3.15) \quad \Delta C_2 = M_1\Phi(\Lambda^\top M_2 + C_2) + (\Lambda^\top - \tilde{\Sigma}^\top - \tilde{\Sigma}^\top M_1\Phi)M_2,$$

$$(3.16) \quad \Xi = (I + M_1\Phi)[M_1(\tilde{\Sigma} - \Lambda) + \Gamma].$$

Recall on the one hand that the matrix ΔK_1 can be expressed in two different ways (in (3.7) and (3.12)), whereas (3.12) can further be simplified by (3.16) so that

$$\Delta K_1 = K_1\Lambda^{-1}\Phi\Lambda^{-\top}K_1 = (\Lambda^\top - \tilde{\Sigma}^\top - \tilde{\Sigma}^\top M_1\Phi)(\Gamma - M_1\Lambda)$$

holds. On the other hand, we can write the middle equation in (3.5) as $\Gamma = M_1\Lambda - \Lambda^{-\top}K_1$. Thus,

$$K_1\Lambda^{-1}\Phi\Lambda^{-\top}K_1 = (\Lambda^\top - \tilde{\Sigma}^\top - \tilde{\Sigma}^\top M_1\Phi)(-\Lambda^{-\top}K_1).$$

It follows that

$$(3.17) \quad \tilde{\Sigma} - \Lambda = \Phi\Lambda^{-\top}(K_1 - \Lambda^\top M_1\tilde{\Sigma}) = \Phi[M_1(\Lambda - \tilde{\Sigma}) - \Gamma].$$

Simplifying (3.16) by using (3.17) in the following way,

$$\Xi = (I + M_1\Phi)[M_1(\tilde{\Sigma} - \Lambda) + \Gamma] = M_1(\tilde{\Sigma} - \Lambda) + \Gamma + M_1(\Lambda - \tilde{\Sigma}) = \Gamma,$$

we see that the two matrices Ξ and Γ are identical. $\square$

The next important result claims that in order to satisfy the three equations (2.1), (2.4), and (2.5) the parameter matrix Φ is uniquely determined.

LEMMA 3.5. *Assume that $\tilde{\Sigma} - \Lambda = T\Sigma T^{-1} - \Lambda$ is nonsingular. Then the parameter matrix Φ is uniquely determined by the formula*

$$(3.18) \quad \Phi = [\Gamma T(\Lambda T - T\Sigma)^{-1} - M_1]^{-1}.$$

Proof. By (3.17), $K_1 - \Lambda^\top M_1\tilde{\Sigma}$ is invertible and

$$(3.19) \quad \Phi = (\tilde{\Sigma} - \Lambda)(K_1 - \Lambda^\top M_1\tilde{\Sigma})^{-1}\Lambda^\top = (T\Sigma - \Lambda T)(K_1 T - \Lambda^\top M_1 T\Sigma)^{-1}\Lambda^\top.$$

Since $\Lambda^{-\top}K_1 T - M_1 T\Sigma = M_1(\Lambda T - T\Sigma) - \Gamma T$ by (3.4) and (3.5), the assertion is proved. $\square$

Note that by Lemma 3.4, (3.9) is equivalent to

$$(3.20) \quad \Sigma^\top T^\top \Gamma T = T^\top \Gamma T \Sigma.$$

Together with the property (3.4), it is clear that the Φ specified in (3.18) is symmetric. In the discussion thus far, we have already developed several necessary conditions which we now summarize in the following theorem.

THEOREM 3.6. *Suppose that the triplet $(\Delta M, \Delta C, \Delta K)$ in the form (3.1) for some symmetric matrix $\Phi \in \mathbb{R}^{k \times k}$ solves the MUP. Assume that the (generic) conditions*

- (a) $Y = XT$ for some nonsingular matrix $T \in \mathbb{R}^{k \times k}$,
- (b) Z is of full row rank,
- (c) the spectra of Υ and Σ are disjoint,
- (d) $T\Sigma T^{-1} - \Lambda$ is nonsingular.

are satisfied by the outgoing eigenpairs (Λ, X) , the updated eigenpairs (Σ, Y) , and the untouched eigenpairs (Υ, Z) . Then it is necessary that

- (1) the matrix $X^\top K_0 X T - \Lambda^\top X^\top M_0 X T \Sigma$ is nonsingular;
- (2) the matrix $T^\top (X^\top C_0 X + \Lambda^\top X^\top M_0 X + X^\top M_0 X \Lambda) T \Sigma$ is symmetric;
- (3) $\Phi = (T\Sigma - \Lambda T)(X^\top K_0 X T - \Lambda^\top X^\top M_0 X T \Sigma)^{-1} \Lambda^\top$;
- (4) $M_0 + \Delta M$ and $K_0 + \Delta K$ are positive definite.

Proof. It is clear from the definition in (3.2) that $M_1 = X^\top M_0 X$, $C_1 = X^\top C_0 X$, and $K_1 = X^\top K_0 X$. The second result above follows from (3.20) if Γ is replaced by (3.3). The first and the third results are established in the proof of Lemma 3.5. The fourth result is inherent in the assumption. $\square$

What is most significant is that the above necessary conditions are also sufficient in the following sense.

THEOREM 3.7. *Suppose that the outgoing eigenpairs (Λ, X) , the updated eigenpairs (Σ, Y) , and the untouched eigenpairs (Υ, Z) satisfy the generic assumptions (a)–(d) in Theorem 3.6. Assume that conditions (1)–(2) are satisfied. Then the triplet $(\Delta M, \Delta C, \Delta K)$ in the form (3.1) with Φ being uniquely given as in condition (3) solves the two equations (2.4) and (2.5). If the resulting $M_0 + \Delta M$ and $K_0 + \Delta K$ are positive definite, then the MUP is solvable.*

Proof. We need to check three things to warrant the assertion. First, condition (1) implies that the specific Φ given by condition (3) is well defined. condition (2) is equivalent to (3.20), where Γ is defined by (3.3), which shows that Φ is symmetric. The triplet $(\Delta M, \Delta C, \Delta K)$ in the form (3.1) with any symmetric matrix Φ already satisfies (2.5). Second, defining $\Xi = \Gamma$ in (3.10), by Lemma 3.2, we then can trace the derivations from (3.11) backward to (3.8) to show that (2.4) is satisfied. Third, the uniqueness of Φ in Lemma 3.5 implies that the conditions (1)–(3) constitute the solvability of the MUP, provided that the resulting $M_0 + \Delta M$ and $K_0 + \Delta K$ are positive definite. $\square$

It should be made clear that since Φ is unique, the incremental matrices $(\Delta M, \Delta C, \Delta K)$ are formed with no further control over the positive definiteness of $M_0 + \Delta M$ and $K_0 + \Delta K$. If any of the resulting matrices $M_0 + \Delta M$ and $K_0 + \Delta K$ in Theorem 3.7 violates the positive definiteness, then it simply implies that the original model (M_0, C_0, K_0) cannot be updated by (Σ, Y) without losing positive definiteness. In short, our conditions ensures a make-or-break decision for the solvability of the MUP.

Example 1. Consider the statically condensed oil rig model (M_0, C_0, K_0) represented by the triplet BCSSTRUC1 in the Harwell–Boeing collection [5]. In this model, M_0 and $K_0 \in \mathbb{R}^{66 \times 66}$ are symmetric and positive definite and $C_0 = 1.55I_{66}$. There are 132 eigenpairs. Suppose we want to replace the eight eigenvalues

$$\begin{aligned}\lambda_1 &= -5.358410088235457, & \lambda_2 &= -3.462830582716281, \\ \lambda_3 &= -3.570946054521908, & \lambda_4 &= -9.276066378899415,\end{aligned}$$

$$\begin{aligned}\lambda_5 &= -7.802118288361733 + 164.3321224340448i, \\ \lambda_6 &= -7.802118288361733 - 164.3321224340448i, \\ \lambda_7 &= -7.755809434339588 + 164.0571880852085i, \\ \lambda_8 &= -7.755809434339588 - 164.0571880852085i\end{aligned}$$

by newly measured eigenvalues

$$\begin{aligned}\sigma_1 &= -5.05, & \sigma_2 &= -3.32, \\ \sigma_3 &= -3.75, & \sigma_4 &= -9.07,\end{aligned}$$

$$\begin{aligned}\sigma_5 &= -7 + 160i, & \sigma_6 &= -7 - 160i, \\ \sigma_7 &= -8 + 170i, & \sigma_8 &= -8 - 170i,\end{aligned}$$

while keeping the corresponding eigenvectors invariant. That is, $Y = X$ and $T = I_8$. We check to see that $X^\top K_0 X - \Lambda^\top X^\top M_0 X \Sigma$ is nonsingular, so condition (1) in Theorem 3.6 is satisfied. Condition (2) is reduced to the symmetry of $\Gamma \Sigma$ which is automatically satisfied. We compute the uniquely described Φ according to condition (3) in Theorem 3.6 and the resulting $(\Delta M, \Delta C, \Delta K)$. We can verify numerically that the computed $M_0 + \Delta M$ and $K_0 + \Delta K$ are positive definite. We thus think that the model has been updated satisfactorily because the residual of the updated model

$$\begin{aligned}\|(M_0 + \Delta M)Y\Sigma^2 + (C_0 + \Delta C)Y\Sigma + (K_0 + \Delta K)Y\|_2 &= 2.830849890712934 \times 10^{-9}, \\ \|(M_0 + \Delta M)Z\Upsilon^2 + (C_0 + \Delta C)Z\Upsilon + (K_0 + \Delta K)Z\|_2 &= 7.620711447892196 \times 10^{-9}\end{aligned}$$

is compatible with the residual of the original model

$$\begin{aligned}\|M_0 X \Lambda^2 + C_0 X \Lambda + K_0 X\|_2 &= 2.873651863994003 \times 10^{-9}, \\ \|M_0 Z \Upsilon^2 + C_0 Z \Upsilon + K_0 Z\|_2 &= 7.617801878019591 \times 10^{-9}.\end{aligned}$$

Example 2. Consider the same data set as above except that $Y = XT$ with

$$T = \text{diag} \left\{ 0.9169, 0.8132, 0.6038, 0.4451, \begin{bmatrix} 0.9318 & 0.4186 \\ 0.4186 & -0.9318 \end{bmatrix}, \begin{bmatrix} 0.5252 & 0.6721 \\ 0.6721 & -0.5252 \end{bmatrix} \right\}.$$

It can be checked that conditions (1) and (2) are satisfied. We compute the unique Φ and the resulting $(\Delta M, \Delta C, \Delta K)$ based on Theorem 3.6. Surely, we find that the updated residuals are reasonably small:

$$\begin{aligned}\|(M_0 + \Delta M)Y\Sigma^2 + (C_0 + \Delta C)Y\Sigma + (K_0 + \Delta K)Y\|_2 &= 2.822913902500785 \times 10^{-9}, \\ \|(M_0 + \Delta M)Z\Upsilon^2 + (C_0 + \Delta C)Z\Upsilon + (K_0 + \Delta K)Z\|_2 &= 7.612797317470159 \times 10^{-9}.\end{aligned}$$

However, it turns out that both $M_0 + \Delta M$ and $K_0 + \Delta K$ are not positive definite. This example demonstrates an important fact that the MUP may not be solvable even if the triplet $(\Delta M, \Delta C, \Delta K)$ satisfies (2.4) and (2.5).

Example 3. Consider the same data set again with $Y = XT$, where

$$T = \text{diag} \left\{ -0.4326, -1.6656, 0.1253, 0.2877, \begin{bmatrix} -1.1465 & 1.1909 \\ -1.1909 & -1.1465 \end{bmatrix}, \begin{bmatrix} 1.1892 & -0.0376 \\ 0.0376 & 1.1892 \end{bmatrix} \right\}.$$

Again, conditions (1) and (2) are satisfied. We compute Φ and $(\Delta M, \Delta C, \Delta K)$. It is found that

$$\begin{aligned}\|(M_0 + \Delta M)Y\Sigma^2 + (C_0 + \Delta C)Y\Sigma + (K_0 + \Delta K)Y\|_2 &= 4.535516693254191 \times 10^{-9}, \\ \|(M_0 + \Delta M)Z\Upsilon^2 + (C_0 + \Delta C)Z\Upsilon + (K_0 + \Delta K)Z\|_2 &= 7.620710821638826 \times 10^{-9},\end{aligned}$$

while $M_0 + \Delta M$ and $K_0 + \Delta K$ are positive definite.

In all these three examples we have used the same set of newly measured, eigenvalues, but different choices of T give rise to different outcomes for the MUP. Clearly a properly selected T plays a critical role in the model update.

Thus far, our theory of solvability has been developed under the assumption that the incremental matrices ΔM , ΔC , and ΔK are of the parametric form (3.1). It is known [10, Theorem 3.5] that for almost all values of the untouched eigenpair (Y, Z) this parametric representation is also necessary. Theorems 3.6 and 3.7 therefore are perhaps the most general results for model updating while preserving both positive definiteness and no spill-over.

4. Solvability under a specific structure. Excluding the nongeneric solutions might be feasible in practice. Still, it is interesting mathematically to ask whether there are other types of solutions $(\Delta M, \Delta C, \Delta K)$ to the MUP which do not assume the generic form. This section provides an affirmative answer to this question for the case $k = n$. We consider the simplest case that $Y = X$ and the measured eigenvalues σ_j 's have the same type of number (real or complex) as the corresponding λ_j 's.

Let the matrix $M_1 = X^\top M_0 X \in \mathbb{R}^{n \times n}$ be partitioned into $t \times t$ blocks,

$$(4.1) \quad M_1 = \begin{bmatrix} 1 & \cdots & 1 & 2 & \cdots & 2 \\ M_1^{(1,1)} & \cdots & M_1^{(1,s)} & M_1^{(1,s+1)} & \cdots & M_1^{(1,t)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M_1^{(s,1)} & \cdots & M_1^{(s,s)} & M_1^{(s,s+1)} & \cdots & M_1^{(s,t)} \\ M_1^{(s+1,1)} & \cdots & M_1^{(s+1,s)} & M_1^{(s+1,s+1)} & \cdots & M_1^{(s+1,t)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M_1^{(t,1)} & \cdots & M_1^{(t,s)} & M_1^{(t,s+1)} & \cdots & M_1^{(t,t)} \end{bmatrix} \begin{matrix} 1 \\ \vdots \\ 1 \\ 2 \\ \vdots \\ 2 \end{matrix},$$

where each block is of the size as indicated along the border. With respect to this partitioning, there exists a permutation matrix $P \in \mathbb{R}^{n \times n}$ among blocks such that

$$(4.2) \quad P^\top M_1 P = \text{diag} \{ \mathcal{M}_{11}, \dots, \mathcal{M}_{pp} \},$$

where each $\mathcal{M}_{ii}$ is block irreducible [31] which is composed of one or more blocks $M_1^{(g,h)}$ (if it exists). If $p = 1$, then M_1 is block irreducible, otherwise, if $p > 1$, then M_1 is block reducible. Let Γ be the same as that defined in (3.3). Denote

$$\begin{aligned} P^\top \Lambda P &= \text{diag} \{ \Lambda_{11}, \dots, \Lambda_{pp} \}, \\ P^\top \Sigma P &= \text{diag} \{ \Sigma_{11}, \dots, \Sigma_{pp} \}, \\ P^\top \Gamma P &= \text{diag} \{ \Gamma_{11}, \dots, \Gamma_{pp} \}, \end{aligned}$$

which is always possible because Λ defined in (2.2) and the corresponding Σ and Γ are block diagonal and have the same partitioning as M_1 .

Indeed, we can further refine the permutation matrix P mentioned above so that the following properties hold.

LEMMA 4.1. *There exists one permutation matrix $P \in \mathbb{R}^{n \times n}$ of blocks specified in (4.1) such that*

1. $\mathcal{M}_{11}$ is either empty or is diagonal with $\mathcal{M}_{11}(\Sigma_{11} - \Lambda_{11}) + \Gamma_{11} = 0$;
2. for $i = 2, \dots, p$, if $\mathcal{M}_{ii} \in \mathbb{R}$, then $\mathcal{M}_{ii}(\Sigma_{ii} - \Lambda_{ii}) + \Gamma_{ii} \neq 0$; otherwise, either $\mathcal{M}_{ii}$ has only one 2×2 block, or $\mathcal{M}_{ii}$ has at least two blocks and is block irreducible.

Proof. Let P be a permutation matrix such that (4.2) holds. For each $i = 1, \dots, p$, the submatrices $\mathcal{M}_{ii}$, Λ_{ii} , Σ_{ii} , and Γ_{ii} are of the same sizes. For each fixed i , there are three possible cases for the four blocks $\mathcal{M}_{ii}$, Λ_{ii} , Σ_{ii} , Γ_{ii} —they are all scalars; or they contain only one 2×2 block from M_1 , Λ , Σ , and Γ , respectively; or they have at least two blocks from M_1 , Λ , Σ , and Γ , respectively, and $\mathcal{M}_{ii}$ is block irreducible. In the first case, we may introduce one more round of permutation on the same partition (4.1) to group those with $\mathcal{M}_{ii}(\Sigma_{ii} - \Lambda_{ii}) + \Gamma_{ii} = 0$ together and rename the resulting block as $\mathcal{M}_{11}$. $\square$

To set forth the exploration of a “new” solution not in the generic form, we first develop some useful identities that any solution $(\Delta M, \Delta C, \Delta K)$ to (2.4) and (2.5) must satisfy. As a necessary condition, it should not be surprising that many of the facts derived in the preceding sections remain valid so long as their cogency is independent of any reference to Φ . For instance, we shall continue using (3.5) with an appropriate symmetric block diagonal matrix $\Gamma \in \mathbb{R}^{n \times n}$.

Since M_0 and X are nonsingular, we can also write

$$\Delta M = M_0 X \underbrace{(M_0 X)^{-1} \Delta M (M_0 X)^{-\top}}_{\Phi} X^{\top} M_0,$$

where Φ is symmetric. Without causing ambiguity, we use the same notation as before,

$$(4.3) \quad \begin{cases} \Delta M_1 &:= X^{\top} \Delta M X = M_1 \Phi M_1, \\ \Delta C_1 &:= X^{\top} \Delta C X, \\ \Delta K_1 &:= X^{\top} \Delta K X, \\ \mathcal{Z} &:= X^{-1} Z, \end{cases}$$

except that ΔC and ΔK are not necessarily in the parametric form in (3.1).

To satisfy (2.4), observe that

$$\begin{aligned} (M_0 + \Delta M) X \Sigma^2 + (C_0 + \Delta C) X \Sigma + (K_0 + \Delta K) X &= 0 \\ \Leftrightarrow (M_1 + M_1 \Phi M_1) \Sigma^2 + (C_1 + \Delta C_1) \Sigma + (K_1 + \Delta K_1) &= 0. \end{aligned}$$

The second equation above enables us to carry through the steps taken in the preceding section with only a few modifications. In particular, by Lemma 3.2, there exists a symmetric matrix Ξ which is block diagonal with a similar structure as that of Σ , satisfies

$$(4.4) \quad \Sigma^{\top} \Xi = \Xi \Sigma,$$

and is such that

$$(4.5) \quad \begin{cases} C_1 + \Delta C_1 &= \Xi - (M_1 + M_1 \Phi M_1) \Sigma - \Sigma^{\top} (M_1 + M_1 \Phi M_1), \\ K_1 + \Delta K_1 &= \Sigma^{\top} (M_1 + M_1 \Phi M_1) \Sigma - \Xi \Sigma = \Sigma^{\top} (M_1 + M_1 \Phi M_1) \Sigma - \Sigma^{\top} \Xi. \end{cases}$$

It follows that (see (3.12))

$$(4.6) \quad \begin{cases} \Delta M_1 &= M_1 \Phi M_1, \\ \Delta C_1 &= M_1 \Phi (\Gamma - M_1 \Lambda) + (\Lambda^{\top} - \Sigma^{\top} - \Sigma^{\top} M_1 \Phi) M_1 \\ &\quad + (M_1 + M_1 \Phi M_1) (\Lambda - \Sigma) + \Xi - \Gamma - M_1 \Phi \Gamma, \\ \Delta K_1 &= (\Lambda^{\top} - \Sigma^{\top} - \Sigma^{\top} M_1 \Phi) (\Gamma - M_1 \Lambda) \\ &\quad - \Sigma^{\top} [(M_1 + M_1 \Phi M_1) (\Lambda - \Sigma) + \Xi - \Gamma - M_1 \Phi \Gamma]. \end{cases}$$

Similarly, we can prove that (see (3.13))

$$(4.7) \quad M_1 \mathcal{Z} \Upsilon + (\Gamma - M_1 \Lambda) \mathcal{Z} = 0$$

and that (see (3.14))

$$(M_1 + M_1 \Phi M_1)(\Lambda - \Sigma) + \Xi - \Gamma - M_1 \Phi \Gamma = 0.$$

Hence, we obtain (see (3.16))

$$(4.8) \quad (I + M_1 \Phi)[M_1(\Sigma - \Lambda) + \Gamma] = \Xi.$$

Applying the permutation matrix P assumed in Lemma 4.1 to both sides of (4.8) and noting that $P^\top \Xi P$ is block diagonal due to the structure of Ξ , we see that

$$(4.9) \quad \begin{bmatrix} I + \mathcal{M}_{11} \Phi_{11} & \cdots & \mathcal{M}_{11} \Phi_{1p} \\ \vdots & \ddots & \vdots \\ \mathcal{M}_{pp} \Phi_{1p}^\top & \cdots & I + \mathcal{M}_{pp} \Phi_{pp} \end{bmatrix} \text{diag} \{ \mathcal{M}_{11}(\Sigma_{11} - \Lambda_{11}) + \Gamma_{11}, \dots, \mathcal{M}_{pp}(\Sigma_{pp} - \Lambda_{pp}) + \Gamma_{pp} \} \\ = \text{diag} \{ \Xi_{11}, \dots, \Xi_{pp} \},$$

where we have denoted the symmetric matrices $P^\top \Phi P$ and $P^\top \Xi P$ by

$$P^\top \Phi P = \begin{bmatrix} \Phi_{11} & \cdots & \Phi_{1p} \\ \vdots & \ddots & \vdots \\ \Phi_{1p}^\top & \cdots & \Phi_{pp} \end{bmatrix}, \\ P^\top \Xi P = \text{diag} \{ \Xi_{11}, \dots, \Xi_{pp} \}.$$

By Lemma 4.1, it must be the case that either Ξ_{11} is empty or $\Xi_{11} = 0$. In the latter, Φ_{11} is arbitrary but symmetric. From (4.9) we also see that the following two equalities hold:

$$(4.10) \quad \mathcal{M}_{11} \begin{bmatrix} \Phi_{12} & \cdots & \Phi_{1p} \end{bmatrix} \text{diag} \{ \mathcal{M}_{22}(\Sigma_{22} - \Lambda_{22}) + \Gamma_{22}, \dots, \mathcal{M}_{pp}(\Sigma_{pp} - \Lambda_{pp}) + \Gamma_{pp} \} = 0, \\ (4.11) \quad \begin{bmatrix} I + \mathcal{M}_{22} \Phi_{22} & \cdots & \mathcal{M}_{22} \Phi_{2p} \\ \vdots & \ddots & \vdots \\ \mathcal{M}_{pp} \Phi_{2p}^\top & \cdots & I + \mathcal{M}_{pp} \Phi_{pp} \end{bmatrix} \text{diag} \{ \mathcal{M}_{22}(\Sigma_{22} - \Lambda_{22}) + \Gamma_{22}, \dots, \mathcal{M}_{pp}(\Sigma_{pp} - \Lambda_{pp}) + \Gamma_{pp} \} \\ = \text{diag} \{ \Xi_{22}, \dots, \Xi_{pp} \}.$$

Now we are ready to characterize necessary conditions for the solvability of the MUP. The emphasis is that $(\Delta M, \Delta C, \Delta K)$ does not assume the parametric form described in (3.1). The first condition is clear—since $K_0 + \Delta K$ is positive definite, the matrix Σ of updated eigenvalues must be nonsingular, thus, Σ_{ii} ($i = 1, \dots, p$) are also nonsingular.

We claim the following intermediate result concerning Ξ .

LEMMA 4.2. *The matrix $\text{diag} \{ \Xi_{22}, \dots, \Xi_{pp} \}$ is nonsingular.*

Proof. We prove by contradiction. Suppose that Ξ_{ii} is singular for some $i = 2, \dots, p$. Recall that Ξ_{ii} is of size either 1×1 or 2×2 .

If $\mathcal{M}_{ii}$ is a scalar, then $\Xi_{ii} = 0$. Since $M_1 + M_1 \Phi M_1 = X^\top (M_0 + \Delta M) X$ is positive definite, the matrix

$$\begin{bmatrix} I + \mathcal{M}_{22} \Phi_{22} & \cdots & \mathcal{M}_{22} \Phi_{2p} \\ \vdots & \ddots & \vdots \\ \mathcal{M}_{pp} \Phi_{2p}^\top & \cdots & I + \mathcal{M}_{pp} \Phi_{pp} \end{bmatrix}$$

is nonsingular. The corresponding entry in (4.11) would force $\mathcal{M}_{ii}(\Sigma_{ii} - \Lambda_{ii}) + \Gamma_{ii} = 0$, which is impossible by the way we choose P in Lemma 4.1.

If $\mathcal{M}_{ii}$ consists of only a 2×2 block, the corresponding Ξ_{ii} is of the form $\Xi_{ii} = \begin{bmatrix} \xi_i & \eta_i \\ \eta_i & -\xi_i \end{bmatrix}$. The singularity of Ξ_{ii} would imply $\Xi_{ii} = 0$. The corresponding block in (4.11) would imply $\mathcal{M}_{ii}(\Sigma_{ii} - \Lambda_{ii}) + \Gamma_{ii} = 0$. By the structures that are inherent in 2×2 matrices Σ_{ii} , Λ_{ii} , and Γ_{ii} , it follows that $\mathcal{M}_{ii} = -(\Sigma_{ii} - \Lambda_{ii})^{-1}\Gamma_{ii}$ would have trace zero, which is impossible because $\mathcal{M}_{ii}$ is positive definite.

Finally, if $\mathcal{M}_{ii}$ consists of at least two blocks and is block irreducible, we may assume without loss of generality that the first diagonal block of Ξ_{ii} is singular which, regardless of its size, must be zero by the argument above. The corresponding first block column of $\mathcal{M}_{ii}(\Sigma_{ii} - \Lambda_{ii}) + \Gamma_{ii}$ therefore would be zero. But this is again impossible, because $\mathcal{M}_{ii}$ is block irreducible, $\Sigma_{ii} - \Lambda_{ii}$ is nonsingular, and both $\Sigma_{ii} - \Lambda_{ii}$ and Γ_{ii} are block diagonal. $\square$

From (4.11), we thus can write

$$\begin{bmatrix} I + \mathcal{M}_{22}\Phi_{22} & \cdots & \mathcal{M}_{22}\Phi_{2p} \\ \vdots & \ddots & \vdots \\ \mathcal{M}_{pp}\Phi_{2p}^\top & \cdots & I + \mathcal{M}_{pp}\Phi_{pp} \end{bmatrix} \\ = \text{diag}\{\Xi_{22}, \dots, \Xi_{pp}\} \text{diag}\{\mathcal{M}_{22}(\Sigma_{22} - \Lambda_{22}) + \Gamma_{22}, \dots, \mathcal{M}_{pp}(\Sigma_{pp} - \Lambda_{pp}) + \Gamma_{pp}\}^{-1},$$

which together with (4.11) implies that

$$(4.12) \quad \begin{cases} \Phi_{ij} = 0 & \text{if } i \neq j, \\ \Phi_{ii} = \mathcal{M}_{ii}^{-1} \{ \Xi_{ii} [\mathcal{M}_{ii}(\Sigma_{ii} - \Lambda_{ii}) + \Gamma_{ii}]^{-1} - I \} & \text{if } i = 2, \dots, p. \end{cases}$$

In other words, we have just proved the following lemma.

LEMMA 4.3. Suppose that ΔM is part of a solution to the MUP. Let $\Phi := (M_0 X)^{-1} \Delta M (M_0 X)^{-\top}$ and let P be the permutation matrix specified in Lemma 4.1. Then

$$P^\top \Phi P = \text{diag}\{\Phi_{11}, \dots, \Phi_{pp}\},$$

where Φ_{11} is arbitrary but $\mathcal{M}_{11} + \mathcal{M}_{11}\Phi_{11}\mathcal{M}_{11}$ must be positive definite and Φ_{ii} , $i = 2, \dots, p$, is given by (4.12).

From the fact that $K_1 = \Lambda^\top M_1 \Lambda - \Gamma \Lambda$, we write

$$P^\top K_1 P = \text{diag}\{\mathcal{K}_{11}, \dots, \mathcal{K}_{pp}\},$$

with

$$(4.13) \quad \mathcal{K}_{ii} := \Lambda_{ii}^\top \mathcal{M}_{ii} \Lambda_{ii} - \Gamma_{ii} \Lambda_{ii} = \Lambda_{ii}^\top \mathcal{M}_{ii} \Lambda_{ii} - \Lambda_{ii}^\top \Gamma_{ii}, \quad i = 1, \dots, p.$$

It follows that for $i = 2, \dots, p$, the matrix

$$(4.14) \quad \Xi_{ii}(\Lambda_{ii}^\top \mathcal{M}_{ii} \Sigma_{ii} - \mathcal{K}_{ii})^{-1} \Lambda_{ii}^\top \mathcal{M}_{ii} = \Xi_{ii}[(\Sigma_{ii} - \Lambda_{ii}) + \mathcal{M}_{ii}^{-1} \Gamma_{ii}]^{-1} = \mathcal{M}_{ii} + \mathcal{M}_{ii} \Phi_{ii} \mathcal{M}_{ii}$$

is positive definite because $M_1 + M_1 \Phi M_1$ is positive definite.

Exploiting the structure of Ξ_{ii} and Γ_{ii} and rearranging the diagonal blocks if necessary, we may assume without loss of generality that

$$\Xi_{ii}^{-1} \Gamma_{ii} = \text{diag}\{\varsigma_1, \dots, \varsigma_{\ell_1}, \Psi_{\ell_1+1}, \dots, \Psi_{\ell_1+\ell_2}\},$$

where $\varsigma_i \in \mathbb{R}$ for $i = 1, \dots, \ell_1$, and $\Psi_{\ell_1+j} = \begin{bmatrix} \varsigma_{\ell_1+j} & \vartheta_{\ell_1+j} \\ -\vartheta_{\ell_1+j} & \varsigma_{\ell_1+j} \end{bmatrix}$ for $j = 1, \dots, \ell_2$. Likewise, let $\mathcal{M}_{ii}$ be partitioned accordingly as

$$\mathcal{M}_{ii} = \begin{bmatrix} \Theta_{1,1} & \cdots & \Theta_{1,\ell_1} & \Theta_{1,\ell_1+1} & \cdots & \Theta_{1,\ell_1+\ell_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Theta_{1,\ell_1}^\top & \cdots & \Theta_{\ell_1,\ell_1} & \Theta_{\ell_1,\ell_1+1} & \cdots & \Theta_{\ell_1,\ell_1+\ell_2} \\ \Theta_{1,\ell_1+1}^\top & \cdots & \Theta_{\ell_1,\ell_1+1}^\top & \Theta_{\ell_1+1,\ell_1+1} & \cdots & \Theta_{\ell_1+1,\ell_1+\ell_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Theta_{1,\ell_1+\ell_2}^\top & \cdots & \Theta_{\ell_1,\ell_1+\ell_2}^\top & \Theta_{\ell_1+1,\ell_1+\ell_2}^\top & \cdots & \Theta_{\ell_1+\ell_2,\ell_1+\ell_2} \end{bmatrix}.$$

Note that by (4.14) and the symmetry of Ξ_{ii} and Γ_{ii} , it is easy to show that for $i = 2, \dots, p$, the matrix $[\Gamma_{ii}\mathcal{M}_{ii}^{-1} + (\Sigma_{ii} - \Lambda_{ii})^\top] \Xi_{ii}$ is symmetric. Since $(\Sigma_{ii} - \Lambda_{ii})^\top \Xi_{ii}$ is symmetric, so is the matrix $\Gamma_{ii}\mathcal{M}_{ii}^{-1}\Xi_{ii}$ and, hence, $\mathcal{M}_{ii}\Xi_{ii}^{-1}\Gamma_{ii}$. It follows that for all $j = 1, \dots, \ell_2$, the diagonal block $\Theta_{\ell_1+j,\ell_1+j}\Psi_{\ell_1+j}$ is symmetric, which can be true only if $\vartheta_{\ell_1+j} = 0$ and, hence, $\Xi_{ii}^{-1}\Gamma_{ii}$ must be a diagonal matrix

$$(4.15) \quad \Xi_{ii}^{-1}\Gamma_{ii} = \text{diag}\{\varsigma_1, \dots, \varsigma_{\ell_1}, \varsigma_{\ell_1+1}I_2, \dots, \varsigma_{\ell_1+\ell_2}I_2\}.$$

We draw the following conclusion which is part of the necessary condition.

LEMMA 4.4. *Let Γ be the matrix defined by (3.3). For $i = 2, \dots, p$,*

1. *if $\Gamma_{ii} \neq 0$, then $\Gamma_{ii}(\Lambda_{ii}^\top \mathcal{M}_{ii} \Sigma_{ii} - \mathcal{K}_{ii})^{-1} \Lambda_{ii}^\top \mathcal{M}_{ii}$ and $\Gamma_{i,i} \Sigma_{i,i} (\Lambda_{ii}^\top \mathcal{M}_{ii} \Sigma_{ii} - \mathcal{K}_{ii})^{-1} \mathcal{K}_{ii}$ are either both positive definite or both negative definite;*
2. *if $\Gamma_{ii} = 0$, then Λ_{ii} is diagonal and $\Lambda_{ii} \Sigma_{ii}$ is positive definite.*

Proof. The facts that $\mathcal{M}_{ii}$ is block irreducible (by Lemma 4.1), (4.15) holds, and $\mathcal{M}_{ii}\Xi_{ii}^{-1}\Gamma_{ii}$ is symmetric imply that $\Xi_{ii}^{-1}\Gamma_{ii} = \eta_i I$ for some scalar η_i . Thus it is always true that

$$(4.16) \quad \Gamma_{ii} = \eta_i \Xi_{ii}.$$

If $\Gamma_{ii} \neq 0$, then $\eta_i \neq 0$ and by (4.14) we see that $\frac{1}{\eta_i} \Gamma_{ii} (\Lambda_{ii}^\top \mathcal{M}_{ii} \Sigma_{ii} - \mathcal{K}_{ii})^{-1} \Lambda_{ii}^\top \mathcal{M}_{ii}$ is positive definite. If $\Gamma_{ii} = 0$, then by (4.13) we have $\mathcal{K}_{ii} = \Lambda_{ii}^\top \mathcal{M}_{ii} \Lambda_{ii}$. By (4.14) again, we see that $\Xi_{ii}(\Sigma_{ii} - \Lambda_{ii})^{-1}$ is positive definite. In this case, suppose that Λ_{ii} is not diagonal; then $\Sigma_{ii} - \Lambda_{ii}$ has at least one nonsingular diagonal block of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Correspondingly, Ξ_{ii} has a diagonal block of the form $\begin{bmatrix} \xi & \eta \\ \eta & -\xi \end{bmatrix}$. It follows that $\Xi_{ii}(\Sigma_{ii} - \Lambda_{ii})^{-1}$ has a diagonal block with trace zero, which contradicts the fact that $\Xi_{ii}(\Sigma_{ii} - \Lambda_{ii})^{-1}$ is positive definite. Hence, Λ_{ii} must be diagonal.

We have one more item to check, namely, the positive definiteness of $K_0 + \Delta K$. From (4.5), it is clear that $P^\top(K_1 + \Delta K_1)P$ is block diagonal and that its i th diagonal block is given by $\Sigma_{ii}^\top(\mathcal{M}_{ii} + \mathcal{M}_{ii}\Phi_{ii}\mathcal{M}_{ii})\Sigma_{ii} - \Sigma_{ii}^\top\Xi_{ii}$, which is positive definite. Obviously, by (3.9), $\Sigma_{ii}^\top\Xi_{ii} = \Xi_{ii}\Sigma_{ii}$ for all i . Using (4.14), we can write

$$(4.17) \quad \begin{aligned} \Sigma_{ii}^\top(\mathcal{M}_{ii} + \mathcal{M}_{ii}\Phi_{ii}\mathcal{M}_{ii})\Sigma_{ii} - \Sigma_{ii}^\top\Xi_{ii} &= \Sigma_{ii}^\top\Xi_{ii}(\Lambda_{ii}^\top \mathcal{M}_{ii} \Sigma_{ii} - \mathcal{K}_{ii})^{-1} \Lambda_{ii}^\top \mathcal{M}_{ii} \Sigma_{ii} - \Sigma_{ii}^\top\Xi_{ii} \\ &= \Xi_{ii}\Sigma_{ii}(\Lambda_{ii}^\top \mathcal{M}_{ii} \Sigma_{ii} - \mathcal{K}_{ii})^{-1} \Lambda_{ii}^\top \mathcal{M}_{ii} \Sigma_{ii} - \Xi_{ii}\Sigma_{ii} \\ &= \Xi_{ii}\Sigma_{ii}(\Lambda_{ii}^\top \mathcal{M}_{ii} \Sigma_{ii} - \mathcal{K}_{ii})^{-1} \mathcal{K}_{ii}, \end{aligned}$$

which is positive definite for $i = 2, \dots, p$. Thus $\frac{1}{\eta_i} \Gamma_{ii} \Sigma_{ii} (\Lambda_{ii}^\top \mathcal{M}_{ii} \Sigma_{ii} - \mathcal{K}_{ii})^{-1} \mathcal{K}_{ii}$ is positive definite when $\Gamma_{ii} \neq 0$. On the other hand, recall from (4.13) that if $\Gamma_{ii} = 0$,

then $\mathcal{K}_{ii} = \Lambda_{ii}^\top \mathcal{M}_{ii} \Lambda_{ii}$. In this case, we can further reduce (4.17) to

$$(4.18) \quad \Sigma_{ii}^\top (\mathcal{M}_{ii} + \mathcal{M}_{ii} \Phi_{ii} \mathcal{M}_{ii}) \Sigma_{ii} - \Sigma_{ii}^\top \Xi_{ii} = \Xi_{ii} \Sigma_{ii} (\Sigma_{ii} - \Lambda_{ii})^{-1} \Lambda_{ii} = \Xi_{ii} (\Sigma_{ii} - \Lambda_{ii})^{-1} \Lambda_{ii} \Sigma_{ii},$$

which is positive definite. By the fact that $\Xi_{ii} (\Sigma_{ii} - \Lambda_{ii})^{-1}$ is positive definite, we conclude that $\Lambda_{ii} \Sigma_{ii}$ is positive definite. $\square$

We summarize the above necessary conditions in the following theorem. The most interesting development is that the necessary conditions are also sufficient.

THEOREM 4.5. *Suppose that $k = n$ and that $Y = X$. Let Γ be defined by (3.3) and P be defined by Lemma 4.1. Then the MUP is solvable if and only if the matrix Σ is nonsingular and for $i = 2, \dots, p$, the following conditions among the corresponding blocks defined by P hold:*

1. *If $\Gamma_{ii} \neq 0$, then $\Gamma_{ii} (\Lambda_{ii}^\top \mathcal{M}_{ii} \Sigma_{ii} - \mathcal{K}_{ii})^{-1} \Lambda_{ii}^\top \mathcal{M}_{ii}$ and $\Gamma_{ii} \Sigma_{ii} (\Lambda_{ii}^\top \mathcal{M}_{ii} \Sigma_{ii} - \mathcal{K}_{ii})^{-1} \mathcal{K}_{ii}$ are either both positive definite or both negative definite.*
2. *If $\Gamma_{ii} = 0$, then Λ_{ii} is diagonal and $\Lambda_{ii} \Sigma_{ii}$ is positive definite.*

Proof. Only the sufficiency needs to be proved. It will be most informative if we prove the sufficiency by constructing the solution $(\Delta M, \Delta C, \Delta K)$.

Clearly, Λ is nonsingular because K_0 is positive definite. We may therefore select a symmetric and positive definite matrix Φ_{11} such that

$$\Delta \mathcal{K}_{11} := \Sigma_{11}^\top (\mathcal{M}_{11} + \mathcal{M}_{11} \Phi_{11} \mathcal{M}_{11}) \Sigma_{11} - \mathcal{K}_{11}$$

is positive definite. Set $\Xi_{11} = 0$. For $i = 2, \dots, p$ and if $\Gamma_{ii} \neq 0$, we can choose by assumption a scalar $\omega_i \in \mathbb{R}$ such that both matrices Φ_{ii} and $\Delta \mathcal{K}_{ii}$ defined by

$$\begin{aligned} \Phi_{ii} &:= \mathcal{M}_{ii}^{-1} [\omega_i \Gamma_{ii} (\Lambda_{ii}^\top \mathcal{M}_{ii} \Sigma_{ii} - \mathcal{K}_{ii})^{-1} \Lambda_{ii}^\top \mathcal{M}_{ii} - \mathcal{M}_{ii}] \mathcal{M}_{ii}^{-1}, \\ \Delta \mathcal{K}_{ii} &:= \omega_i \Gamma_{ii} \Sigma_{ii} (\Lambda_{ii}^\top \mathcal{M}_{ii} \Sigma_{ii} - \mathcal{K}_{ii})^{-1} \mathcal{K}_{ii} - \mathcal{K}_{ii} \end{aligned}$$

are positive definite. Set $\Xi_{ii} = \omega_i \Gamma_{ii}$. Similarly, if $\Gamma_{ii} = 0$, we may choose a diagonal matrix Ξ_{ii} such that both matrices Φ_{ii} and $\Delta \mathcal{K}_{ii}$ defined by

$$\begin{aligned} \Phi_{ii} &:= \mathcal{M}_{ii}^{-1} [\Xi_{ii} (\Sigma_{ii} - \Lambda_{ii})^{-1} - \mathcal{M}_{ii}] \mathcal{M}_{ii}^{-1}, \\ \Delta \mathcal{K}_{ii} &:= \Xi_{ii} (\Sigma_{ii} - \Lambda_{ii})^{-1} \Lambda_{ii} \Sigma_{ii} - \mathcal{K}_{ii} \end{aligned}$$

are positive definite. The matrix

$$\Phi := P \text{diag} \{ \Phi_{11}, \dots, \Phi_{pp} \} P^\top$$

is positive definite. By construction and the definition (4.13), (4.8) is satisfied. Furthermore, defining

$$(4.19) \quad \begin{cases} \Delta M := M_0 X \Phi X^\top M_0, \\ \Delta C := X^{-\top} [-X^\top M_0 X \Phi \Lambda^{-\top} X^\top K_0 X + (\Lambda^\top - \Sigma^\top - \Sigma^\top X^\top M_0 X \Phi) X^\top M_0 X] X^{-1}, \\ \Delta K := -X^{-\top} [(\Lambda^\top - \Sigma^\top - \Sigma^\top X^\top M_0 X \Phi) \Lambda^{-\top} X^\top K_0 X] X^{-1}, \end{cases}$$

we find by using (3.5) and (4.8) that the corresponding $(\Delta M_1, \Delta C_1, \Delta K_1)$ defined in (4.19) can be expressed as

$$\begin{cases} \Delta M_1 &= M_1 \Phi M_1, \\ \Delta C_1 &= \Xi - (M_1 + M_1 \Phi M_1) \Sigma - \Sigma^\top (M_1 + M_1 \Phi M_1) - C_1, \\ \Delta K_1 &= \Sigma^\top (M_1 + M_1 \Phi M_1) \Sigma - \Sigma^\top \Xi - K_1, \end{cases}$$

showing that

$$(M_0 + \Delta M) \begin{bmatrix} X & Z \end{bmatrix} \begin{bmatrix} \Sigma & \\ & \Upsilon \end{bmatrix}^2 + (C_0 + \Delta C) \begin{bmatrix} X & Z \end{bmatrix} \begin{bmatrix} \Sigma & \\ & \Upsilon \end{bmatrix} + (K_0 + \Delta K) \begin{bmatrix} X & Z \end{bmatrix} = 0.$$

Note that $M_0 + \Delta M = M_0 + M_0 X \Phi X^T M_0$ is positive definite. Since ΔK satisfies

$$P^T \Delta K_1 P = \text{diag} \{ \Delta \mathcal{K}_{11}, \dots, \Delta \mathcal{K}_{pp} \},$$

which by construction is positive definite, we conclude that $K_0 + \Delta K$ is also positive definite and that the MUP is solved. $\square$

We stress again that our goal in this section is to explore a solution that is not in the parametric form (3.1) assumed in the preceding section. It is important to note that the “parameter matrix” Φ in the above proof does not enter the solution $(\Delta M, \Delta C, \Delta K)$ in (4.19) in the same way as that in the generic form characterized by (3.1).

5. Conclusions. Updating a real symmetric quadratic model while preserving positive definiteness and no spill-over is a quadratic inverse eigenvalue problem and remains a fundamental challenge in the field. In this paper, we have made some advances toward this challenge. Our main contributions of the present work are twofold:

1. Theorems 3.6 and 3.7 provide necessary and sufficient solvability conditions for the underlying problem, when the triplet $(\Delta M, \Delta C, \Delta K)$ assumes the parametric form (3.1). The parametric representation is known to be generic. Our results therefore solve the MUP from a practical point of view.
2. Theorem 4.5 characterizes another necessary and sufficient solvability condition of the underlying problem for the case $k = n$ and $Y = X$ while not using the parametric form.

We remark that the techniques developed in section 4 give a complete answer to the MUP, but only under the assumption that precisely n eigenvalues are to be updated. Theorem 4.5 therefore includes and generalizes Theorems 3.6 and 3.7 only under this special circumstance. The techniques cannot be carried through under other scenarios such as fewer than n updated eigenpairs.

Our study represents some new steps toward the understanding of the MUP. Many questions remain open, including the preservation of semidefiniteness or skewness of the damping matrix C and the structured problem.

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