

Theory and Methodology

A periodic review inventory system with two supply modes

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Abstract

We describe a periodic review inventory system in which there are two modes of resupply, namely a regular mode and an emergency mode. Orders placed through the emergency channel have a shorter supply lead time but are subject to higher ordering costs compared to orders placed through the regular channel. We analyze this problem within the framework of an order-up-to- R inventory control policy. At each review epoch, the inventory manager must decide which of the two supply modes to use and then order enough units to raise the inventory position to a level R . We show that given any non-negative order-up-to level, either only the regular supply mode is used, or there exists an indifference inventory level such that if the inventory position at the review epoch is below the indifference inventory level, the emergency supply mode is used. We also develop procedures for solving for the two policy parameters, i.e., the order-up-to level and the indifference inventory level.

Keywords: Inventory

1. Introduction

Emergency or expedited shipments are common in inventory systems. When inventory positions get dangerously low relative to the demand anticipated until the next replenishment arrives, inventory managers are willing to pay the higher replenishment costs associated with faster delivery times in order to avoid unduly large shortages. In general, the costs associated with an emergency shipment may include a combination of higher unit prices and/or higher transportation costs.

When the items must be manufactured to order, production may have to be expedited in order to meet tighter shipping due dates. Even if manufacturing costs do not increase for the supplier, the tighter shipping schedule strengthens the bargaining position of the supplier which typically will demand a higher price per unit for the emergency shipment. On the other hand, if the items are commodities or standardized products that are manufactured to stock, the price per unit of the emergency order will not be likely to increase.

Faster delivery times often require different, more expensive modes of transportation. If regular orders are shipped by rail, expedited orders may have to be shipped by truck or air at a cost premium. In some cases, fixed order set-up costs are incurred due to the structure of freight rates. Increasingly companies are negotiating

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simplified trucking rates: a flat rate per mile regardless of whether the truck is full or half-empty [16]. Regular shipments may include a mix of products in the same truck. However, if a shipment must be expedited, a dedicated truck may be used and its cost is charged to the emergency order. In these cases, small variations in the order size will not affect the transportation cost of the order.

The higher costs of an emergency order can be modeled as a higher cost per unit, a higher set-up cost, or both. In this paper, we consider the case in which the cost per unit of an emergency order does not increase, but there is a set-up cost associated with each emergency order. (See [5] for the case in which the set-up cost of an emergency order is the same as that of a regular order, but the cost per unit of an emergency order is larger.) This model will be generally applicable for standardized items or commodities that are manufactured to stock. The emergency order set-up costs are due to the flat freight rates. Expediting an order often requires a dedicated truck for the shipment, thus incurring a fixed cost whose magnitude is given by the truck flat rate.

Studies in this area date back to Barankin [3] who develops an one-product single-period inventory model where two supply options are available with fixed time lag of one and zero periods respectively. Daniel [6], Neuts [14], Bulinskaya [4], Fukuda [7], and Veinott [18] extend the analysis to the n -period case and derive similar forms of the optimal policy. Fukuda and Veinott also consider the situation where a fixed order set-up cost is present. Wright [20] further investigates the inventory system with multiple products. However, they all restrict attention to the case where the expedited and regular lead times differ by one period only. Whittmore and Saunders [19] analyze the n -period inventory problem by allowing the expedited and regular lead times to be of arbitrary lengths. Unfortunately, the form of the optimal policy they derive is extremely complex. They are able to obtain explicit results only for the case where the two lead times differ by one period. Different models for the periodic review system have also been considered by Gross and Soriano [8] and Rosenshine and Obee [15]. For the continuous review system, Allen and D'Esopo [2] propose an ordering policy which is of the familiar reorder-point–order-quantity type, except that an additional level called the expediting level is specified, and Moinszadeh and Nahmias [12] extend Allen and D'Esopo's analysis and develop an approximate model where an expedited order is placed if the on-hand inventory reduces to the expediting level. Other models are considered by Hadley and Whitin [9], Moinszadeh and Schmidt [13], and Aggarwal and Moinszadeh [1].

In this paper, we study in detail a periodic review inventory system in which there is a faster (emergency) supply channel in addition to the slow (regular) supply channel. Earlier works mentioned above on periodic review models have focused on the situation where supply lead times is a multiple of a review period. Such models could be regarded as an approximation of continuous review inventory models, as the review periods can be modeled as small as, say, one working day. Different from the earlier works, our model considers periodic systems where supply lead times could be smaller than the review periods. For example, A retailer may place regular replenishment orders biweekly or monthly while the supply lead time is of the order of one week. To the best of our knowledge the utilization of emergency shipments for this class of problems has not been studied previously. However, this class of problems is important since periodic systems with relatively long review periods (with respect to supply lead times) are commonly used to achieve economies in the inventory review as well as in the coordination and consolidation of orders for different items. A common way to implement the coordination mechanism is through a periodic review system in which on a given date the inventory level of all items from a same supplier are reviewed and the appropriate orders are placed.

We concentrate on the case in which the purchasing cost per unit is the same between the two supply modes, and there is a positive fixed order set-up cost S associated with the emergency supply mode. The fixed set-up cost of S represents the extra expense of making a special arrangement with the supplier – for example, using a specialized vehicle to deliver only the emergency order. The fixed set-up cost for the regular mode is usually zero, since a regular order for an item is part of a joint order that includes a mix of products shipped in the same truck. If there is a positive set-up cost associated with the regular mode, then S is the *additional* expense for using the emergency mode.

We analyze the problem within the framework of an order-up-to- R inventory control policy, i.e., we assume that every time period of length T , we review the inventory and place an order which will raise the inventory

level to R (see, e.g., [10]). Such an operating policy, known as a replenishment cycle system, is in common use (see, e.g., [17]), especially for relatively long review periods (see, e.g., [11, p. 234]). When only a single supply mode exists, the order is placed using that single mode periodically. However, if two supply modes exist, the order may be placed using either supply mode, depending on the inventory position (inventory on hand plus inventory on order minus backorder) H at the review epoch.

We will show that given any $R \geq 0$, either only the regular supply mode is employed, or there exists an indifference inventory level I such that (i) if $H \geq I$, the regular supply mode is used and (ii) otherwise if $H < I$, the emergency supply mode is used. As a result, there are three policy parameters: (T, R, I) . In this paper, we assume that T is not a decision variable and is exogenously determined. Thus, our goal is to determine the optimal combination of R and I by minimizing the average total cost per review period of length T .

The control policy we derive is a natural extension of the ordinary (R, T) policy. It is intuitively appealing that a faster supply mode may need to be employed if the inventory position at the review epoch is low. We show that the savings of the proposed policy are realized from less inventory holding or reduced shortages, compared to the ordinary one-supply-mode (R, T) policy.

2. Assumptions and notation

Suppose that we are now reviewing the inventory at time kT (k is an integer) and we want to decide which supply mode to use. There are two supply modes available for which the unit item cost is the same. The first (regular) supply mode has a (deterministic) lead time τ_1 and the second (emergency) mode τ_2 , where $\tau_2 < \tau_1$. Assume that costs are levied against:

1. Holding inventory at a rate of h per unit held per unit time.
2. Stockout at π per unit of unsatisfied demand, charged at the time it is satisfied.
3. Fixed ordering cost of S per order with the emergency supply mode.

We assume that the lead time for the emergency supply mode is less than the length of review periods, i.e., $\tau_2 < T$, and that there is no order cross-over, i.e., $\tau_3 = \tau_1 - \tau_2 < T$. Also, we assume that the same order-up-to level $R \geq 0$ is used for either supply mode (based on the same purchasing cost per unit). Thus, the inventory process restarts itself at the inventory position R at every review epoch (see Fig. 1) and decisions can be made period by period at review epochs $T, 2T, \dots, kT, \dots$, based on the expected cost for each supply mode over the upcoming time interval $[kT + \tau_2, (k+1)T + \tau_2)$ of length T (note that the expected cost incurred before time $kT + \tau_2$ has been determined by the decision made in the previous review).

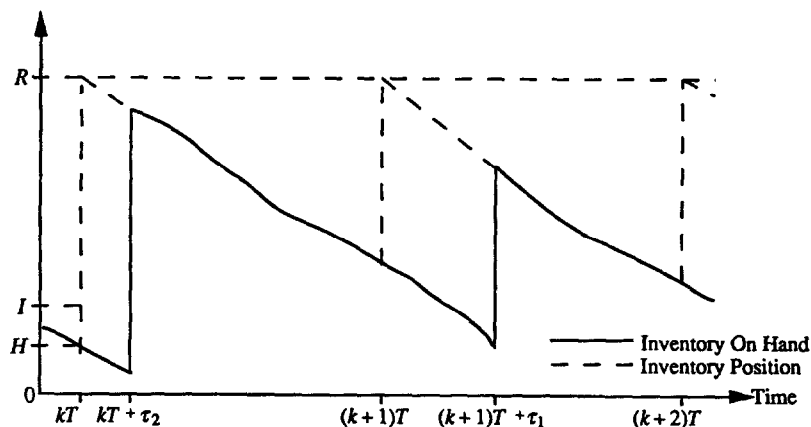


Fig. 1. Inventory position as function of time.

Finally, it is assumed that demand is non-negative and independently distributed in disjoint time intervals, and that demand X in a time period of length τ has mean $\lambda\tau$, where λ is the mean demand rate per unit time. For convenience of notation, demand is assumed to be continuous for the remainder of the presentation.

In addition to the cost and policy parameters defined above, we will refer to the following notation.

$(\cdot)^+ = \max\{\cdot, 0\}$.

Df = the first derivative of the function f .

X_0 = demand during the previous time interval $[(k-1)T, kT)$. $X_0 = R - H$.

X_1 = demand during the upcoming time interval $[kT, kT + \tau_1)$.

X_2 = demand during the upcoming time interval $[kT, kT + \tau_2)$.

X_3 = demand during the time period $[kT + \tau_2, kT + \tau_1)$. $X_3 = X_1 - X_2$ and $E(X_3) = \lambda\tau_3$.

X_4 = demand during the time period $[kT + \tau_1, (k+1)T + \tau_2)$. $E(X_4) = \lambda(T - \tau_3)$.

$f_i(X_i)$ = the probability density function (p.d.f.) of X_i , $i = 0, 1, \dots, 4$.

$F_i(X_i)$ = the cumulative probability distribution of X_i , $i = 0, 1, \dots, 4$.

$C_n(H, R)$ (resp. $C_f(H, R)$) = the expected cost for the upcoming time interval $[kT + \tau_2, (k+1)T + \tau_2)$ given that the current inventory position is H and only the regular (resp. emergency) supply mode is used.

$C_n(R)$ (resp. $C_f(R)$) = the expected cost for the upcoming time interval $[kT + \tau_2, (k+1)T + \tau_2)$ given that only the regular (resp. emergency) supply mode is used. $C_n(R) = E_{x_0}[C_n(R - X_0, R)]$ and $C_f(R) = E_{x_0}[C_f(R - X_0, R)]$.

$C_m(R)$ = the expected cost for the upcoming time interval $[kT + \tau_2, (k+1)T + \tau_2)$ given that either supply mode can be used. $C_m(R)$ can be expressed as

$$C_m(R) = E_{x_0}[\min\{C_n(R - X_0, R), C_f(R - X_0, R)\}]. \quad (1)$$

3. Development of the model

In this section, we develop explicit expressions for the cost functions $C_n(R - X_0, R)$ and $C_f(R - X_0, R)$. Notice that the on-hand inventory will be larger during the upcoming period $[kT + \tau_2, kT + \tau_1)$ if the emergency supply mode is used; however, it will be the same for both supply modes during the period $[kT + \tau_1, (k+1)T + \tau_2)$. Thus, in deriving inventory cost expressions for $C_n(R - X_0, R)$ and $C_f(R - X_0, R)$, we need to consider the two disjoint time intervals $[kT + \tau_2, kT + \tau_1)$ and $[kT + \tau_1, (k+1)T + \tau_2)$ separately.

We first evaluate the inventory holding cost during the period $[kT + \tau_2, kT + \tau_1)$ for the regular supply mode. If $H \leq X_2$, then the on-hand inventory is zero for any time during the period $[kT + \tau_2, kT + \tau_1)$. However, if $H > X_2$ and the demand X_3 during the lead time τ_3 is less than or equal to $H - X_2$, then the average inventory holding cost over the period $[kT + \tau_2, kT + \tau_1)$ is $h\tau_3(H - X_2 - 0.5X_3)$. On the other hand, if $H > X_2$ and $X_3 > H - X_2$, the average inventory holding cost is $0.5h\tau_3(H - X_2)^2/X_3$. As the variable X_3 appears in the denominator, this exact expression will substantially complicate the subsequent analysis. Instead, our approach in this paper is to bound $0.5h\tau_3(H - X_2)^2/X_3$. A lower bound for this expression is $h\tau_3(H - X_2 - 0.5X_3)$. A simple upper bound for it is $0.5h\tau_3(H - X_2)$. To summarize, if the regular supply mode is used and $H > X_2$, a lower bound for the average inventory holding cost over the period $[kT + \tau_2, kT + \tau_1)$ is $h\tau_3(H - X_2 - 0.5X_3)$ and when taking the expected value of X_3 , becomes $h\tau_3(H - X_2 - 0.5\lambda\tau_3)$, and an upper bound is $h\tau_3[H - X_2 - 0.5\min\{X_3, H - X_2\}]$.

Likewise, if the emergency supply mode is used and if $R > X_2$, a lower bound for the average inventory holding cost over the period $[kT + \tau_2, kT + \tau_1)$ is $h\tau_3(R - X_2 - 0.5\lambda\tau_3)$ and an upper bound is $h\tau_3[R - X_2 - 0.5\min\{X_3, R - X_2\}]$.

Similarly, we can develop inventory cost approximations for both supply modes during the upcoming period $[kT + \tau_1, (k+1)T + \tau_2)$.

As a result of the approximations used for the inventory cost expression, we have a lower bound problem as well as an upper bound problem, in addition to the exact problem. In what follows, we use an underbar (resp. upperbar) wherever it is appropriate to indicate the lower (resp. upper) bound problem. Our approach is to analyze and solve the lower bound problem $\underline{C}_m(R)$ instead of the considerably more complex exact problem $C_m(R)$. Once $\underline{C}_m(R)$ is solved and the optimal R (denoted by R_m) is obtained, we compute $\bar{C}_m(R_m)$ to bound the exact optimal cost denoted by C_m^* , as $\underline{C}_m(R_m) \leq C_m^* \leq \bar{C}_m(R_m)$. Notice also that $\underline{C}_m(R_m) \leq C_m^* \leq C_m(R_m) \leq \bar{C}_m(R_m)$.

Next, we develop the shortage cost expression for $C_n(H, R)$. We consider two possible cases if shortage occurs: first, the regular order which arrives at the upcoming epoch $kT + \tau_1$ will clear the backorder (i.e., if $X_1 \leq R$), and second, the regular order fails to clear the backorder (i.e., if $X_1 > R$). In the former case, the amount of shortage is $(X_1 - H)^+$, while in the latter case, the amount of shortage is the quantity ordered, i.e., $R - H$ (recall that we assume that shortage costs are charged at the time shortage is satisfied). Similarly, the shortage cost expression for $C_f(H, R)$ is obtained by considering the upcoming epoch $kT + \tau_2$ when the emergency order arrives.

Combining the above analysis, we obtain

$$\begin{aligned} \underline{C}_n(H, R) = & h \left\{ \int_0^{H^+} \tau_3 (H - X_2 - 0.5\lambda\tau_3) f_2(X_2) dX_2 \right. \\ & + \int_0^R (T - \tau_3) [R - X_1 - 0.5\lambda(T - \tau_3)] f_1(X_1) dX_1 \Big\} \\ & + \pi \left\{ \int_{H^+}^R (X_1 - H) f_1(X_1) dX_1 + (R - H) \int_R^\infty f_1(X_1) dX_1 \right\} \\ = & h \left\{ \int_0^{H^+} \tau_3 (H - X_2 - 0.5\lambda\tau_3) f_2(X_2) dX_2 \right. \\ & + \int_0^R (T - \tau_3) [R - X_1 - 0.5\lambda(T - \tau_3)] f_1(X_1) dX_1 \Big\} \\ & + \pi \left\{ \int_{H^+}^\infty (X_1 - H) f_1(X_1) dX_1 - \int_R^\infty (X_1 - R) f_1(X_1) dX_1 \right\}. \end{aligned} \quad (2)$$

$$\begin{aligned} \underline{C}_f(H, R) = & S + h \left\{ \int_0^R \tau_3 (R - X_2 - 0.5\lambda\tau_3) f_2(X_2) dX_2 \right. \\ & + \int_0^R (T - \tau_3) [R - X_1 - 0.5\lambda(T - \tau_3)] f_1(X_1) dX_1 \Big\} \\ & + \pi \left\{ \int_{H^+}^\infty (X_2 - H) f_2(X_2) dX_2 - \int_R^\infty (X_2 - R) f_2(X_2) dX_2 \right\}. \end{aligned} \quad (3)$$

4. Nature of the optimal policy

We are interested in characterizing the form of the optimal selection of a supply mode for any given (H, R) , $R \geq 0$. In particular, we are interested in finding conditions which guarantee that (a) the emergency supply mode will never be used, or (b) there exists an indifference inventory level I such that if $H < I$, the emergency mode is used. The main results of this section are expressed in Theorem 1, which shows that if it is cheaper to use the regular supply mode when the inventory position at the review epoch is zero, then the emergency mode will never be used, and otherwise, both supply modes need to be used.

Define $\underline{\Delta}(H, R)$ as

$$\begin{aligned}\underline{\Delta}(H, R) &\equiv \underline{C}_f(H, R) - \underline{C}_n(H, R) \\ &= S + h \int_0^R \tau_3(R - X_2 - 0.5\lambda\tau_3)f_2(X_2) dX_2 - h \int_0^{H^+} \tau_3(H - X_2 - 0.5\lambda\tau_3)f_2(X_2) dX_2 \\ &\quad + \pi \left\{ \int_{H^+}^{\infty} (X_2 - H)f_2(X_2) dX_2 - \int_{H^+}^{\infty} (X_1 - H)f_1(X_1) dX_1 \right. \\ &\quad \left. - \int_R^{\infty} (X_2 - R)f_2(X_2) dX_2 + \int_R^{\infty} (X_1 - R)f_1(X_1) dX_1 \right\}.\end{aligned}\quad (4)$$

Clearly, the regular supply mode is preferred whenever $\underline{\Delta} \geq 0$. Also as we see, our model allows the situation where there is a positive set-up cost for the regular mode, since $\underline{\Delta}(H, R)$ considers only the difference between the two supply mode set-up costs. It follows from (4) that for any given R , $\underline{\Delta}(H, R)$ is constant and independent of H if $H \leq 0$.

For $H > 0$, the first and second derivatives of $\underline{\Delta}$ with respect to H are

$$\partial \underline{\Delta} / \partial H = h\tau_3 \left\{ 0.5\lambda\tau_3 f_2(H) - \int_0^H f_2(X_2) dX_2 \right\} + \pi \left\{ \int_H^{\infty} f_1(X_1) dX_1 - \int_H^{\infty} f_2(X_2) dX_2 \right\}, \quad (5)$$

and

$$\partial^2 \underline{\Delta} / \partial H^2 = h\tau_3 \{ 0.5\lambda\tau_3 Df_2(H) - f_2(H) \} + \pi \{ f_2(H) - f_1(H) \}. \quad (6)$$

Note from (4) that $\underline{\Delta}(0, R)$ is given by

$$\begin{aligned}\underline{\Delta}(0, R) &= S + h \int_0^R \tau_3(R - X_2 - 0.5\lambda\tau_3)f_2(X_2) dX_2 \\ &\quad + \pi \left\{ \lambda\tau_2 - \lambda\tau_1 - \int_R^{\infty} (X_2 - R)f_2(X_2) dX_2 + \int_R^{\infty} (X_1 - R)f_1(X_1) dX_1 \right\}.\end{aligned}\quad (7)$$

Observe that $\underline{\Delta}$ is not monotonic in H . To see this, notice from (5) that as H approaches 0 from the right, $\partial \underline{\Delta} / \partial H$ approaches $0.5h\tau_3\lambda\tau_3 f_2(0) \geq 0$, while for large enough values of H , $\partial \underline{\Delta} / \partial H = -h\tau_3 < 0$. We assume throughout the rest of the analysis that there is at most one positive H , denoted by H_1 , equating (6) to zero. This assumption holds for Normal or Poisson demand (see Appendix A). It can be shown that it also holds for Gamma and Geometric demand. If H_1 does not exist, then $\partial^2 \underline{\Delta} / \partial H^2$ does not change sign. Hence, $\underline{\Delta}$ is either convex or concave on $H > 0$. On the other hand, if H_1 exists, it can be shown that $\underline{\Delta}$ is convex for $H \in (0, H_1)$ and concave for $H \in (H_1, R)$ (if $H_1 < R$). Also notice from (4) that for a given R , as H approaches R , $\underline{\Delta}$ approaches $S > 0$. Combining these facts, we can see that $\underline{\Delta}(H, R)$ will have either of the two possible forms

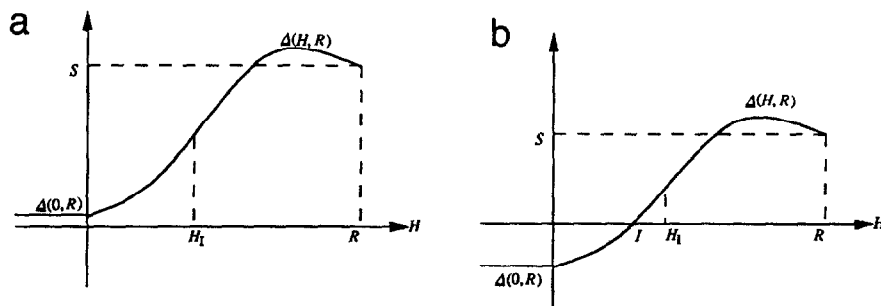


Fig. 2. (a) $\underline{\Delta}(H, R)$ with $\underline{\Delta}(0, R) \geq 0$. (b) $\underline{\Delta}(H, R)$ with $\underline{\Delta}(0, R) < 0$.

described in Figure 2(a) or Figure 2(b), depending on whether $\underline{\Delta}(0, R) \geq 0$ or $\underline{\Delta}(0, R) < 0$. The following theorem is immediate from the above discussion.

Theorem 1. For any $R \geq 0$, if $\underline{\Delta}(0, R) \geq 0$, only the regular supply mode is used. Otherwise if $\underline{\Delta}(0, R) < 0$, then there exists an indifference inventory level $I(R) > 0$ such that (i) if $H \geq I$, it is optimal to place an order using the regular supply mode and (ii) if $H < I$, it is optimal to place an order using the emergency supply mode.

Notice that the point $I(R)$ is obtained by solving $\underline{\Delta}(I, R) = 0$. Theorem 1 states that for a given non-negative order-up-to level, if it is cheaper to use the regular supply mode when the inventory position at the review epoch is zero (or less than zero), then the emergency supply mode will never be used (see Fig. 2(a)). Otherwise, there exists an indifference inventory level that equates the cost of using one supply mode to the other such that the emergency supply mode should be used if the inventory position at the review epoch is less than the indifference inventory level (see Fig. 2(b)).

Next, we take the first and second derivative of $\underline{\Delta}(0, R)$ with respect to R to investigate when $\underline{\Delta}(0, R) < 0$.

$$\partial \underline{\Delta}(0, R) / \partial R = h\tau_3 \left\{ \int_0^R f_2(X_2) dX_2 - 0.5\lambda\tau_3 f_2(R) \right\} + \pi \left\{ \int_R^\infty f_2(X_2) dX_2 - \int_R^\infty f_1(X_1) dX_1 \right\}, \quad (8)$$

$$\partial^2 \underline{\Delta}(0, R) / \partial R^2 = h\tau_3 \{ f_2(R) - 0.5\lambda\tau_3 Df_2(R) \} + \pi \{ f_1(R) - f_2(R) \}. \quad (9)$$

Comparing (8) to (5) and (9) to (6), we see that negating (8) and (9) give (5) and (6) respectively. Thus there is at most one R , denoted by R_1 , which equates (9) to zero (as we assumed for (6)). If R_1 exists, then $R_1 = H_1$. In this case, it can be shown that $\underline{\Delta}(0, R)$ is concave for $R \in (0, R_1)$ and convex for $R > R_1$. Otherwise if R_1 does not exist, then $\underline{\Delta}(0, R)$ is either convex or concave on $R > 0$. Note that (8) is non-positive when R approaches zero and $h\tau_3$ when R becomes very large. Thus, if R_1 exists, the non-zero unique R which minimizes $\underline{\Delta}(0, R)$, denoted by R_M , can be obtained by solving $\partial \underline{\Delta}(0, R) / \partial R = 0$. Consequently, if $\underline{\Delta}(0, R_M) < 0$, then there exist R_L and R_U with $\underline{\Delta}(0, R_L) = \underline{\Delta}(0, R_U) = 0$ such that for $R \in (R_L, R_U)$, both supply modes should be employed (see Fig. 3).

5. Characterizations of optimal order-up-to levels

In this section, we show (in Theorem 2) that the indifference inventory level I is a decreasing function of the order-up-to level R . That is, as R increases, we will be less willing to use the emergency supply mode. We also show (in Theorem 3) that the optimal order-up-to level for the proposed dual-supply-mode model lies between the optimal order-up-to level for the regular-supply-mode model and the optimal order-up-to level for the emergency-supply-mode model. Finally, we develop algorithms to find the optimal order-up-to level for the proposed model.

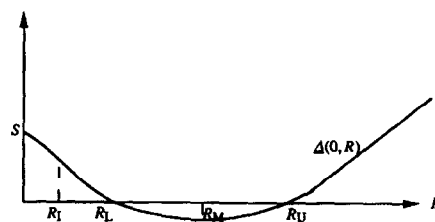


Fig. 3. $\underline{\Delta}(0, R)$ with $\underline{\Delta}(0, R_M) < 0$.

We first derive and analyze the cost functions $\underline{C}_n(R)$, $\underline{C}_f(R)$, and $\underline{C}_m(R)$ corresponding respectively to the regular, emergency, and dual-supply-mode models. By definition, we have

$$\begin{aligned}\underline{C}_n(R) &= \int_0^\infty \underline{C}_n(R - X_0, R) f_0(X_0) dX_0 \\ &= h \left\{ \int_0^R \int_0^{R-X_0} \tau_3 (R - X_0 - X_2 - 0.5\lambda\tau_3) f_2(X_2) dX_2 f_0(X_0) dX_0 \right. \\ &\quad \left. + \int_0^R (T - \tau_3) [R - X_1 - 0.5\lambda(T - \tau_3)] f_1(X_1) dX_1 \right\} \\ &\quad + \pi \left\{ \int_R^\infty (X_0 - R) f_0(X_0) dX_0 + \lambda\tau_1 \int_R^\infty f_0(X_0) dX_0 - \int_R^\infty (X_1 - R) f_1(X_1) dX_1 \right. \\ &\quad \left. + \int_0^R \int_{R-X_0}^\infty (X_0 + X_1 - R) f_1(X_1) dX_1 f_0(X_0) dX_0 \right\} \quad (10)\end{aligned}$$

and

$$\begin{aligned}\underline{C}_f(R) &= \int_0^\infty \underline{C}_f(R - X_0, R) f_0(X_0) dX_0 \\ &= S + h \left\{ \int_0^R \tau_3 (R - X_2 - 0.5\lambda\tau_3) f_2(X_2) dX_2 \right. \\ &\quad \left. + \int_0^R (T - \tau_3) [R - X_1 - 0.5\lambda(T - \tau_3)] f_1(X_1) dX_1 \right\} \\ &\quad + \pi \left\{ \int_R^\infty (X_0 - R) f_0(X_0) dX_0 + \lambda\tau_2 \int_R^\infty f_0(X_0) dX_0 - \int_R^\infty (X_2 - R) f_2(X_2) dX_2 \right. \\ &\quad \left. + \int_0^R \int_{R-X_0}^\infty (X_0 + X_2 - R) f_2(X_2) dX_2 f_0(X_0) dX_0 \right\}. \quad (11)\end{aligned}$$

The objective is to minimize the average cost for the upcoming time period $[kT + \tau_2, (k+1)T + \tau_2]$. For a given T , the value of R minimizing $\underline{C}_n(R)$ or $\underline{C}_f(R)$ must satisfy

$$\begin{aligned}D\underline{C}_n(R) = 0 &= h\tau_3 \int_0^R \left(\int_0^{R-X_0} f_2(X_2) dX_2 - 0.5\lambda\tau_3 f_2(R - X_0) \right) f_0(X_0) dX_0 \\ &\quad + h(T - \tau_3) \left(\int_0^R f_1(X_1) dX_1 - 0.5\lambda(T - \tau_3) f_1(R) \right) \\ &\quad + \pi \left\{ \int_R^\infty f_1(X_1) dX_1 - \int_R^\infty f_0(X_0) dX_0 - \int_0^R \int_{R-X_0}^\infty f_1(X_1) dX_1 f_0(X_0) dX_0 \right\}, \quad (12)\end{aligned}$$

and

$$\begin{aligned}D\underline{C}_f(R) = 0 &= h\tau_3 \left(\int_0^R f_2(X_2) dX_2 - 0.5\lambda\tau_3 f_2(R) \right) \\ &\quad + h(T - \tau_3) \left(\int_0^R f_1(X_1) dX_1 - 0.5\lambda(T - \tau_3) f_1(R) \right) \\ &\quad + \pi \left\{ \int_R^\infty f_2(X_2) dX_2 - \int_R^\infty f_0(X_0) dX_0 - \int_0^R \int_{R-X_0}^\infty f_2(X_2) dX_2 f_0(X_0) dX_0 \right\}. \quad (13)\end{aligned}$$

Let $\underline{R}_n$ and $\underline{R}_f$ be the minimum order-up-to levels satisfying $\Pr(X_1 \geq R) = 1 - F_1(R) \leq \varphi$ and $\Pr(X_2 \geq R) = 1 - F_2(R) \leq \varphi$ respectively, where φ is a maximum allowable probability that the order placed fails to clear any backorder ($\varphi = 0.01$, for example). Although $\underline{C}_n(R)$ and $\underline{C}_f(R)$ are not convex in general, it can be shown that $\underline{C}_n(R)$ is convex for $R \geq \max\{\underline{R}_n, \lambda(T + \tau_2)\}$ and $\underline{C}_f(R)$ is convex for $R \geq \max\{\underline{R}_f, \lambda\tau_1\}$ (see Appendix B). Intuitively, for the regular supply mode (resp. the emergency supply mode), R should be set large enough to cover demand over a review period plus the regular supply lead time τ_1 (resp. the emergency supply lead time τ_2) since it will take time $T + \tau_1$ (resp. $T + \tau_2$) for the next regular (resp. emergency) order to arrive. As we would expect non-negative safety stock in practice, the assumption of $R \geq \max\{\underline{R}_n, \lambda(T + \tau_2)\}$ for $\underline{C}_n(R)$ and $R \geq \max\{\underline{R}_f, \lambda\tau_1\}$ for $\underline{C}_f(R)$ is not restrictive at all.

Next, we derive $\underline{C}_m(R)$. As noted before, if R_1 exists and $\underline{\Delta}(0, R_M) < 0$ (which is true for all problems we solve in the computation experiment), then there exist R_L and R_U with $\underline{\Delta}(0, R_L) = \underline{\Delta}(0, R_U) = 0$ such that for $R \geq R_U$ or $R \leq R_L$, $\underline{C}_m(R) = \underline{C}_n(R)$, i.e., only the regular supply mode is used, and for $R \in (R_L, R_U)$, both supply modes need to be used and $\underline{C}_m(R)$ is given by

$$\begin{aligned}\underline{C}_m(R) &= \int_{R-I}^{\infty} \underline{C}_f(R - X_0, R) f_0(X_0) dX_0 + \int_0^{R-I} \underline{C}_n(R - X_0, R) f_0(X_0) dX_0 \\ &= \text{OC} + \text{IC}_{\text{LB}} + \text{SC},\end{aligned}\quad (14)$$

where OC, IC_{LB} , and SC represent respectively the ordering, inventory holding, and shortage cost component of $\underline{C}_m(R)$, and are expressed by

$$\begin{aligned}\text{OC} &= S \int_{R-I}^{\infty} f_0(X_0) dX_0, \\ \text{IC}_{\text{LB}} &= h \left\{ \int_0^{R-I} \int_0^{R-X_0} \tau_3 (R - X_0 - X_2 - 0.5\lambda\tau_3) f_2(X_2) dX_2 f_0(X_0) dX_0 \right. \\ &\quad + \int_{R-I}^{\infty} f_0(X_0) dX_0 \cdot \int_0^R \tau_3 (R - X_2 - 0.5\lambda\tau_3) f_2(X_2) dX_2 \\ &\quad \left. + \int_0^R (T - \tau_3) [R - X_1 - 0.5\lambda(T - \tau_3)] f_1(X_1) dX_1 \right\}, \\ \text{SC} &= \pi \left\{ \int_R^{\infty} (X_0 - R) f_0(X_0) dX_0 + \lambda\tau_2 \int_R^{\infty} f_0(X_0) dX_0 - \int_{R-I}^{\infty} f_0(X_0) dX_0 \cdot \int_R^{\infty} (X_2 - R) f_2(X_2) dX_2 \right. \\ &\quad - \int_0^{R-I} f_0(X_0) dX_0 \cdot \int_R^{\infty} (X_1 - R) f_1(X_1) dX_1 \\ &\quad + \int_{R-I}^R \int_{R-X_0}^{\infty} (X_0 + X_2 - R) f_2(X_2) dX_2 f_0(X_0) dX_0 \\ &\quad \left. + \int_0^{R-I} \int_{R-X_0}^{\infty} (X_0 + X_1 - R) f_1(X_1) dX_1 f_0(X_0) dX_0 \right\}.\end{aligned}$$

The value of R minimizing $\underline{C}_m(R)$ must satisfy

$$\begin{aligned}\text{DC}_{\underline{C}_m}(R) = 0 &= h \left\{ (T - \tau_3) \left(\int_0^R f_1(X_1) dX_1 - 0.5\lambda(T - \tau_3) f_1(R) \right) \right. \\ &\quad \left. + \int_{R-I}^{\infty} f_0(X_0) dX_0 \cdot \tau_3 \left(\int_0^R f_2(X_2) dX_2 - 0.5\lambda\tau_3 f_2(R) \right) \right\}\end{aligned}$$

$$\begin{aligned}
& + \tau_3 \int_0^{R-I} \left(\int_0^{R-X_0} f_2(X_2) dX_2 - 0.5\lambda\tau_3 f_2(R-X_0) \right) f_0(X_0) dX_0 \Big\} \\
& + \pi \left\{ \int_{R-I}^\infty f_0(X_0) dX_0 \cdot \int_R^\infty f_2(X_2) dX_2 + \int_0^{R-I} f_0(X_0) dX_0 \cdot \int_R^\infty f_1(X_1) dX_1 \right. \\
& \quad - \int_R^\infty f_0(X_0) dX_0 - \int_{R-I}^R \int_{R-X_0}^\infty f_2(X_2) dX_2 f_0(X_0) dX_0 \\
& \quad \left. - \int_0^{R-I} \int_{R-X_0}^\infty f_1(X_1) dX_1 f_0(X_0) dX_0 \right\}. \tag{15}
\end{aligned}$$

(note $\underline{\Delta}(I, R) = 0$ in taking the derivative). If we differentiate again, we see that the second derivative is not necessarily positive, implying that $\underline{C}_m(R)$ is not a convex function. Thus, the search for the optimal order-up-to level cannot be totally based on (15). Note that $\underline{C}_m(R)$ is continuous at R_U (resp. R_L) as $\underline{C}_m(R_U) - \underline{C}_n(R_U) = \Pr(X_0 \geq R_U) \underline{\Delta}(0, R_U) = 0$ (resp. $\underline{C}_m(R_L) - \underline{C}_n(R_L) = \Pr(X_0 \geq R_L) \underline{\Delta}(0, R_L) = 0$).

If we require that $R \geq \max\{\underline{R}_n, \lambda(T + \tau_2)\}$ for $\underline{C}_m(R)$, then I is a decreasing function of R . This is shown in the following theorem.

Theorem 2. *If it is required that $R \geq \max\{\underline{R}_n, \lambda(T + \tau_2)\}$ for $\underline{C}_m(R)$, then I is a decreasing function of R .*

Proof. See Appendix C.

Theorem 2 basically shows that as R increases, $\Pr(X_0 > R - I)$ decreases, i.e., it becomes less likely to use the emergency supply mode. Consequently, if R is large enough (not so large as R_U) such that $\Pr(X_0 > R - I)$ is close to zero, then the emergency supply mode almost will never be used. Conversely, if R is low enough (not as low as $\max\{\underline{R}_n, \lambda(T + \tau_2)\}$) such that $\Pr(X_0 > R - I)$ is close to one, then the regular supply mode will seldom be used.

Let R_n , R_f , and R_m be the optimal order-up-to levels for the regular, emergency, and dual-supply-mode models respectively. Intuitively, we would expect that $R_m \leq R_n$. This is because the option of using a faster supply mode is open depending on the inventory position at the review epoch, and thus R_m can be lower to avoid unnecessary holding of safety stock. The same reasoning holds for $R_f \leq R_m$. In the following theorem, we assume that $R_U > \max\{\underline{R}_n, \lambda(T + \tau_2)\}$ (otherwise, only the regular supply mode is used).

Theorem 3. *If it is required that $R \geq \max\{\underline{R}_n, \lambda(T + \tau_2)\}$ for $\underline{C}_n(R)$ and $\underline{C}_m(R)$, and $R \geq \max\{\underline{R}_f, \lambda\tau_1\}$ for $\underline{C}_f(R)$, then $R_f \leq R_m \leq R_n$.*

Proof. See Appendix C.

In the proof of Theorem 3, we also have shown that if $R_n \leq R_U$, the strict inequality holds, i.e., $R_f < R_m < R_n$. Moreover, if $R_U \leq R_f$, $R_m = R_n$. However, if $R_f < R_U < R_n$, it is not clear whether or not $R_m = R_n$ because of (A5). In summary, we distinguish among three cases in the search for R_m (assuming $R_U > \max\{\underline{R}_n, \lambda(T + \tau_2)\}$): $R_n \leq R_U$, $R_U \leq R_f$, and $R_f < R_U < R_n$. For the case of $R_n \leq R_U$, we propose the following algorithm.

Step 0. Find R_n by using (12) and R_f by (13).

Step 1. (Notice in the proof of Theorem 3 that $D\underline{C}_m(R_n) \geq 0$.) Using (15), decrease R from R_n by an interval of 5 (depending on the magnitude of data) until $R = R_{mr}$ where $D\underline{C}_m(R_{mr}) = 0$. (Notice that in case of overshooting and thus $D\underline{C}_m(R) < 0$, increase R back by an interval of 1.)

Step 2. (Notice that $D\bar{C}_m(R_f) \leq 0$.) Increase R from R_f until $R = R_{ml}$ where $D\bar{C}_m(R_{ml}) = 0$.

Step 3. If $R_{mr} = R_{ml}$, then $R_m = R_{mr} = R_{ml}$ clearly. Otherwise, determine $\bar{C}_m(R_{mr})$ and $\bar{C}_m(R_{ml})$ and select the one with lower expected cost.

Note that for a given R , I is obtained by solving $\Delta(I, R) = 0$ in (4) by binary search (between 0 and R). If there are only one sign change (i.e., one unique local minimum R_m) or three sign changes (i.e., two distinct local minimums R_{ml} and R_{mr}) between R_f and R_n , then the above algorithm is optimal. For the case of $R_f < R_U < R_n$, we propose the same algorithm, except that Step 1 is omitted (as R_n is a local minimum). Note from (A3) that R_U is (approximately) equal to $(\pi\lambda/h) + 0.5\lambda(\tau_1 + \tau_2) - (S/h\tau_3)$, which usually is a relatively large number. Thus, $R_n \leq R_U$ holds in general.

To illustrate the proposed algorithm, suppose that $\lambda = 250$, $\sigma^2 = 6,250$, $T = 1.0$, $\tau_1 = 0.6$, $\tau_2 = 0.2$, $h = \$1.0$, $S = \$40.0$, $\pi = \$70.0$. We first find $R_f = 490$ and $R_n = 619$. We then decrease R from R_n to find $R_{mr} = 532$ and increase R from R_f to find $R_{ml} = 532$. Thus, $R_m = R_{mr} = R_{ml} = 532$ clearly.

6. Computational results

Recall that the problem we solve is the lower bound problem. However, we use the value of R (as well as the value of I) obtained in the algorithm to determine $\bar{C}_m(R)$, thus bounding the exact C_m^* . $\bar{C}_m(R)$ is given by

$$\bar{C}_m(R) = OC + IC_{UB} + SC, \quad (16)$$

where

$$\begin{aligned} IC_{LB} = h & \left\{ \int_{R-I}^{\infty} f_0(X_0) dX_0 \cdot \tau_3 \int_0^R \left(\int_0^{R-X_2} (R - X_2 - 0.5X_3) f_3(X_3) dX_3 \right. \right. \\ & \quad \left. \left. + \int_{R-X_2}^{\infty} 0.5(R - X_2) f_3(X_3) dX_3 \right) f_2(X_2) dX_2 \right. \\ & \quad \left. + \tau_3 \int_0^{R-I} \int_0^{R-X_0} \left(\int_0^{R-X_0-X_2} (R - X_0 - X_2 - 0.5X_3) f_3(X_3) dX_3 \right. \right. \\ & \quad \left. \left. + \int_{R-X_0-X_2}^{\infty} 0.5(R - X_0 - X_2) f_3(X_3) dX_3 \right) f_2(X_2) dX_2 f_0(X_0) dX_0 \right. \\ & \quad \left. + (T - \tau_3) \int_0^R \left(\int_0^{R-X_1} (R - X_1 - 0.5X_4) f_4(X_4) dX_4 \right. \right. \\ & \quad \left. \left. + \int_{R-X_1}^{\infty} 0.5(R - X_1) f_4(X_4) dX_4 \right) f_1(X_1) dX_1 \right\}, \end{aligned}$$

and OC and SC are given in (14).

Demand is assumed to be Normal with mean $\lambda\tau$ and variance $\sigma^2\tau$ for a time period of length τ . All problems we solve entail the use of two supply modes and the average CPU time for solving $\bar{C}_m(R)$ is 0.607 seconds on an IBM 3081. The performance of the proposed model is compared to that of the regular-supply-mode model as well as that of the best-supply-mode model (i.e. using always the supply mode that gives the lower expected cost). In addition, we compare the performance of the proposed model to that of a heuristic, which simply uses R_n and the indifference inventory level I_h given by

$$I_h = \lambda\tau_1 + K\sigma_{x_1},$$

Table 1

Computational results: Sensitivity to demand variance, unit shortage cost, and lead time ($\lambda = 250.0$, $\tau_2 = 0.2$, $T = 1.0$, $h = 1.0$, $S = 40$)

Factors			Average cost				% Savings ^a			Upper bound	
σ^2	τ_1	π	$\underline{C}_n(R_n)$	$\underline{C}_r(R_r)$	$\underline{C}_m(R_m)$	$\underline{C}_h(R_h)$	w.r.t. R.	w.r.t. B.	w.r.t. H.	$\bar{C}_m(R_m)$	% error
6,250	0.4	10.0	289.5	317.0	268.4	281.6	7.26	7.26	4.91	269.5	0.41
		40.0	343.7	367.5	308.8	328.8	10.14	10.14	6.45	309.2	0.12
		70.0	362.8	385.2	323.3	346.7	10.88	10.88	7.22	323.6	0.09
		100.0	374.3	395.8	332.2	357.6	11.26	11.26	7.65	332.4	0.06
	0.6	10.0	300.6	317.0	275.4	304.6	8.39	8.39	10.58	276.5	0.40
		40.0	358.8	367.5	317.1	354.4	11.61	11.61	11.75	317.6	0.15
		70.0	379.2	385.2	331.7	372.2	12.54	12.54	12.24	332.1	0.13
		100.0	391.5	395.8	340.4	383.2	13.06	13.06	12.56	340.8	0.12
	0.8	10.0	311.2	317.0	296.7	334.6	4.64	4.64	12.78	298.4	0.56
		40.0	373.0	367.5	346.2	383.8	7.19	5.80	10.88	346.9	0.21
		70.0	394.6	385.2	363.0	401.3	8.01	5.75	10.54	363.6	0.17
		100.0	407.7	395.8	373.1	412.4	8.48	5.74	10.53	373.7	0.16
	1.0	10.0	321.2	317.0	314.5	366.6	2.10	0.86	16.59	316.8	0.72
		40.0	386.4	367.5	365.6	414.4	5.37	0.50	13.32	365.7	0.02
		70.0	409.2	385.2	383.2	432.5	6.36	0.51	12.87	383.2	0.00
		100.0	423.0	395.8	393.8	443.3	6.90	0.52	12.58	393.8	0.00
1,250	0.4	10.0	198.5	233.0	189.3	203.2	4.63	4.63	7.37	189.7	0.21
		40.0	222.8	255.6	206.0	221.2	7.55	7.55	7.39	206.3	0.14
		70.0	231.4	263.5	211.7	228.7	8.51	8.51	8.06	211.9	0.12
		100.0	236.5	268.2	215.1	232.2	9.05	9.05	7.96	215.4	0.12
	0.6	10.0	203.5	233.0	199.7	223.9	1.86	1.86	12.10	200.3	0.29
		40.0	229.6	255.6	220.7	239.5	3.88	3.88	8.55	221.0	0.16
		70.0	238.7	263.5	227.7	246.3	4.63	4.63	8.17	228.0	0.16
		100.0	244.2	268.2	231.8	249.4	5.08	5.08	7.60	232.2	0.16
	0.8	10.0	208.3	233.0	206.9	247.0	0.65	0.65	19.39	207.7	0.38
		40.0	235.9	255.6	231.3	260.5	1.97	1.97	12.65	231.6	0.16
		70.0	245.6	263.5	239.4	265.2	2.52	2.52	10.79	239.8	0.15
		100.0	251.4	268.2	244.2	268.3	2.86	2.86	9.86	244.6	0.15
	1.0	10.0	212.8	233.0	212.3	271.4	0.23	0.23	27.83	213.3	0.47
		40.0	241.9	255.6	239.4	281.3	1.04	1.04	17.49	239.8	0.16
		70.0	252.1	263.5	248.5	285.9	1.44	1.44	15.05	248.8	0.14
		100.0	258.3	268.2	253.9	288.9	1.69	1.69	13.77	254.2	0.13
250	0.4	10.0	157.8	195.4	156.7	170.2	0.73	0.73	8.61	156.9	0.12
		40.0	168.7	205.5	165.2	174.1	2.10	2.10	5.37	165.4	0.11
		70.0	172.6	209.0	168.0	175.8	2.66	2.66	4.64	168.2	0.12
		100.0	174.9	211.2	169.6	177.7	3.02	3.02	4.78	169.8	0.13
	0.6	10.0	160.1	195.4	159.9	186.8	0.14	0.14	16.87	160.1	0.16
		40.0	171.8	205.5	170.3	187.0	0.83	0.83	9.80	170.5	0.12
		70.0	175.8	209.0	173.7	187.8	1.20	1.20	8.12	173.9	0.12
		100.0	178.3	211.2	175.8	188.9	1.44	1.44	7.51	176.0	0.13
	0.8	10.0	162.3	195.4	162.2	208.6	0.03	0.03	28.65	162.5	0.20
		40.0	174.6	205.5	174.0	204.5	0.36	0.36	17.53	174.2	0.11
		70.0	178.9	209.2	177.9	202.2	0.57	0.57	13.68	178.1	0.11
		100.0	181.5	211.2	180.2	202.3	0.73	0.73	12.26	180.4	0.11
	1.0	10.0	164.2	195.4	164.2	225.7	0.00	0.00	37.40	164.7	0.26
		40.0	177.3	205.5	177.0	220.8	0.15	0.15	24.73	177.2	0.10
		70.0	181.8	209.0	181.3	217.3	0.28	0.28	19.86	181.5	0.09
		100.0	184.6	211.2	183.9	216.4	0.38	0.38	17.70	184.1	0.09

^a Note: R., B., and H. denote the regular and best-supply-mode models, and the heuristic respectively.

Table 2

Computational results: order-up-to levels and indifference inventory levels for the 48 problems in Table 1

σ^2	τ_1	π	R_n	R_f	R_m	Code ^a	I	I_h
6,250	0.4	10.0	470	411	438	U	124	164
		40.0	533	470	488	U	163	197
		70.0	555	490	506	U	175	209
		100.0	568	501	516	U	183	216
	0.6	10.0	528	411	469	U	170	228
		40.0	596	470	515	U	220	270
		70.0	619	490	532	U	236	284
		100.0	633	501	542	U	246	292
	0.8	10.0	586	411	541	U	210	290
		40.0	658	470	591	U	272	338
		70.0	682	490	608	R	292	354
		100.0	697	501	616	R	304	364
	1.0	10.0	643	411	624	R	245	351
		40.0	719	470	473	L	355	404
		70.0	745	490	493	L	375	423
		100.0	761	501	505	L	387	434
1,250	0.4	10.0	404	350	385	U	99	128
		40.0	432	376	403	U	120	143
		70.0	442	385	410	U	126	149
		100.0	447	390	414	U	130	151
	0.6	10.0	457	350	448	U	141	184
		40.0	488	376	471	U	169	203
		70.0	498	385	478	U	178	210
		100.0	504	390	482	U	183	213
	0.8	10.0	511	350	507	R	183	240
		40.0	543	376	534	R	218	262
		70.0	554	385	542	R	228	269
		100.0	560	390	547	R	235	273
	1.0	10.0	564	350	563	R	224	295
		40.0	598	376	593	R	266	319
		70.0	609	385	602	R	279	327
		100.0	616	390	608	R	286	332
250	0.4	10.0	374	322	372	U	92	112
		40.0	387	334	381	U	104	119
		70.0	391	338	384	U	108	121
		100.0	394	340	385	U	110	123
	0.6	10.0	426	322	425	R	136	165
		40.0	439	334	437	R	153	173
		70.0	444	338	440	R	157	176
		100.0	447	340	442	R	160	178
	0.8	10.0	477	322	477	R	179	218
		40.0	492	334	490	R	201	228
		70.0	496	338	495	R	207	230
		100.0	499	340	497	R	210	232
	1.0	10.0	529	322	529	R	222	270
		40.0	544	334	543	R	249	281
		70.0	549	338	548	R	256	284
		100.0	552	340	551	R	260	286

^a Note: U, R, and L indicate the unique local minimum, the local minimum on the right, and the local minimum on the left respectively.

where $K = [(R_n - \lambda(T + \tau_1)) / \sigma_{x_0+x_1}]$ and $\sigma_{x_0+x_1}$ is the standard deviation of demand during a review period plus the regular supply lead time. Notice that this value of K corresponds to the safety factor implied by the policy that uses only the regular supply mode. If the inventory position at the review epoch is less than I_n , an emergency order is placed. This heuristic seems intuitively appealing, since it basically employs the regular supply mode, and uses the emergency mode only if the safety stock as seen at the review epoch is lower than expected. The expected cost per review period of this heuristic is denoted by $\underline{C}_h(R_n)$.

The validity of the proposed algorithm is also checked after the experiment is run. It is found that there are one or three sign changes between R_f and R_n for all problems solved, indicating that there exist at most two local minimums and thus the order-up-to level obtained for each of the problems solved is optimal.

Five parameters are hypothesized to have an effect on the performance of the proposed model: unit shortage cost, demand variance, difference between the regular and emergency-mode lead times, mean demand rate, and the fixed ordering cost for the emergency supply mode. Table 1 gives the sensitivity results for the first three parameters and Table 2 shows the respective order-up-to levels and indifference levels.

It is clear from Table 1 that as π increases (other things being equal), the percentage savings with respect to the regular-supply-mode model increases, i.e., the proposed model becomes more attractive. This result agrees with the intuition that a faster supply mode may be worthwhile to use to avoid stockouts if shortages are costly.

The same behavior is also observed by the changes in the demand variance. Other things being equal, the percentage savings with respect to the regular-supply-mode model increases as demand variance increases. This is because a relatively small amount of safety stock may be enough to avoid stockouts if demand variability is low. However, the use of a faster supply mode may also be needed to reduce the cost incurred by carrying a large amount of safety stock if demand variability is high.

The effect of $\tau_1 - \tau_2$ on the performance of the proposed model is investigated next. As shown in Table 2, when $\tau_1 - \tau_2$ is small, $\underline{C}_m(R)$ tends to have only one local minimum. As $\tau_1 - \tau_2$ increases (other things

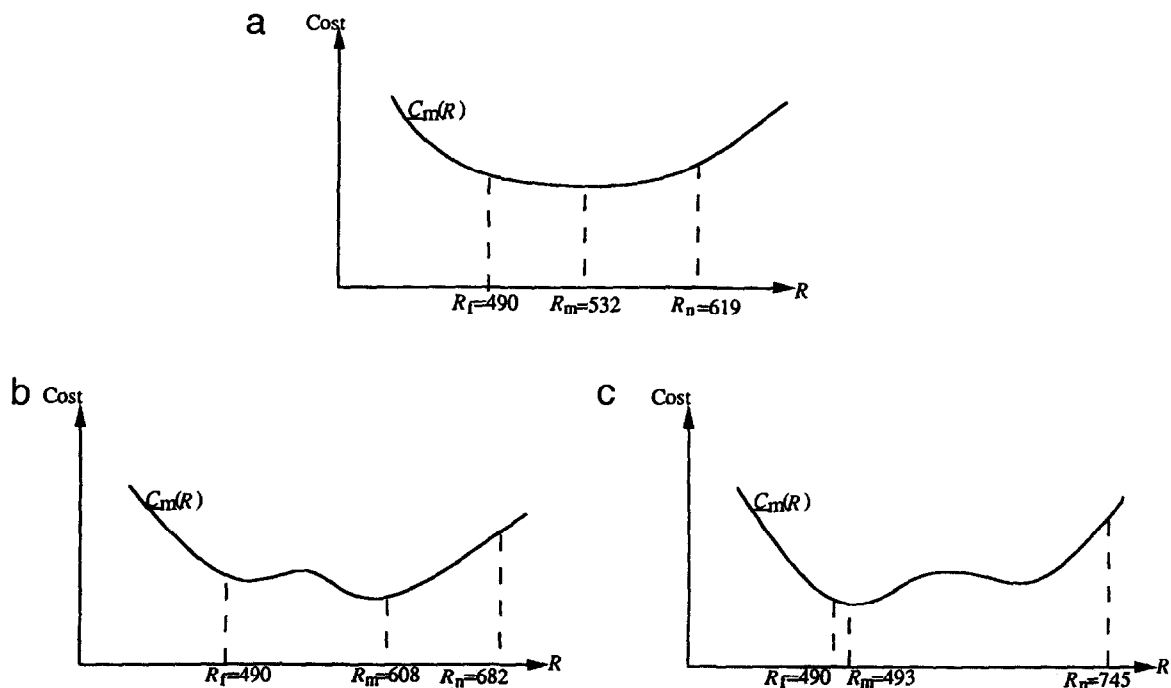


Fig. 4. (a) $\underline{C}_m(R)$ for $\tau_1 - \tau_2 = 0.4$, $\pi = 70$, and $\sigma^2 = 6,250$. (b) $\underline{C}_m(R)$ for $\tau_1 - \tau_2 = 0.6$, $\pi = 70$, and $\sigma^2 = 6,250$. (c) $\underline{C}_m(R)$ for $\tau_1 - \tau_2 = 0.8$, $\pi = 70$, and $\sigma^2 = 6,250$.

Table 3

Computational results: Break-even fixed ordering cost S_B for different emergency supply lead time τ_2 ($\lambda = 250$, $\sigma^2 = 6,250$, $\tau_1 = 0.8$, $T = 1.0$, $h = 1.0$)

π	τ_2	S_B	$\underline{C}_m(R_m)$	R_m	Code	I	$P_r(X_0 > R - I)$
10.0	0.60	20.0	292.7	553	U	255	0.272
	0.50	44.9	292.7	547	U	235	0.216
	0.40	52.6	292.7	540	U	224	0.202
	0.30	42.3	292.7	530	L	219	0.220
	0.20	25.5	292.7	438	L	233	0.715
40.0	0.60	20.0	343.1	615	U	305	0.224
	0.50	60.6	343.1	606	U	286	0.188
	0.40	69.1	343.1	597	U	278	0.191
	0.30	53.4	343.1	586	U	276	0.224
	0.20	30.3	343.1	499	L	285	0.676
70.0	0.60	20.0	360.9	636	U	322	0.209
	0.50	67.6	360.9	627	U	302	0.171
	0.40	76.7	360.9	618	U	295	0.178
	0.30	59.3	360.9	608	U	294	0.209
	0.20	32.4	360.9	519	L	303	0.666
100.0	0.60	20.0	371.7	649	U	332	0.198
	0.50	72.2	371.7	639	U	313	0.168
	0.40	81.8	371.7	630	U	306	0.175
	0.30	63.3	371.7	622	R	304	0.195
	0.20	33.9	371.7	531	L	314	0.662

being equal), $\underline{C}_m(R)$ starts to develop two local minimums and the global minimum R_m tends to be the local minimum on the right, i.e., R_m seems to be closer to the optimal order-up-to level R_n for the regular-supply-mode model. However, if $\tau_1 - \tau_2$ keeps increasing and demand variance is large, the global minimum R_m may migrate to the local minimum on the left, which is of course closer to the optimal order-up-to level R_f for the emergency-supply-mode model (specific examples of migrating minimums in the top portion of Table 2 are shown in Fig. 4). As a result, when $\tau_1 - \tau_2$ is small and $\underline{C}_m(R)$ has only one local minimum, the percentage savings of the proposed model tends to be large compared to the best-supply-mode model. As $\tau_1 - \tau_2$ becomes larger and two local minimums start to develop, the percentage savings of the proposed model seems to decrease. For example, as we see in the top portion of Table 1, if $\tau_1 - \tau_2 = 0.8$, the percentage savings with respect to the best-supply-mode model are all below 1% for four different values of π . In these situations, it is probably all right to use only the regular supply mode or the emergency mode.

Related to this result is the following finding. Given a regular supplier for the regular mode, we may need to choose and employ one emergency supplier among several available ones for the emergency mode. To investigate this issue, we perform the computation as follows: (i) initializing τ_1 and τ_2 and other base case

Table 4(a)

Computational results: Sensitivity to mean demand rate ($\tau_1 = 0.6$, $\tau_2 = 0.2$, $T = 1.0$, $h = 1.0$, $\pi = 40$, $S = 40$)

σ^2	λ	Average cost				% Savings		
		$\underline{C}_n(R_n)$	$\underline{C}_f(R_f)$	$\underline{C}_m(R_m)$	$\underline{C}_h(R_h)$	w.r.t. R.	w.r.t. B.	w.r.t. H.
250	100	96.8	130.5	93.8	103.1	3.02	3.02	9.90
	250	171.8	205.5	170.3	187.0	0.83	0.83	9.80
	500	296.8	330.5	296.2	326.8	0.20	0.20	10.36
100	100	79.6	115.6	78.6	88.4	1.25	1.25	12.52
	250	154.6	190.6	154.2	172.2	0.27	0.27	11.72
	500	279.6	315.6	279.5	311.9	0.04	0.04	11.61

Table 4(b)

Computational results: Sensitivity to mean demand rate ($\tau_1 = 0.6$, $\tau_2 = 0.2$, $T = 1.0$, $h = 1.0$, $\pi = 40$, $S = 40$)

scv	λ	Average cost				% Savings		
		$\underline{C}_n(R_n)$	$\underline{C}_f(R_f)$	$\underline{C}_m(R_m)$	$\underline{C}_h(R_n)$	w.r.t. R.	w.r.t. B.	w.r.t. H.
0.025	100	96.8	130.5	93.8	103.1	3.02	3.02	9.90
	250	241.9	266.2	230.6	251.1	4.66	4.66	8.88
	500	483.8	492.5	456.9	495.8	5.56	5.56	8.52
0.01	100	79.6	115.6	78.6	88.4	1.25	1.25	12.52
	250	198.9	229.0	194.7	211.8	2.12	2.12	8.77
	500	397.9	418.1	387.7	418.9	2.56	2.56	8.06

parameters, compute $\underline{C}_m(R_m)$ and denote it by C_0 , and (ii) for each decreasing τ_2 , find the value of S (call it S_B) such that $\underline{C}_m(R_m) = C_0$. We are interested in the behavior of the break-even set-up cost S_B . As shown in Table 3, for example, given $\tau_1 = 0.8$, $\pi = \$10.0$, $\tau_2 = 0.6$, and other base case parameters, S_B first increases but then decreases as τ_2 decreases. There are two possible reasons for this. First, as mentioned above, the proposed model becomes less attractive as the difference between τ_1 and τ_2 becomes larger. Hence, for a given τ_1 , S_B may need to decrease as τ_2 decreases below a certain level. Second (perhaps more importantly), as τ_2 decreases, R_m decreases and the probability of using the emergency supplier, i.e., $\Pr(X_0 > R - I)$, decreases but then increases for small levels of τ_2 (see Table 3). In this range, S_B is smaller, but the emergency supply mode is used more frequently. For example, in the top portion of Table 3, if $\tau_2 = 0.2$ (and total cost equals \$292.7), S_B equals \$25.5, which enables the emergency mode to be used with probability 0.715. (Note that if $S = \$40$ for $\tau_2 = 0.2$ and other parameters being equal, as we see from Tables 1 and 2, $\Pr(X_0 > R - I) = \Pr(X_0 > 541 - 210)$ is only 0.153 and total cost is \$296.7.) Since we expect in practice that the set-up cost increases as the lead time of an emergency order decreases, suppliers with both shorter lead time and smaller set-up cost should not be available.

Also clear from Table 1 is the fact that the heuristic suggested above performs pretty badly compared to the proposed model, especially if $\tau_1 - \tau_2$ is large. Finally, we can observe that the percentage error of approximating the exact C_m^* by using a lower bound is negligible (as shown in the last column).

Next, we examine the effect of the mean demand rate. As shown in Table 4(a), other things being equal (including the demand variance), the percentage savings (as well as the dollar savings) of the proposed model

Table 5

Computational results: Sensitivity to the fixed ordering cost for the emergency supply mode ($\lambda = 250.0$, $\sigma^2 = 1,250$, $\tau_2 = 0.2$, $T = 1.0$, $h = 1.0$, $\pi = 40$)

τ_1	S	Average Cost				% Savings		
		$\underline{C}_n(R_n)$	$\underline{C}_f(R_f)$	$\underline{C}_m(R_m)$	$\underline{C}_h(R_n)$	w.r.t. R.	w.r.t. B.	w.r.t. H.
0.4	20.0	222.8	235.6	201.9	218.5	9.38	9.38	8.22
	40.0	222.8	255.6	206.0	221.2	7.55	7.55	7.39
	60.0	222.8	275.6	208.8	223.9	6.29	6.29	7.23
0.6	20.0	229.6	235.6	219.0	236.3	4.62	4.62	7.92
	40.0	229.6	255.6	220.7	239.5	3.88	3.88	8.55
	60.0	229.6	275.6	221.9	242.8	3.33	3.33	9.38
0.8	20.0	235.9	235.6	230.6	256.7	2.27	2.12	11.33
	40.0	235.9	255.6	231.3	260.5	1.97	1.97	12.65
	60.0	235.9	275.6	231.8	264.3	1.73	1.73	14.01
1.0	20.0	241.9	235.6	235.6	277.1	2.62	0.00	17.64
	40.0	241.9	255.6	239.4	281.3	1.04	1.04	17.49
	60.0	241.9	275.6	239.7	285.4	0.93	0.93	19.07

decreases with respect to the regular-supply-mode model as mean demand increases. Basically, this is because the squared coefficient of variation decreases as mean demand increases. If the squared coefficient of variation is maintained as mean demand increases, then the proposed model becomes more attractive with larger mean demand, as shown in Table 4(b).

Finally, it is clear from Table 5 that the proposed model becomes less attractive compared to the regular-supply-mode model as S increases, which is intuitively reasonable as a large fixed ordering cost would make a faster supply mode less worthwhile to use.

7. Conclusion

In this paper, we describe a periodic review inventory system where a faster resupply mode could be used under emergency conditions. We concentrate on the case in which the purchasing cost per unit is the same between the two supply modes. We show that given any non-negative order-up-to level, either only the regular supply mode is used, or there exists an indifference inventory level such that if the inventory position at the review epoch is below the indifference inventory level, the emergency supply mode is used. We also develop procedures for solving for the two policy parameters, i.e., the order-up-to level and the indifference inventory level.

The computational results indicate that the model we propose is attractive when the unit shortage cost is large, demand variance is high, or the fixed order set-up cost for the emergency supply mode is small. On the other hand, the proposed model is only marginally useful in cases where the unit shortage penalty is small, demand variability is low, difference between the two supply lead times is large, or the fixed set-up cost for the emergency mode is large. Additional results suggest that an intuitively appealing heuristic might perform very badly in many cases, which of course illustrates the superiority of the proposed model. Given that very little computational time is needed to compute the policy parameters, we think that the proposed model can be incorporated routinely into inventory control systems in practice.

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Appendix A

We show that for Normal or Poisson demand, there is at most one positive H , denoted by H_1 , equating (6) to zero.

For Normal demand (with mean $\lambda\tau$ and variance $\sigma^2\tau$ for a time period of length τ), (6) reduces to

$$\begin{aligned}\partial^2\Delta/\partial H^2 &= h\tau_3 f_2(H) \left\{ \left(\lambda\tau_3(\lambda\tau_2 - H)/2\sigma^2\tau_2 \right) - 1.0 \right\} + \pi \{ f_2(H) - f_1(H) \} \\ &= \pi f_2(H) \left\{ 1.0 - (h\tau_3/\pi) + \left(h\tau_3\lambda\tau_3(\lambda\tau_2 - H)/2\sigma^2\tau_2\pi \right) - (f_1(H)/f_2(H)) \right\} \\ &= \pi f_2(H) \left\{ 1.0 - (h\tau_3/\pi) + \left(h\tau_3\lambda\tau_3(\lambda\tau_2 - H)/2\sigma^2\tau_2\pi \right) \right. \\ &\quad \left. - (c_1/c_2)(\tau_2/\tau_1)^{0.5} \exp\{0.5\tau_3(H^2 - \lambda^2\tau_1\tau_2)/\sigma^2\tau_1\tau_2\} \right\},\end{aligned}$$

where c_1 and c_2 are normalizing constants that are greater than 1 as demand is non-negative. As the expression within the big parentheses decreases on H , there exists at most one positive H that can equate $\partial^2\Delta/\partial H^2$ to zero.

For Poisson demand, (6) (interpreted as the second difference) is expressed by

$$\begin{aligned} h\tau_3 P_2(H) \{ & (\tau_3(\lambda\tau_2 - H)/2\tau_2) - 1.0 \} + \pi \{ P_2(H) - P_1(H) \} \\ & = \pi P_2(H) \{ 1.0 - (h\tau_3/\pi) + (h\tau_3\tau_3(\lambda\tau_2 - H)/2\tau_2\pi) - (P_1(H)/P_2(H)) \} \\ & = \pi P_2(H) \{ 1.0 - (h\tau_3/\pi) + (h\tau_3\tau_3(\lambda\tau_2 - H)/2\tau_2\pi) - (\tau_1/\tau_2)^H \exp\{-\lambda\tau_3\} \}, \end{aligned}$$

where $P_1(\cdot)$ (resp. $P_2(\cdot)$) is the Poisson density function for demand during the time interval $[kT, kT + \tau_1)$ (resp. $[kT, kT + \tau_2)$). As before, the expression within the big parentheses decreases on H . Thus, there exists at most one positive H that can equate the second difference of $\underline{A}$ zero.

Appendix B

We show that under certain conditions which are usually true in practice, $\underline{C}_n(R)$ and $\underline{C}_f(R)$ are both convex on R .

Note that (12) and (13) can be rewritten as follows

$$\begin{aligned} D\underline{C}_n(R) &= h\tau_3 \left(\int_0^R f_{0+2}(X) dX - 0.5\lambda\tau_3 f_{0+2}(R) \right) \\ &\quad + h(T - \tau_3) \left(\int_0^R f_1(X_1) dX_1 - 0.5\lambda(T - \tau_3) f_1(R) \right) \\ &\quad + \pi \left\{ \int_R^\infty f_1(X_1) dX_1 - \int_R^\infty f_{0+1}(X) dX \right\}, \\ D\underline{C}_f(R) &= h\tau_3 \left(\int_0^R f_2(X_2) dX_2 - 0.5\lambda\tau_3 f_2(R) \right) + h(T - \tau_3) \left(\int_0^R f_1(X_1) dX_1 - 0.5\lambda(T - \tau_3) f_1(R) \right) \\ &\quad + \pi \left\{ \int_R^\infty f_2(X_2) dX_2 - \int_R^\infty f_{0+2}(X) dX \right\}, \end{aligned}$$

where $f_{0+1}(\cdot)$ (resp. $f_{0+2}(\cdot)$) is the p.d.f. for demand during the period $[(k-1)T, kT + \tau_1)$ (resp. $[(k-1)T, kT + \tau_2)$). Let DDf denote the second derivative of the function f . Then,

$$\begin{aligned} DD\underline{C}_n(R) &= h\tau_3(f_{0+2}(R) - 0.5\lambda\tau_3 Df_{0+2}(R)) + h(T - \tau_3)(f_1(R) - 0.5\lambda(T - \tau_3) Df_1(R)) \\ &\quad + \pi(f_{0+1}(R) - f_1(R)), \\ DD\underline{C}_f(R) &= h\tau_3(f_2(R) - 0.5\lambda\tau_3 Df_2(R)) + h(T - \tau_3)(f_1(R) - 0.5\lambda(T - \tau_3) Df_1(R)) \\ &\quad + \pi(f_{0+2}(R) - f_2(R)). \end{aligned}$$

For Normal demand, $DD\underline{C}_n(R)$ and $DD\underline{C}_f(R)$ are respectively given by

$$\begin{aligned} DD\underline{C}_n(R) &= h\tau_3 f_{0+2}(R) \{ 1.0 + (\lambda\tau_3(R - \lambda T - \lambda\tau_2)/2\sigma^2(T + \tau_2)) \} \\ &\quad + h(T - \tau_3) f_1(R) \{ 1.0 + (\lambda(T - \tau_3)(R - \lambda\tau_1)/2\sigma^2\tau_1) \} + \pi(f_{0+1}(R) - f_1(R)), \\ DD\underline{C}_f(R) &= h\tau_3 f_2(R) \{ 1.0 + (\lambda\tau_3(R - \lambda\tau_2)/2\sigma^2\tau_2) \} \\ &\quad + h(T - \tau_3) f_1(R) \{ 1.0 + (\lambda(T - \tau_3)(R - \lambda\tau_1)/2\sigma^2\tau_1) \} + \pi(f_{0+2}(R) - f_2(R)). \end{aligned}$$

Hence, the restriction $R \geq \underline{R}_n$ (resp. $R \geq \underline{R}_f$) would make $f_1(R)$ (resp. $f_2(R)$) approximately zero. If we require that $R \geq \max\{\underline{R}_n, \lambda(T + \tau_2)\}$ for $\underline{C}_n(R)$ and $R \geq \max\{\underline{R}_f, \lambda\tau_1\}$ for $\underline{C}_f(R)$, then both $DD\underline{C}_n(R)$ and $DD\underline{C}_f(R)$ are positive.

Similarly for Poisson demand, the second difference of $\underline{C}_n(R)$ and $\underline{C}_f(R)$ can be written as follows respectively

$$\begin{aligned} & h\tau_3 P_{0+2}(R) \{1.0 + (\tau_3(R - \lambda T - \lambda\tau_2)/2(T + \tau_2))\} \\ & + h(T - \tau_3) P_1(R) \cdot \{1.0 + ((T - \tau_3)(R - \lambda\tau_1)/2\tau_1)\} + \pi(P_{0+1}(R) - P_1(R)), \\ & h\tau_3 P_2(R) \{1.0 + (\tau_3(R - \lambda\tau_2)/2\tau_2)\} + h(T - \tau_3) P_1(R) \{1.0 + ((T - \tau_3)(R - \lambda\tau_1)/2\tau_1)\} \\ & + \pi(P_{0+2}(R) - P_2(R)), \end{aligned}$$

where $P_{0+1}(\cdot)$ (resp. $P_{0+2}(\cdot)$) is the Poisson density function for demand during the period $[(k-1)T, kT + \tau_1]$ (resp. $[(k-1)T, kT + \tau_2]$). Thus, $R \geq \max\{\underline{R}_n, \lambda(T + \tau_2)\}$ and $R \geq \max\{\underline{R}_f, \lambda\tau_1\}$ would make the above two expressions positive respectively.

Appendix C

Proof of Theorem 2. Evaluating by parts the first and last two integrals of (4) gives

$$\begin{aligned} \int_0^R (R - X_2 - 0.5\lambda\tau_3) f_2(X_2) dX_2 &= (R - 0.5\lambda\tau_3) F_2(R) - \lambda\tau_2 + \int_R^\infty X_2 f_2(X_2) dX_2 \\ &= (R - 0.5\lambda\tau_3) F_2(R) - \lambda\tau_2 + R - R F_2(R) + \int_R^\infty (1 - F_2(X)) dX, \\ \int_R^\infty (X_1 - R) f_1(X_1) dX_1 - \int_R^\infty (X_2 - R) f_2(X_2) dX_2 &= \int_R^\infty (F_2(X) - F_1(X)) dX. \end{aligned}$$

By assumption, $F_1(R)$ and $F_2(R)$ are approximately one. Hence, the above two equations can be approximated as

$$\int_0^R (R - X_2 - 0.5\lambda\tau_3) f_2(X_2) dX_2 \approx R - \lambda\tau_2 - 0.5\lambda\tau_3, \quad (A1)$$

$$\int_R^\infty (X_1 - R) f_1(X_1) dX_1 - \int_R^\infty (X_2 - R) f_2(X_2) dX_2 \approx 0. \quad (A2)$$

Thus, (4) reduces to

$$\begin{aligned} \underline{\Delta}(H, R) &\approx S + h\tau_3(R - \lambda\tau_2 - 0.5\lambda\tau_3) - h \int_0^{H^+} \tau_3(H - X_2 - 0.5\lambda\tau_3) f_2(X_2) dX_2 \\ &+ \pi \left\{ \int_{H^+}^\infty (X_2 - H) f_2(X_2) dX_2 - \int_{H^+}^\infty (X_1 - H) f_1(X_1) dX_1 \right\}, \end{aligned} \quad (A3)$$

which is increasing in R for every H . This implies that I will decrease as R increases (see Fig. 2(b)). $\square$

Proof of Theorem 3. Part I: $R_m \leq R_n$.

$\underline{C}_n(R)$ is convex on R for $R \geq \max\{\underline{R}_n, \lambda(T + \tau_2)\}$. Thus, $D\underline{C}_n(R) \geq 0$ for $R \geq R_n$. To prove that $R_m \leq R_n$, it is sufficient to show that $D\underline{C}_m(R) \geq D\underline{C}_n(R)$ for $R \geq R_n$.

Case 1: $R > R_U$. Then clearly $D\underline{C}_m(R) = D\underline{C}_n(R)$.

Case 2: $R \in (R_L, R_U)$. It follows from (10), (14), (A1), and (A2) that

$$D\underline{C}_m(R) - D\underline{C}_n(R) \approx h\tau_3 \left\{ \int_{R-I}^\infty f_0(X_0) dX_0 - \int_{R-I}^R \int_0^{R-X_0} f_2(X_2) dX_2 f_0(X_0) dX_0 \right\}$$

$$\begin{aligned}
& + \int_{R-I}^R 0.5\lambda\tau_3 f_2(R-X_0)f_0(X_0) dX_0 \Big\} \\
& + \pi \int_{R-I}^R \left(\int_{R-X_0}^{\infty} f_1(X_1) dX_1 - \int_{R-X_0}^{\infty} f_2(X_2) dX_2 \right) f_0(X_0) dX_0 > 0 \quad (A4)
\end{aligned}$$

(note $\underline{A}(I, R) = 0$ in taking the derivative of $\underline{C}_m(R) - \underline{C}_n(R)$).

Case 3: $R = R_U$. We show that it is not possible that $R_m = R_U$. Let $D^- \underline{C}_n(R_U)$ (resp. $D^- \underline{C}_m(R_U)$) be the first derivative of $\underline{C}_n(R)$ (resp. $\underline{C}_m(R)$) as R approaches R_U from the left, and $D^+ \underline{C}_n(R_U)$ the first derivative of $\underline{C}_n(R)$ as R approaches R_U from the right. Then,

$$D^- \underline{C}_m(R_U) - D^+ \underline{C}_n(R_U) = D^- \underline{C}_m(R_U) - D^- \underline{C}_n(R_U) > 0, \quad (A5)$$

which follows from convexity of $\underline{C}_n(R)$ and (A4). This implies that R_m cannot occur at R_U .

Case 4: $R \leq R_L$. It is not possible that $R_m \leq R_L$. If we require that $R \geq \max\{\underline{R}_n, \lambda(T + \tau_2)\}$ for $\underline{C}_m(R)$, then $\underline{A}(0, R)$ is increasing on R as indicated by (A3), implying that $R_L < \max\{\underline{R}_n, \lambda(T + \tau_2)\}$ (see also Fig. 3).

Part II: $R_f \leq R_m$.

$\underline{C}_f(R)$ is convex on R for $R \geq \max\{\underline{R}_f, \lambda\tau_1\}$. Thus, $D\underline{C}_f(R) \leq 0$ for $R \leq R_f$. To prove that $R_f \leq R_m$, it is sufficient to show that $D\underline{C}_m(R) \leq D\underline{C}_f(R)$ for $R \leq R_f$.

Case 1: $R > R_U$. It follows from (10), (11), (A1), and (A2) that

$$\begin{aligned}
D\underline{C}_m(R) - D\underline{C}_f(R) &= D\underline{C}_n(R) - D\underline{C}_f(R) \\
&\approx h\tau_3 \left(\int_0^{R-X_0} f_2(X_2) dX_2 f_0(X_0) dX_0 - 1.0 - \int_0^R 0.5\lambda\tau_3 f_2(R-X_0)f_0(X_0) dX_0 \right) \\
&+ \pi \int_0^R \left(\int_{R-X_0}^{\infty} f_2(X_2) dX_2 - \int_{R-X_0}^{\infty} f_1(X_1) dX_1 \right) f_0(X_0) dX_0 < 0.
\end{aligned}$$

Case 2: $R \in (R_L, R_U)$. It follows from (11), (14), (A1), and (A2) that

$$\begin{aligned}
D\underline{C}_m(R) - D\underline{C}_f(R) &\approx h\tau_3 \left\{ \int_0^{R-I} \int_0^{R-X_0} f_2(X_2) dX_2 f_0(X_0) dX_0 - \int_0^{R-I} f_0(X_0) dX_0 \right. \\
&\quad \left. - \int_0^{R-I} 0.5\lambda\tau_3 f_2(R-X_0)f_0(X_0) dX_0 \right\} \\
&+ \pi \int_0^{R-I} \left(\int_{R-X_0}^{\infty} f_2(X_2) dX_2 - \int_{R-X_0}^{\infty} f_1(X_1) dX_1 \right) f_0(X_0) dX_0 < 0.
\end{aligned}$$

(Note $\underline{A}(I, R) = 0$ in taking the derivative of $\underline{C}_m(R) - \underline{C}_f(R)$.)

Case 3: $R = R_U$. It is not possible as shown above that $R_m = R_U$.

Case 4: $R \leq R_L$. $R_m \leq R_L$ is not possible as explained above. $\square$

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