

ON MINIMUM CRITICALLY n -EDGE-CONNECTED GRAPHS*

MARGARET B. COZZENS† AND SHU-SHIH Y. WU‡

Abstract. Let n be an integer with $n \geq 2$. A graph G is called critically n -edge-connected if the edge-connectivity $\lambda(G) = n$ and for any vertex v of G , $\lambda(G - v) = n - 1$. The sizes of critically n -edge-connected graphs are important and interesting in applications in communication networks. The maximum graphs with this property have been characterized [2]. In this paper, we first discuss some properties of minimum graphs, then show that the problem of finding a minimum critically n -edge-connected spanning subgraph of a given graph G is NP-complete.

Key words. graph theory, edge-connectivity $\lambda(G)$, connectivity $\kappa(G)$, NP-completeness

AMS(MOS) subject classification. 05C41

1. Introduction. Let n be a fixed integer with $n \geq 2$. A graph G shall be called n -edge-connected if the edge-connectivity $\lambda(G) = n$. A graph G is called *critically n -edge-connected* if G is n -edge-connected and for any vertex v of G , $\lambda(G - v) = n - 1$. A graph G is called n -connected if the vertex connectivity, $\kappa(G) = n$. A graph G is called *critically n -connected* if G is n -connected and for any vertex v in G , $\kappa(G - v) = n - 1$. A graph G is a *minimum (maximum) critically n -edge-connected graph* if no critically n -edge-connected graphs with the same number of vertices has fewer (more) edges than G .

In a communication network and circuit design, reliability is often determined by the connectivity and edge-connectivity of the corresponding graph. Therefore it is important to investigate, for fixed n , critically n -connected graphs ([3], [7]), and critically n -edge-connected graphs. We characterized the maximum graphs in a subset of critically n -edge-connected graphs, for each $n \geq 2$ in [2]. Here we investigate the minimum critically n -edge-connected graphs.

We use $\{x\}$ to denote the least integer greater than or equal to x , and $[x]$ the greatest integer less than or equal to x .

2. An example of a minimum critically n -edge-connected graph. For any fixed integers $n, m, m \geq n + 1$, Harary [5] constructed classes of graphs $H_{n,m}$, that are minimum n -connected. These same graphs are minimum critically n -edge-connected graph with order m . $H_{n,m}$ is constructed as follows:

Case 1. n is even. Let $n = 2r$. Then $H_{2r,m}$ has vertices $0, 1, 2, 3, \dots, m - 1$ and two vertices i and j are adjacent if $i - r \leq j \leq i + r$ (where addition is taken modulo m). $H_{4,8}$ is shown in Fig. 1.

Case 2. n is odd ($n > 1$), m is even. Let $n = 2r + 1$ ($r > 0$). Then $H_{2r+1,m}$ is constructed by first drawing $H_{2r,m}$, and then adding edges joining vertex i to vertex $i + m/2$ for $1 \leq i < m/2$. $H_{5,8}$ is shown in Fig. 2.

Case 3. n is odd ($n > 1$), m is odd. Let $n = 2r + 1$ ($r > 0$). Then $H_{2r+1,m}$ is constructed by first drawing $H_{2r,m}$, and then adding edges $[0, (m - 1)/2]$ and $[0, (m + 1)/2]$, and $[i, i + (m + 1)/2]$ for $1 \leq i < (m - 1)/2$. $H_{5,9}$ is shown in Fig. 3.

In Case 1 and Case 2, $\deg_{H_{n,m}}(i) = n$, for all $i \in V(H_{n,m})$ so that $|E(H_{n,m})| = \frac{1}{2} \sum_{i \in V(H_{n,m})} \deg_{H_{n,m}}(i) = \frac{1}{2} n \cdot m$.

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† Department of Mathematics, Northeastern University, Boston, Massachusetts 02115.

‡ Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan, Republic of China.

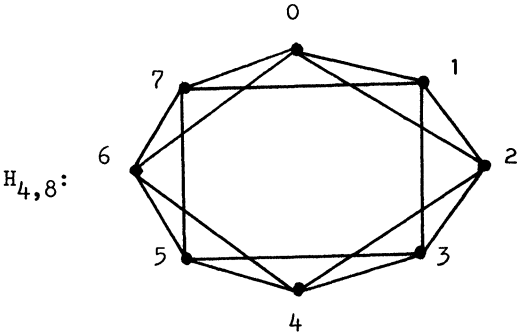


FIG. 1

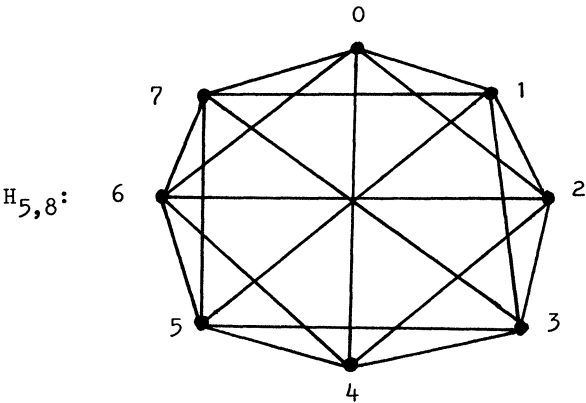


FIG. 2

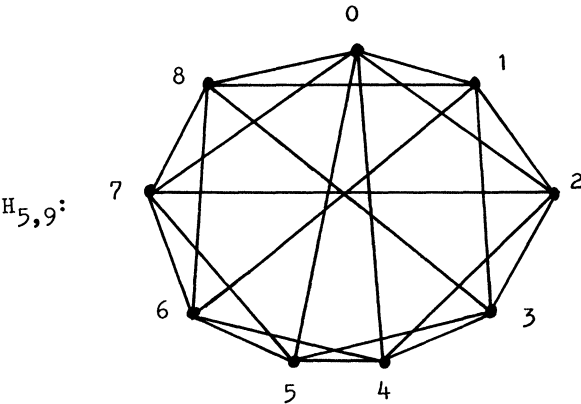


FIG. 3

In Case 3, $\deg_{H_{n,m}}(i) = n$, for $i = 1, 2, \dots, m-1$, and $\deg_{H_{n,m}}(0) = n+1$. So that $\sum_{i \in V(H_{n,m})} \deg_{H_{n,m}}(i) = nm + 1 = 2 \cdot |E|$. So $|E(H_{n,m})| = (nm + 1)/2$.

Therefore for any fixed integers $n, m, m \geq n+1$, $|E(H_{n,m})| = \{nm/2\}$.

Now we show that $H_{n,m}$ is a minimum critically n -edge-connected graph. $\delta(G)$ is the least degree over all vertices of G .

THEOREM 1. *The graph $H_{n,m}$ is n -connected [5].*

From the construction of $H_{n,m}$, it is clear that $\delta(H_{n,m}) = n$, and since

$$n \leq \kappa(H_{n,m}) \leq \lambda(H_{n,m}) \leq \delta(H_{n,m}) = n,$$

we have $\lambda(H_{n,m}) = \delta(H_{n,m}) = n$. Therefore, we have the following theorem.

THEOREM 2. *The graph $H_{n,m}$ is n -edge-connected.*

For vertices j and k in a graph G , a (j, k) -cutset of G is a vertex cutset T such that j and k are in different components of $G - T$.

THEOREM 3. *The graph $H_{n,m} - \{i\}$ is $(n-1)$ -connected, for any vertex i in $H_{n,m}$.*

Proof. Let $n = 2r$ if n is even, $2r+1$ if n is odd. The minimum degree, $\delta(H_{n,m} - \{i\})$, is $n-1$ so there exists a vertex cutset of size $n-1$. We will show that there is no vertex cutset with fewer than $n-1$ vertices.

Suppose there exists a vertex cutset T such that $2 \leq |T| < n-1$. Let j and k be vertices belonging to different components of $(H_{n,m} - \{i\}) - T$ such that if i is between j and k then $0 \leq k < i < j$, and if i is not between j and k then $j < k$. Define two vertex sets A and B in $H_{n,m} - \{i\}$ (addition is modulo m):

$$A = \{j, j+1, j+2, \dots, k-1, k\},$$

$$B = \{k, k+1, k+2, \dots, i-1, i+1, \dots, j-1, j\}.$$

Note that $A \cup B = V(H_{n,m} - \{i\})$ and $A \cap B = \{j, k\}$. Since $|T| < n-1$, $|T| < 2r$. Therefore not both $T \cap A$ and $T \cap B$ can have r or more elements.

Case 1. $|T \cap A| < r$. $A - T = A - (A \cap T)$ so no more than $r-1$ consecutive elements are removed from A by T . Hence $A - T$ has a sequence of distinct vertices starting with j and ending with k with no difference greater than r between any pair of consecutive vertices. This sequence is a (j, k) -path in $(H_{n,m} - \{i\}) - T$, a contradiction to T being a (j, k) -cutset.

Case 2. $|T \cap B| < r$.

Subcase (i). $|T \cap B| < r-1$. As in Case 1, no more than $r-2$ consecutive elements are removed from B by T . Hence $B - T$ has a sequence of distinct vertices starting with k and ending with j , and the difference between any two consecutive vertices is at most $(r-1)+1 = r$. (There is an additional 1 in the gap between $i-1$ and $i+1$.) This sequence is a (k, j) -path of $(H_{n,m} - \{i\}) - T$, a contradiction to T being a (j, k) -cutset.

Subcase (ii). $|T \cap B| = r-1$. Since j and k are not in T ,

$$|T \cap A| = |T| - |T \cap B| < n-1 - (r-1) = n-r < r+1.$$

If $|T \cap A| < r$ then Case 1 applies. Therefore $|T \cap A| = r$. $|A| + |B| = (m+2) - 1 = m+1$. Therefore not both of $|A|$ and $|B|$ can be greater than $\{(m+1)/2\}$, but at least one is greater than or equal to $\{(m+1)/2\}$.

Suppose $|A| \geq \{(m+1)/2\}$. If there exists a sequence of vertices in $A - T$ beginning with j and ending with k such that no pair of consecutive terms has a difference $\geq r+1$, then this sequence is a (j, k) -path in $(H_{n,m} - \{i\}) - T$, a contradiction to T being

a (j, k) -cutset. Thus we may assume that every sequence of vertices in $A - T$ beginning with j and ending with k has a pair of consecutive terms with difference $\geq r + 1$. In fact, since $|T \cap A| = r$, this difference is exactly $r + 1$, and there is only one such consecutive pair with difference $r + 1$. All other consecutive pairs have a difference of 1. Call the pair of vertices with difference $r + 1$, s and $s + r + 1$ in the sequence $A - T$. Thus we can write $A - T$ as $\{j, j + 1, j + 2, \dots, s - 1, s, s + r + 1, \dots, k - 1, k\}$. (Note that j can be s .) Split $A - T$ into two parts:

$$A_1 = \{j, j + 1, \dots, s - 1, s\} \quad \text{and} \quad A_2 = \{s + r + 1, s + r + 2, \dots, k - 1, k\}.$$

The difference in consecutive terms in each A_i is 1, so there is an edge in $(H_{n,m} - \{i\}) - T$ between them. But $m \geq n + 1 \geq 2r + 1$ implies $m/2 \geq r + \frac{1}{2} > r$ if m is even, and $(m + 1)/2 \geq r + 1 > r$ if m is odd. Thus there are some $a_1 \in A_1$ and $a_2 \in A_2$ such that $a_2 = a_1 + [(m + 1)/2]$. The sequence $\{j, j + 1, \dots, a_1, a_2, \dots, k - 1, k\}$ is a (j, k) -path in $(H_{n,m} - \{i\}) - T$, a contradiction to T being a (j, k) -cutset.

If $|B| \geq \{(m + 1)/2\}$ then the same argument applies since $n - 1 > |T| \geq 2$ implies $n \geq 4$, hence $r \geq 2$, so there is an edge between $i - 1$ and $i + 1$ in $H_{n,m} - \{i\}$.

All that remains is to show that no vertex cutset of only one vertex exists for $H_{n,m} - \{i\}$. Suppose $T = \{p\}$ is a vertex cutset of $H_{n,m} - \{i\}$. Since $|T| < n - 1$, $n \geq 3$.

Case 1. If $p = i - 1$ (equivalently $i = p + 1$), then $i + 1, i + 2, \dots, m - 1, 0, \dots, i - 2$ is a path containing all the vertices of $H_{n,m} - \{i, p\}$, a contradiction to T being a cutset of $H_{n,m} - \{i\}$.

Case 2. $p \neq i - 1$ and $p \neq i + 1$. Without loss of generality assume $i < p \leq m - 1$. Now $P_1 = p + 1, p + 2, \dots, m - 1, 0, 1, \dots, i - 1$ is a path and $P_2 = i + 1, i + 2, \dots, p - 1$ is a path in $(H_{n,m} - \{i\}) - \{p\}$. If n is even then $r \geq 2$ and

$$\{i - 1, i + 1\} \in E(H_{n,m})$$

so there is only one component of $(H_{n,m} - \{i\}) - \{p\}$. If n is odd then there exists an edge between some x in P_1 and $x + [(m + 1)/2]$ in P_2 , again contradicting $T = \{p\}$ being a cutset of $H_{n,m} - \{i\}$. Therefore, there exists no cutset with only one vertex, and the theorem is proved. QED

Since $n - 1 \leq \kappa(H_{n,m} - \{i\}) \leq \lambda(H_{n,m} - \{i\}) \leq \delta(H_{n,m} - \{i\}) = n - 1$, we have $\lambda(H_{n,m} - \{i\}) = \delta(H_{n,m} - \{i\}) = n - 1$. Therefore, we have the following theorem.

THEOREM 4. *The graph $H_{n,m} - \{i\}$ is $(n - 1)$ -edge-connected, for any vertex i in $H_{n,m}$.*

Now we can show the main theorem of this section.

THEOREM 5. *For any given positive integers m, n , $m \geq n + 1$, there exists a minimum critically n -edge-connected graph with order m .*

Proof. By Theorem 2 and Theorem 4, $H_{n,m}$ is critically n -edge-connected. $|E(H_{n,m})| = \{nm/2\}$ and $|V(H_{n,m})| = m$.

Let $G = (V, E)$ be a critically n -edge-connected graph with $|V| = m$. Thus $\lambda(G) = n$, and for any vertex v in G , $\lambda(G) \leq \delta(G) \leq \deg_G v$. Hence

$$2 \cdot |E| = \sum_{v \in V(G)} \deg_G v \geq m \cdot \delta(G) = m \cdot n.$$

So $|E| \geq mn/2$. $|E|$ is an integer, hence $|E| \geq \{mn/2\} = |E(H_{n,m})|$. So no critically n -edge-connected graph with m vertices has fewer edges than $H_{n,m}$. Therefore $H_{n,m}$ is a minimum critically n -edge-connected graph with order m . QED

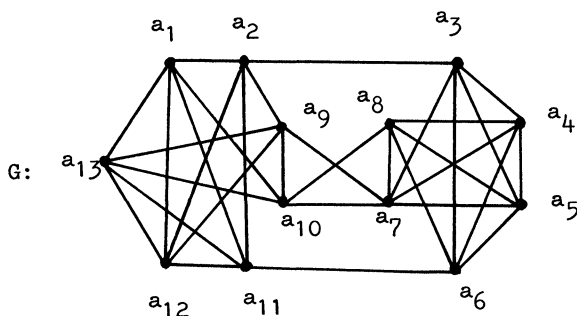


FIG. 4

3. Characterizations of minimum critically n -edge-connected graphs. In addition to $H_{n,m}$, there are other minimum critically n -edge-connected graphs. First we discuss some properties of minimum critically n -edge-connected graphs.

From the discussion of the graph $H_{n,m}$, it is easy to obtain the following lemma.

LEMMA 6. *If G is a minimum critically n -edge-connected graph with order m , then $|E(G)| = \{mn/2\}$.*

A graph G is called *almost regular of degree n* if there is at most one vertex of degree $n + 1$ and all other vertices have degree n . Clearly, an n -regular graph is almost regular of degree n .

THEOREM 7. *If $G = (V, E)$ is a minimum critically n -edge-connected graph, then G is almost regular of degree n . The proof follows from Lemma 6.*

The converse of Theorem 7 is not true. G , as shown in Fig. 4, is almost regular of degree 5, but G is not critically 5-edge-connected, since $\lambda(G) = 5$, and $\lambda(G - a_{10}) = 3 \neq 5 - 1$.

If G is n -edge-connected, then the order of G , m , is such that $m \geq n + 1$. For $n + 1 \leq m \leq 2n$, we have a characterization of minimum critically n -edge-connected graphs.

THEOREM 8. *Let the order of G be m . For any n such that $n + 1 \leq m \leq 2n$, $G = (V, E)$ is a minimum critically n -edge-connected graph if and only if G is almost regular of degree n .*

To prove Theorem 8, we will use the following lemma.

LEMMA 9. *If G has m vertices and $\delta(G) \geq \lceil m/2 \rceil$, then $\lambda(G) = \delta(G) - 1$.*

Proof of Theorem 8. By Theorem 7, if G is a minimum critically n -edge-connected graph, then G is almost regular of degree n .

Conversely, if G is almost regular of degree n , then $\delta(G) = n \geq m/2 \geq \lceil m/2 \rceil$. By Lemma 9, we have $\lambda(G) = \delta(G) = n$. For any vertex $u \in V(G)$, $\delta(G - u) = n - 1 \geq m/2 - 1$. Since $n - 1$ is an integer, $n - 1 \geq \lceil m/2 - 1 \rceil$.

Case 1. m is odd.

$$\delta(G - u) = n - 1 \geq \left\lceil \frac{m}{2} - 1 \right\rceil, \quad n - 1 \geq \frac{m + 1}{2} - 1 = \frac{m - 1}{2} = \left\lceil \frac{m - 1}{2} \right\rceil.$$

Case 2. m is even.

$$\delta(G - u) = n - 1 \geq \left\lceil \frac{m}{2} - 1 \right\rceil = \frac{m - 2}{2} = \left\lceil \frac{m - 1}{2} \right\rceil.$$

By Lemma 9, we have $\lambda(G - u) = \delta(G - u) = n - 1$.

$$\begin{aligned} |E(G)| &= \frac{1}{2} \sum_{v \in V(G)} \deg_G v \\ &= \begin{cases} \frac{mn}{2}, \text{ or} \\ \frac{1}{2}((m-1)n + n + 1) = \frac{1}{2}(mn + 1) \end{cases} \\ &= \left\lfloor \frac{mn}{2} \right\rfloor. \end{aligned}$$

Therefore, G is a minimum critically n -edge-connected graph. QED

The reader should note that G need not be n -connected in Theorem 8.

In general, the converse of Theorem 7 is not true, but if the vertex connectivity $\kappa(G) = n$, then we can give a characterization of minimum critically n -edge-connected graphs.

THEOREM 10. *Let $\kappa(G) = n$. $G = (V, E)$ is a minimum critically n -edge-connected graph if and only if G is almost regular of degree n .*

Proof. Let the order of G be m . By Theorem 7, we obtain the “only if part.”

Conversely, if G is almost regular of degree n , then $\delta(G) = n$. Since $n = \kappa(G) \leq \lambda(G) \leq \delta(G) = n$, we have $\lambda(G) = n$.

For any vertex u in G , $\kappa(G - u) \leq \lambda(G - u) \leq \delta(G - u) = n - 1$. Suppose that $\lambda(G - u) < \delta(G - u)$, for some vertex u in G , then $\kappa(G - u) \leq \lambda(G - u) < n - 1$. Thus, the connectivity $\kappa(G) < n$, a contradiction. So for any vertex u in G , we have $\lambda(G - u) = \delta(G - u) = n - 1$.

G is almost regular of degree n , so by the proof of Theorem 8, $|E(G)| = \{mn/2\}$.

Therefore, G is a minimum critically n -edge-connected graph. QED

The condition $\kappa(G) = n$ in Theorem 10 is necessary, since we can find a graph G , the one shown in Fig. 4, which is almost regular of degree n with $\kappa(G) < n$, G is a minimum n -edge-connected graph, but G is not critical with respect to $\lambda(G)$. Here $\kappa(G) = 4$, since $\{a_2, a_9, a_{10}, a_{11}\}$ is a vertex cutset.

For $m \geq 2n + 1$, we can give some characterizations of minimum critically n -edge-connected graphs.

THEOREM 11. *For any given positive integers m, n , $m \geq 2n + 1$, and $|V(G)| = m$, $G = (V, E)$ is a minimum critically n -edge-connected graph if and only if G is almost regular of degree n , and for each vertex u in a vertex cutset T with $|T| \leq n - 1$, $\lambda(G - u) \geq n - 1$.*

Proof. By Theorem 7, if G is a minimum critically n -edge-connected graph, then G is almost regular of degree n . Since G is critical with respect to $\lambda(G)$, for each vertex u in G , $\lambda(G - u) = n - 1$. So “the only if part” is complete.

Conversely, if G is almost regular of degree n , then $\delta(G) = n$. Since $\lambda(G - u) \geq n - 1$ for some vertex u in G , and $\delta(G) = n$, we have $\lambda(G) \geq n - 1$.

Suppose $\lambda(G) = n - 1$. Let E_1 be a minimum edge-cutset and G_1, G_2 be two components of $G - E_1$. $\delta(G) = n$ and $|E_1| = n - 1$, so $|V(G_1)| \geq 2$ and $|V(G_2)| \geq 2$. Since $m \geq 2n + 1$, without loss of generality, we may let $|V(G_1)| \geq n + 1$. Let A be the set of vertices in G_1 which are incident with E_1 . $|A| \leq n - 1$, since $|E_1| = n - 1$. So A is a vertex cutset with $|A| \leq n - 1$, and for any vertex u in A , $\lambda(G - u) \leq n - 2$, a contradiction.

Therefore $\lambda(G) > n - 1$. $n - 1 < \lambda(G) \leq \delta(G) = n$, so $\lambda(G) = n$. Therefore G is n -edge-connected. We show next that G is critically n -edge-connected.

For each vertex u in G , we consider the following two cases for a cutset containing it.

Case 1. u is in a vertex cutset T with $|T| \leq n - 1$, then $\lambda(G - u) \geq n - 1$. Since $\lambda(G - u) \leq \delta(G - u) = n - 1$, we have $\lambda(G - u) = n - 1$.

Case 2. Every vertex cutset containing u has at least n vertices. Suppose $\lambda(G - u) < n - 1$. Let $\hat{E}$ be a minimum edge-cutset of $G - u$, and H_1, H_2 be two components of $(G - u) - \hat{E}$. $|V(H_1)| + |V(H_2)| = m - 1 \geq (2n + 1) - 1 = 2n$. Without loss of generality, let $|V(H_1)| \geq n$. Since $|\hat{E}| < n - 1$, u must be adjacent to some vertices in H_1 and some vertices in H_2 , as shown in Fig. 5.

Let T_1 be the set of vertices in H_1 which are incident with $\hat{E}$. $|T_1| < n - 1$, since $|\hat{E}| < n - 1$. $|V(H_1) - T_1| > 1$. Thus $T_1 \cup \{u\}$ is a vertex cutset of G and $|T_1 \cup \{u\}| \leq n - 1$, a contradiction to the assumption of this case. So $\lambda(G - u) \geq n - 1$. Since $\lambda(G - u) \leq \delta(G - u) = n - 1$, we have $\lambda(G - u) = n - 1$. Therefore G is critical with respect to $\lambda(G)$.

G is almost regular of degree n , by the proof of Theorem 8, $|E(G)| = \{mn/2\}$, where m is the order of G . Therefore, G is a minimum critically n -edge-connected graph with order m . QED

A vertex u of a graph G is called *critical* if u is contained in a minimum vertex cutset. Thus, we have the following lemma.

LEMMA 12. A vertex u in graph G is critical if and only if $\kappa(G - u) = \kappa(G) - 1$.

COROLLARY 13. For any given positive integers m, n , such that $m \geq 2n + 1$, $|V(G)| = m$, and $\kappa(G) \geq n - 1$, $G = (V, E)$ is a minimum critically n -edge-connected graph if and only if G is almost regular of degree n , and for any critical vertex u , $\lambda(G - u) \geq n - 1$.

Next, we give some examples to illustrate Theorem 11 and Corollary 13.

Example 1. G is shown in Fig. 6.

G is almost regular of degree 5, $\kappa(G) = 3$. For any vertex u in a vertex cutset T with $|T| \leq 4$, $\lambda(G - u) \geq 4$. By Theorem 11, G is a minimum critically 5-edge-connected graph.

Example 2. G is shown in Fig. 7.

G is almost regular of degree 5, $\kappa(G) = 4$, and for any critical vertex u , $\lambda(G - u) \geq 4$. By Corollary 13, G is a minimum critically 5-edge-connected graph.

Example 3. G is shown in Fig. 8.

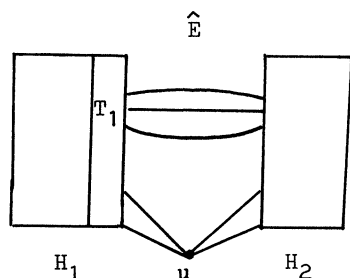


FIG. 5

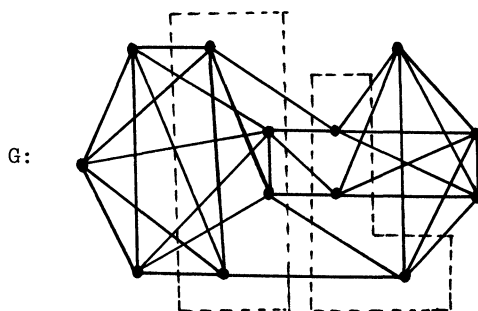


FIG. 6

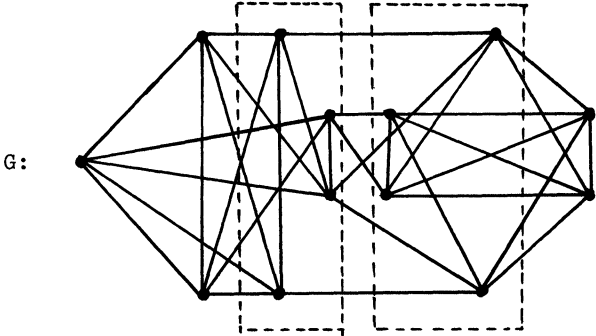


FIG. 7

G is almost regular of degree 5, $\kappa(G) = 3$, u is in a vertex cutset S , $|S| = 4$, $\lambda(G - u) = 3 < 5 - 1$. By Theorem 11 G is not a minimum critically 5-edge-connected graph. In fact, G is not critical with respect to $\lambda(G)$.

Example 4. G is shown in Fig. 9.

G is almost regular of degree 5, $\kappa(G) = 4$, u is a critical vertex, but $\lambda(G - u) = 3 < 5 - 1$. By Corollary 13, G is not a minimum critically 5-edge-connected graph. In fact, G is not critical with respect to $\lambda(G)$.

COROLLARY 14. For positive integers m, n , $m \geq 2n + 1$, at least one of n or m is even, and $|V(G)| = m$, $G = (V, E)$ is a minimum critically n -edge-connected graph if and only if G is regular of degree n , and for any vertex u in a vertex cutset T with $|T| \leq n - 1$, $\lambda(G - u) \geq n - 1$.

Proof. By Theorem 11, G is a minimum critically n -edge-connected graph if and only if G is almost regular of degree n , and for any vertex u in a vertex cutset T with $|T| \leq n - 1$, $\lambda(G - u) \geq n - 1$.

Now, suppose that G is not regular of degree n , but G is almost regular of degree n . Then $\sum_{v \in V(G)} \deg_G v = n(m - 1) + (n + 1) = nm + 1$ is odd, since nm is even. But $\sum_{v \in V(G)} \deg_G v = 2 \cdot |E|$, so we obtain a contradiction. Conversely, if G is regular of degree n , then G is almost regular of degree n . QED

4. NP-completeness. A problem is in the class NP if some nondeterministic machine could, in every instance, find the answer in a number of steps which is bounded by some fixed polynomial in the length of the input data. A problem is NP-complete if it is in NP, and the existence of a deterministic polynomial algorithm, for it would imply the

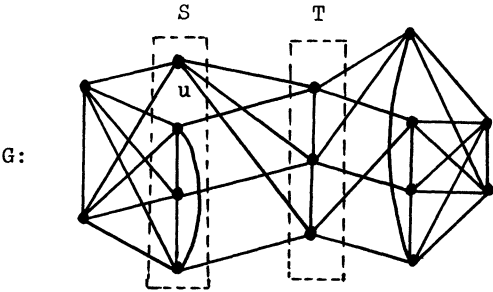


FIG. 8

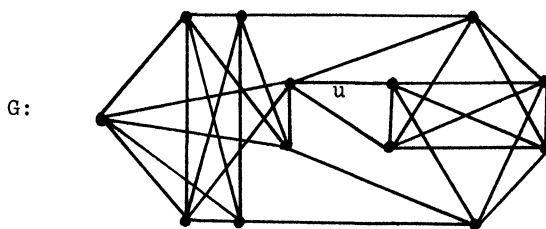


FIG. 9

existence of a deterministic polynomial algorithm for all NP problems. The proof technique for NP-completeness in this section uses the restriction technique. An NP-completeness proof by *restriction* for a given problem $Q \in \text{NP}$ consists simply of showing that Q contains a known NP-complete problem R as a special case.

The main problem in this section is as follows:

Problem n -EDGE.

Instance: $G = (V, E)$, a positive integer n , $1 < n \leq |V| - 1$.

Question: Is there a minimum critically n -edge-connected subgraph $G' = (V, E')$ of G ?

We shall show that Problem n -EDGE is NP-complete. To do this, we will use the NP-complete problem, the *Hamiltonian Circuit Problem* (HC).

Problem HC.

Instance: Graph $G = (V, E)$.

Question: Does G contain a Hamiltonian circuit?

LEMMA 15. $G' = (V, E')$ is a connected spanning subgraph of $G = (V, E)$ and G' is almost regular of degree 2 if and only if G' is a Hamiltonian circuit of G .

Lemma 15 is proved by using the facts that the number of vertices of odd degree for any graph is even, a connected graph with no vertices of odd degree is Eulerian, and an Eulerian circuit in a 2-regular graph must be a Hamiltonian circuit.

There are many polynomial time algorithms for computing the number of components of a graph $G = (V, E)$ including the one given in [8].

Now we consider Problem AR_n .

Problem AR_n .

Instance: $G = (V, E)$, a positive integer n , $1 < n \leq |V| - 1$.

Question: Is there a spanning connected subgraph $G' = (V, E')$, such that G' is almost regular of degree n ?

THEOREM 16. Problem AR_n is NP-complete.

Proof. First, we prove that Problem AR_n is in NP: Given a yes solution (called certificate) to Problem AR_n , we give a polynomial checking algorithm:

Certificate: a subgraph G' of G .

CERTIFICATE-CHECKING ALGORITHM (*Procedure I*):

Begin

1. If $V(G') \neq V(G)$
Then return "No"
Else
2. If $c(G')$ (the number of components of G') ≥ 2
Then return "No"
Else

3. Sort degrees of vertices in G' , such that

$$d_1 \leq d_2 \leq d_3 \leq \cdots \leq d_m;$$
 4. If $(d_1 = d_2 = d_3 = \cdots = d_{m-1} = n)$ and $(d_m = n \text{ or } d_m = n + 1)$
 Then return "Yes"
 Else return "No";
- End.

Step 2 is a polynomial procedure. Step 3 is a sorting procedure, so it also runs in polynomial time. Therefore, the certificate-checking algorithm runs in polynomial time, Problem AR n is in NP.

Let $n = 2$. Problem AR n is reduced to Problem HC by Lemma 15. So a specified type of instance of Problem AR n is NP-complete. By the "restriction technique," Problem AR n is NP-complete. QED

Problem n -EDGE- T .

Instance: $G = (V, E)$, a positive integer n , $1 < |V|/2 \leq n \leq |V| - 1$.

Question: Is there a minimum critically n -edge-connected subgraph $G' = (V, E')$ of G ?

THEOREM 17. *Problem n -EDGE- T is NP-complete.*

Proof. By Theorem 8, Problem n -EDGE- T is the same as Problem AR n . So Problem n -EDGE- T is NP-complete. QED

Problem MENS (Minimum n -edge-connected subgraph).

Instance: $G = (V, E)$ and positive integers $n \leq |V|$ and $b \leq |E|$.

Question: Is there a subset $E' \subseteq E$ with $|E'| \leq b$ such that $G' = (V, E')$ is n -edge-connected?

COROLLARY 18. *Problem MENS is NP-complete [4].*

Therefore, if $G' = (V, E')$ is a certificate, then there is a polynomial time certificate-checking algorithm for Problem MENS, we call it "Procedure II."

THEOREM 19. *Problem n -EDGE is NP-complete.*

Proof. First, we show that Problem n -EDGE is in NP.

Certificate: A subgraph G' of G .

CERTIFICATE-CHECKING ALGORITHM:

Begin

1. If G' is not a spanning connected subgraph of G or G' is not almost regular of degree n —(Call Procedure I)
 Then return "No"
 Else
2. If G' is not n -edge-connected—(Call Procedure II)
 Then return "No"
 Else
3. For $I := 1$ to $|V|$.
 Construct $H' = G' - v_I$, $H = G - v_I$;
 If H' is not $(n - 1)$ -edge-connected—(Call Procedure II (Instance: H , $n - 1$))
 Then return "No" and go to 5.
 Else go to loop 3;
4. Return "Yes";
5. End.

In step 1, Procedure I runs in polynomial time P_1 .

In step 2, Procedure II runs in polynomial time P_2 .

In step 3, the number of computation steps is $O(P_2 \cdot |V|)$.

Therefore, the certificate-checking algorithm runs in polynomial time, Problem n -EDGE is NP.

If we use instance n , $|V|/2 \leq n \leq |V| - 1$, Theorem 17 and the "restriction technique," Problem n -EDGE is NP-complete. QED

We have shown that the problem of finding a minimum critically n -edge-connected spanning subgraph of G is NP-complete. If we place any restrictions on graph G other than the ones imposed in Theorems 8, 10, 11 and Corollary 13 does the problem become easier?

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